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On Nonlinear Periodic Problems with Caputo's Exponential Fractional Derivative

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Abstract

In this article, we employ Mawhin's theory of degree of coincidence to provide an existence result for a class of problems involving non-linear implicit fractional differential equations with the exponentially fractional derivative of Caputo. Two examples are provided to demonstrate the applicability of our results.

Keywords: Caputo's exponential fractional derivative implicit fractional differential equations fractional integral existence coincidence degree theory.
2010 MSC: 34A08, 34B10, 34B40.

1. Introduction

Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary order. For additional information check, for example, the books ([1, 2, 8, 9, 33]), the papers [12, 15, 17, 22, 25, 26, 38, 37, 36, 35, 3, 4, 5, 6, 7] and the references therein.

Fractional differential equations have emerged naturally in a variety of fields in recent years including fractals, chaotic dynamics, modeling and control theory, signal processing, bio-engineering and biomedical applications, and so on. Fractional derivatives are an ideal tool for describing memory and hereditary characteristics of diverse materials and processes. We recommend that the reader examine the books

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[10, 11, 21, 23, 24, 27, 31, 32, 34, 20] and the references therein.

In [18], the authors considered the following fractional-order equations with a new boundary value condition in $\mathbb{R}^n$:

$$
\begin{gathered}
{}^cD_{0+}^{\alpha}u(t)=f\left(t,u(t),{}^cD_{0+}^{\alpha-1}u\right),\quad t\in(0,1),\\
u(0)=Bu(\xi),\quad u(1)=Cu(\eta),
\end{gathered}
$$

where $0<\eta,\xi<1, 1<\alpha\leqslant 2; B, C$ are two n-order nonzero square matrices, ${}^cD_{0+}^{\alpha}$ represents the Caputo differentiation, and $f:[0,1]\times\mathbb{R}^{2n}\rightarrow\mathbb{R}^n$ satisfies Carathéodory conditions.

In [14], the authors investigated some existence result for periodic solutions to the following nonlinear implicit fractional differential equations with Caputo fractional derivatives:

$$
\begin{cases}
{}^cD^{\alpha}\varphi(\vartheta)=\Psi\left(\vartheta,\varphi(\vartheta),{}^cD^{\alpha}\varphi(\vartheta)\right),\quad \vartheta\in\Xi:=[0,\Theta],\Theta>0,0<\alpha\leq 1,\\
\varphi(0)=\varphi(\Theta),
\end{cases}
$$

where $\Psi:\Xi\times\mathbb{R}\times\mathbb{R}\rightarrow\mathbb{R}$ is a continuous function. They based their arguments on the coincidence degree theory.

In [16], Bouriah *et al.* considered the following nonlinear pantograph fractional equation with ψ-Caputo fractional derivative:

$$
\begin{cases}
{}^c\mathfrak{D}_{0^+}^{\alpha;\psi}\varphi(\vartheta)=\Psi(\vartheta,\varphi(\vartheta),\varphi(\varepsilon\vartheta)),\vartheta\in\Xi:=[0,\Theta],\\
\varphi(0)=\varphi(\Theta),
\end{cases}
$$

where ${}^c\mathfrak{D}_{0^+}^{\alpha;\psi}$ denotes the ψ-Caputo fractional derivative of order $0<\alpha<1,\varepsilon\in(0,1)$, and $\Psi:\Xi\times\mathbb{R}\times\mathbb{R}\rightarrow\mathbb{R}$ is a continuous function.

The authors of [19] studied the nonlinear problem:

$$
\begin{cases}
{}^H\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}\varphi(\xi)=\mathcal{F}\left(\xi,\varphi(\theta\xi),I_{0^+}^{\alpha,\psi}\varphi(\varepsilon\xi)\right),\xi\in(0,\Theta],\\
I_{0^+}^{1-v,\psi}\varphi(0)=I_{0^+}^{1-v,\psi}\varphi(\Theta),
\end{cases}
$$

where ${}^H\mathfrak{D}_{0^+}^{\alpha,\beta;\psi}$ denote the ψ-Hilfer fractional derivative of order $0<\alpha\leq 1$ and type $\beta\in[0,1]$. $I_{0^+}^{1-v,\psi}$ is the ψ-Riemann-Liouville fractional integral of order $1-v$, $(v=\alpha+\beta-\alpha\beta)$ and $\theta,\varepsilon\in(0,1]$. Moreover, $\mathcal{F}:\mathfrak{J}\times\mathbb{R}^2\rightarrow\mathbb{R}$ is given continuous function.

Motivated by the mentioned papers, in this paper, we are concerned with the existence of solutions for the following nonlinear fractional differential equation:

$$
{}_{\mathfrak{c}}^{\mathfrak{c}}\mathfrak{D}_{\kappa_1+}^{\delta}\varphi(\vartheta)=g(\vartheta,\varphi(\vartheta),{}_{\mathfrak{c}}^{\mathfrak{c}}\mathfrak{D}_{\kappa_1+}^{\mu}\varphi(\vartheta)),\ \vartheta\in\Xi:=[\kappa_1,\kappa_2], \tag{1}
$$

with the periodic conditions:

$$
\varphi(\kappa_1)=\varphi(\kappa_2)\quad\text{and}\quad {}^{\mathfrak{c}}\mathfrak{D}_{\kappa_1+}^{1}\varphi(\kappa_1)={}^{\mathfrak{c}}\mathfrak{D}_{\kappa_1+}^{1}\varphi(\kappa_2), \tag{2}
$$

where ${}_{\mathfrak{c}}^{\mathfrak{c}}\mathfrak{D}_{\kappa_1+}^{\delta}$and ${}_{\mathfrak{c}}^{\mathfrak{c}}\mathfrak{D}_{\kappa_1+}^{\mu}$ are the Caputo's exponential fractional derivative of orders $1<\delta\leq 2$ and $0<\mu\leq 1$, $g:\Xi\times\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$ is a continuous function.

The following is how this paper is arranged. In Section 2, several remarks are introduced, and some introductory concepts concerning fractional calculus and auxiliary results are addressed. Section 3 presents proof of the main result utilizing Mawhin's degree of coincidence. Finally, two examples are provided in the last section to demonstrate the applicability of our results. Our findings are mostly based on papers [13, 14, 16, 19], where we employed the coincidence degree theory.

2. Preliminaries

This section introduces the notations and definitions that will be utilized throughout the study. Let $\Xi := [\kappa_1, \kappa_2]$ such that $-\infty < \kappa_1 < \kappa_2 < +\infty$.

By $C(\Xi) := C(\Xi, \mathbb{R})$, we denote the Banach space of all continuous functions φ from Ξ into $\mathbb{R}$ with the supremum norm

$$\|\varphi\|_\infty = \sup\{|\varphi(\vartheta)| : \vartheta \in \Xi\}.$$

By $L^1(\Xi)$ we denote the space of Lebesgue-integrable functions $\varphi : \Xi \to \mathbb{R}$ with the norm

$$\|\varphi\|_{L^1} = \int_{\kappa_1}^{\kappa_2} |\varphi(\vartheta)| d\vartheta.$$

As usual, $\mathcal{AC}(\Xi)$ denote the space of absolutely continuous function from Ξ into $\mathbb{R}$. We denote by $\mathcal{AC}^{\mathfrak{n}}_{\mathfrak{e}}(\Xi)$ the space defined by

$$\mathcal{AC}^{\mathfrak{n}}_{\mathfrak{e}}(\Xi) := \left\{ \varphi : \Xi \to \mathbb{R} : {}^{\mathfrak{e}}\mathfrak{D}^{\mathfrak{n}-1}\varphi(\vartheta) \in \mathcal{AC}(\Xi), {}^{\mathfrak{e}}\mathfrak{D} = \mathfrak{e}^{-\vartheta}\frac{d}{d\vartheta} \right\},$$

where $\mathfrak{n} = [\delta]+1$ and $[\delta]$ is the integer part of δ. In particular, if $0 < \delta \leq 1$ then $\mathfrak{n} = 1$ and $\mathcal{AC}^1_{\mathfrak{e}}(\Xi) := \mathcal{AC}_{\mathfrak{e}}(\Xi)$.

Definition 2.1 ([29])**.** *The exponential type fractional integral of order $\delta > 0$ of a function $\mathfrak{g} \in L^1(\Xi)$ is defined by*

$${}^{\mathfrak{e}}J^{\delta}_{\kappa_1+}\mathfrak{g}(\vartheta) = \frac{1}{\Gamma(\delta)} \int_{\kappa_1}^{\vartheta} (\mathfrak{e}^{\vartheta} - \mathfrak{e}^{\varrho})^{\delta-1} \mathfrak{g}(\varrho)\mathfrak{e}^{\varrho} d\varrho, \quad \text{for each } \vartheta \in \Xi.$$

Definition 2.2 ([29])**.** *Let $\delta \geq 0$ and $\mathfrak{g} \in \mathcal{AC}^{\mathfrak{n}}_{\mathfrak{e}}(\Xi)$. The exponential derivatives of Riemann-Liouville type of order $\delta > 0$ is defined by*

$$({}^{\mathfrak{e}}\mathfrak{D}^{\delta}_{\kappa_1+}\mathfrak{g})(\vartheta) := \frac{1}{\Gamma(\mathfrak{n}-\delta)} \left(\mathfrak{e}^{-\vartheta}\frac{d}{d\vartheta}\right)^{\mathfrak{n}} \int_{\kappa_1}^{\vartheta} \left(\mathfrak{e}^{\vartheta} - \mathfrak{e}^{\varrho}\right)^{\mathfrak{n}-\delta-1} \mathfrak{g}(\varrho)\mathfrak{e}^{\varrho} d\varrho, \vartheta \in \Xi,$$

where $\mathfrak{n} = [\delta] + 1$ and $[\delta]$ denotes the integer part of the real number δ.

Definition 2.3 ([29])**.** *Let $\delta \geq 0$ and $\mathfrak{g} \in \mathcal{AC}^{\mathfrak{n}}_{\mathfrak{e}}(\Xi)$. The Caputo's exponential type fractional derivatives of order δ is defined by*

$$({}^{\mathfrak{e}}_{\mathfrak{c}}\mathfrak{D}^{\delta}_{\kappa_1+}\mathfrak{g})(\vartheta) = \frac{1}{\Gamma(n-\delta)} \int_{\kappa_1}^{\vartheta} \left(\mathfrak{e}^{\vartheta} - \mathfrak{e}^{\varrho}\right)^{\mathfrak{n}-\delta-1} \left(\mathfrak{e}^{-\vartheta}\frac{d}{d\vartheta}\right)^{\mathfrak{n}} \mathfrak{g}(\varrho)\frac{d\varrho}{\mathfrak{e}^{-\varrho}},$$

for each $\vartheta \in \Xi$.

Properties 2.4 ([29])**.** If $\gamma, \beta > 0$, then for each $\vartheta \in \Xi$ we have

1. ${}^{\mathfrak{e}}J^{\gamma}_{\mathfrak{a}+}(\mathfrak{e}^{\vartheta} - \mathfrak{e}^{\mathfrak{a}})^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\gamma+\beta+1)}(\mathfrak{e}^{\vartheta} - \mathfrak{e}^{\mathfrak{a}})^{\gamma+\beta}$.
2. ${}^{\mathfrak{e}}J^{\gamma}_{\mathfrak{a}+}(c) = \frac{(\mathfrak{e}^{\vartheta}-\mathfrak{e}^{\mathfrak{a}})^{\gamma}}{\gamma}$.
3. ${}^{\mathfrak{e}}\mathfrak{D}^{\gamma}_{\mathfrak{a}+}(\mathfrak{e}^{\vartheta} - \mathfrak{e}^{\mathfrak{a}})^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\gamma++1)}(\mathfrak{e}^{\vartheta} - \mathfrak{e}^{\mathfrak{a}})^{\beta-\gamma}$.

Lemma 2.5 ([29])**.** *Let $\gamma > 0$ and $\mathfrak{h} \in \mathcal{AC}^{\mathfrak{n}}_{\mathfrak{e}}(\Xi)$. Then,*

$${}^{\mathfrak{e}}J^{\gamma}_{\mathfrak{a}+}({}^{\mathfrak{e}}_{\mathfrak{c}}\mathfrak{D}^{\gamma}_{\mathfrak{a}+}\mathfrak{h}(\vartheta)) = \mathfrak{h}(\vartheta) - \sum_{\mathfrak{i}=0}^{\mathfrak{n}-1} \frac{(\mathfrak{e}^{\vartheta} - \mathfrak{e}^{\mathfrak{a}})^{\mathfrak{i}}}{\mathfrak{i}!} {}^{\mathfrak{e}}\mathfrak{D}^{\mathfrak{i}}\mathfrak{h}(\mathfrak{a}),$$

where $\mathfrak{n} = [\gamma] + 1$.

Lemma 2.6 ([29]). *Let $\gamma > 0$. Then, the differential equation:*

$$ {}^{\mathfrak{e}}_{\mathfrak{c}}\mathfrak{D}^{\gamma}_{\mathfrak{a}+}\mathfrak{h}(\vartheta) = 0, $$

has a solution:

$$ \mathfrak{h}(\vartheta) = \varpi_0 + \varpi_1(\mathfrak{e}^{\vartheta} - \mathfrak{e}^{\mathfrak{a}}) + \varpi_2(\mathfrak{e}^{\vartheta} - \mathfrak{e}^{\mathfrak{a}})^2 + \ldots + \varpi_{\mathfrak{n}-1}(\mathfrak{e}^{\vartheta} - \mathfrak{e}^{\mathfrak{a}})^{\mathfrak{n}-1}, $$

where $\varpi_i \in \mathbb{R}$, $i = 1, \ldots, \mathfrak{n}$ are constants and $\mathfrak{n} = [\gamma] + 1$.

We shall provide definitions and the coincidence degree theory, both of which are required for demonstrations of our outcomes. For more information, see [28].

Definition 2.7 ([28]). *Let Ω and Θ be normed spaces. The linear operator $\mathcal{K} : \mathfrak{Dom}(\mathcal{K}) \subset \Omega \to \Theta$ is called a Fredholm operator of index zero where*

a) $\mathfrak{dim}\ker\mathcal{K} = \mathfrak{codim}\mathfrak{Img}\mathcal{K} < +\infty$.

b) $\mathfrak{Img}\mathcal{K}$ *is a closed subset of* Θ.

Definition 2.7 implies the existence of continuous projectors $\Psi : \Theta \to \Theta$ and $\Upsilon : \Omega \to \Omega$ verifying

$$ \mathfrak{Img}\mathcal{K} = \ker\Psi, \qquad \ker\mathcal{K} = \mathfrak{Img}\Upsilon, \qquad \Theta = \mathfrak{Img}\Psi \oplus \mathfrak{Img}\mathcal{K}, \qquad \Omega = \ker\Upsilon \oplus \ker\mathcal{K}. $$

Then, the restriction of $\mathcal{K}$ to $\mathfrak{Dom}\mathcal{K} \cap \ker\Upsilon$, denoted by $\mathcal{K}_{\Upsilon}$, is an isomorphism onto its image.

Definition 2.8 ([28]). *Let $\mathfrak{G} \subseteq \Omega$ be a bounded subset and $\mathcal{K}$ be a Fredholm operator of index zero with $\mathfrak{Dom}\mathcal{K} \cap \mathfrak{G} \neq \emptyset$. Then, $\mathcal{S} : \overline{\mathfrak{G}} \to \Theta$ is said to be $\mathcal{K}-$compact in $\overline{\mathfrak{G}}$ if*

a) $\Psi\mathcal{S} : \overline{\mathfrak{G}} \to \Theta$ *is continuous and* $\Psi\mathcal{S}\left(\overline{\mathfrak{G}}\right) \subseteq \Theta$ *is bounded.*

b) $(\mathcal{K}_{\Upsilon})^{-1}(id - \Psi)\mathcal{S} : \overline{\mathfrak{G}} \to \Omega$ *is completely continuous.*

Lemma 2.9 ([30]). *Let Ω, and Θ be Banach spaces, $\mathfrak{G} \subset \Omega$ a bounded open set and symmetric with $0 \in \mathfrak{G}$. Assume that $\mathcal{K} : \mathfrak{Dom}\mathcal{K} \subset \Omega \to \Theta$ is a Fredholm operator of index zero with $\mathfrak{Dom}\mathcal{K} \cap \overline{\mathfrak{G}} \neq \emptyset$ and $\mathcal{S} : \Omega \to \Theta$ is a $\mathcal{K}-$compact operator on $\overline{\mathfrak{G}}$. Moreover,*

$$ \mathcal{K}\mu - \mathcal{S}\mu \neq -\varepsilon(\mathcal{K}\mu + \mathcal{S}(-\mu)), $$

for any $\mu \in \mathfrak{Dom}\mathcal{K} \cap \partial\mathfrak{G}$ and any $\varepsilon \in (0, 1]$, where $\partial\mathfrak{G}$ is the boundary of $\mathfrak{G}$ with respect to Ω. If these requirements are met, then there exist at least one solution of $\mathcal{K}\mu = \mathcal{S}\mu$ on $\mathfrak{Dom}\mathcal{K} \cap \overline{\mathfrak{G}}$.

3. Existence of Solutions

We consider the Banach space

$$ \Omega = \{\varphi \in C(\Xi, \mathbb{R}) : \varphi(\vartheta) = {}^{\mathfrak{e}}J^{\delta}_{\kappa_1+}\eta(\vartheta) : \eta \in C(\Xi, \mathbb{R}), \vartheta \in \Xi\}, $$

with the norm

$$ \|\varphi\|_{\Omega} = \max\{\|\varphi\|_{\infty}, \|{}^{\mathfrak{e}}_{\mathfrak{c}}\mathfrak{D}^{\delta}_{\kappa_1+}\varphi\|_{\infty}, \|{}^{\mathfrak{e}}_{\mathfrak{c}}\mathfrak{D}^{\mu}_{\kappa_1+}\varphi\|_{\infty}\}. $$

More details can be found in [16, 13].
Consider the Banach space $\Theta = C(\Xi, \mathbb{R})$ with the norm

$$ \|\eta\|_{\Theta} = \sup\{|\eta(\vartheta)| : \vartheta \in \Xi\}. $$

Consider the linear operator $\mathcal{K} : \mathfrak{Dom}\mathcal{K} \subseteq \Omega \to \Theta$ by

$$ \mathcal{K}\varphi := {}^{\mathfrak{e}}_{\mathfrak{c}}\mathfrak{D}^{\delta}_{\kappa_1+}\varphi, \ 1 < \delta < 2, \tag{3} $$

where

$$\mathfrak{Dom}\mathcal{K} = \Big\{\varphi \in \Omega : {}^{\mathfrak{e}}_{\mathfrak{c}}\mathfrak{D}^{\delta}_{\kappa_1+}\varphi \in \Theta; \varphi(\kappa_1) = \varphi(\kappa_2) \text{ and } {}^{\mathfrak{e}}\mathfrak{D}^{1}_{\kappa_1+}\varphi(\kappa_1) = {}^{\mathfrak{e}}\mathfrak{D}^{1}_{\kappa_1+}\varphi(\kappa_2)\Big\}.$$

Define $\Phi : \Omega \to \Theta$ by

$$\Phi\varphi(\vartheta) := g\left(\vartheta, \varphi(\vartheta), {}^{\mathfrak{e}}_{\mathfrak{c}}\mathfrak{D}^{\mu}_{\kappa_1+}\varphi(\vartheta)\right), \vartheta \in \Xi \text{ and } 0 < \mu < 1. \tag{4}$$

Then the problem (1)-(2) can be rewritten as $\mathcal{K}\varphi = \Phi\varphi$.

Lemma 3.1. *Let $\mathcal{K}$ be defined by (3). Then,*

$$\ker \mathcal{K} = \{\varphi \in \Omega : \varphi(\vartheta) = \varphi(\kappa_1),\ \vartheta \in \Xi\}$$

and

$$\mathfrak{Img}\mathcal{K} = \left\{\omega \in \Theta : \frac{1}{\Gamma(\delta-1)}\int_{\kappa_1}^{\kappa_2}(\mathfrak{e}^{\kappa_2} - \mathfrak{e}^{\varrho})^{\delta-2}\omega(\varrho)\mathfrak{e}^{\varrho}d\varrho = 0\right\}.$$

Proof. By Lemma 2.6, for $\vartheta \in \Xi$, $\mathcal{K}\varphi(\vartheta) = {}^{\mathfrak{e}}_{\mathfrak{c}}\mathfrak{D}^{\delta}_{\kappa_1+}\varphi(\vartheta) = 0$ has the following solution

$$\varphi(\vartheta) = \varpi_0 + \varpi_1(\mathfrak{e}^{\vartheta} - \mathfrak{e}^{\kappa_1});\ \vartheta \in \Xi,$$

where $\varpi_0 = \varphi(\kappa_1)$ and $\varpi_1 = {}^{\mathfrak{e}}\mathfrak{D}^{1}_{\kappa_1+}\varphi(\kappa_1)$.
Since $\varphi \in \mathfrak{Dom}\mathcal{K}$, we can get that

$$\varphi(\vartheta) = \varphi(\kappa_1),\ \vartheta \in \Xi.$$

Then,

$$\ker \mathcal{K} = \{\varphi \in \Omega : \varphi(\vartheta) = \varphi(\kappa_1),\ \vartheta \in \Xi\}.$$

For $\omega \in \mathfrak{Img}\mathcal{K}$, there exists $\varphi \in \mathfrak{Dom}\mathcal{K}$ such that $\omega = \mathcal{K}\varphi \in \Theta$. By Lemma 2.5, we have for each $\vartheta \in \Xi$

$$\varphi(\vartheta) = \varphi(\kappa_1) + {}^{\mathfrak{e}}\mathfrak{D}^{1}_{\kappa_1+}\varphi(\kappa_1)(\mathfrak{e}^{\vartheta} - \mathfrak{e}^{\kappa_1}) + \frac{1}{\Gamma(\delta)}\int_{\kappa_1}^{\vartheta}(\mathfrak{e}^{\vartheta} - \mathfrak{e}^{\varrho})^{\delta-1}\omega(\varrho)\mathfrak{e}^{\varrho}d\varrho.$$

Then,

$${}^{\mathfrak{e}}\mathfrak{D}^{1}_{\kappa_1+}\varphi(\vartheta) = {}^{\mathfrak{e}}\mathfrak{D}^{1}_{\kappa_1+}\varphi(\kappa_1) + \frac{1}{\Gamma(\delta-1)}\int_{\kappa_1}^{\vartheta}(\mathfrak{e}^{\vartheta} - \mathfrak{e}^{\varrho})^{\delta-2}\omega(\varrho)\mathfrak{e}^{\varrho}d\varrho.$$

Since $\varphi \in \mathfrak{Dom}\mathcal{K}$, we can get that ω satisfies

$$\frac{1}{\Gamma(\delta-1)}\int_{\kappa_1}^{\kappa_2}(\mathfrak{e}^{\kappa_2} - \mathfrak{e}^{\varrho})^{\delta-2}\omega(\varrho)\mathfrak{e}^{\varrho}d\varrho = 0.$$

On the other hand, suppose $\omega \in \Theta$ and satisfies

$$\int_{\kappa_1}^{\kappa_2}(\mathfrak{e}^{\kappa_2} - \mathfrak{e}^{\varrho})^{\delta-2}\omega(\varrho)\mathfrak{e}^{\varrho}d\varrho = 0.$$

Then, we get

$$\omega(\vartheta) = {}^{\mathfrak{e}}_{\mathfrak{c}}\mathfrak{D}^{\delta}_{\kappa_1+}\varphi(\vartheta)$$

and

$${}^{\mathfrak{e}}\mathfrak{D}^{1}_{\kappa_1+}\varphi(\vartheta) = {}^{\mathfrak{e}}J^{\delta-1}_{\kappa_1+}\omega(\vartheta).$$

Therefore,

$$\varphi(\kappa_1) = \varphi(\kappa_2)$$

and

$$ {}^{\mathfrak{e}}\mathfrak{D}^{1}_{\kappa_1+}\varphi(\kappa_1) = {}^{\mathfrak{e}}\mathfrak{D}^{1}_{\kappa_1+}\varphi(\kappa_2), $$

which implies that $\varphi \in \mathfrak{Dom}\mathcal{K}$. Thus, $\omega \in \mathfrak{Img}\mathcal{K}$.
Then, we have

$$ \mathfrak{Img}\mathcal{K} = \left\{ \omega \in \Theta : \int_{\kappa_1}^{\kappa_2} (\mathfrak{e}^{\kappa_2} - \mathfrak{e}^{\varrho})^{\delta-2}\omega(\varrho)e^{\varrho}d\varrho = 0 \right\}. $$

□

Lemma 3.2. *Let $\mathcal{K}$ be defined by (3). Then, $\mathcal{K}$ is a Fredholm operator of index zero, and the linear continuous projector operators $\psi_1 : \Omega \to \Omega$ and $\psi_2 : \Theta \to \Theta$ can be defined as*

$$ \psi_1\eta(\vartheta) = \eta(\kappa_1) + {}^{\mathfrak{e}}\mathfrak{D}^{1}_{\kappa_1+}\eta(\kappa_1)(\mathfrak{e}^{\vartheta} - \mathfrak{e}^{\kappa_1}) $$

and

$$ \psi_2 v(\vartheta) = \frac{\delta - 1}{(\mathfrak{e}^{\kappa_2} - \mathfrak{e}^{\kappa_1})^{\delta-1}} \int_{\kappa_1}^{\kappa_2} (\mathfrak{e}^{\kappa_2} - \mathfrak{e}^{\varrho})^{\delta-2} v(\varrho)\mathfrak{e}^{\varrho}d\varrho. $$

Furthermore, the operator $\mathcal{K}^{-1}_{\psi_1} : \mathfrak{Img}\mathcal{K} \to \Omega \cap \ker\psi_1$ can be given by

$$ \mathcal{K}^{-1}_{\psi_1}(v)(\vartheta) = {}^{\mathfrak{e}}\Xi^{\delta}_{\kappa_1+}v(\vartheta). $$

Proof. Clearly, $\mathfrak{Img}\psi_1 = \ker\mathcal{K}$ and $\psi_1^2 = \psi_1$. It follows for each $\eta \in \Omega, \eta = (\eta - \psi_1\eta) + \psi_1\eta$ that $\Omega = \ker\psi_1 + \ker\mathcal{K}$.
Also, we have $\ker\psi_1 \cap \ker\mathcal{K} = 0$. Therefore,

$$ \Omega = \ker\psi_1 \oplus \ker\mathcal{K}. $$

Similarly, for each $v \in \Theta$, $\psi_2^2 v = \psi_2 v$ and $v = (v - \psi_2(v)) + \psi_2(v)$, where $(v - \psi_2(v)) \in \ker\psi_2 = \mathfrak{Img}\mathcal{K}$.
Since $\mathfrak{Img}\mathcal{K} = \ker\psi_2$ and $\psi_2^2 = \psi_2$, then $\mathfrak{Img}\psi_2 \cap \mathfrak{Img}\mathcal{K} = 0$. Thus,

$$ \Theta = \mathfrak{Img}\mathcal{K} \oplus \mathfrak{Img}\psi_2. $$

Hence,

$$ \mathfrak{dim}\ker\mathcal{K} = \mathfrak{dim}\mathfrak{Img}\psi_2 = \mathfrak{codim}\mathfrak{Img}\mathcal{K}, $$

which implies that $\mathcal{K}$ is a Fredholm operator of index zero.
We will demonstrate that $\mathcal{K}^{-1}_{\psi_1} = {}^{\mathfrak{e}}J^{\delta}_{\kappa_1+}$ is the inverse of $\mathcal{K}|_{\mathfrak{Dom}\mathcal{K}\cap\ker\psi_1}$. In fact, for $v \in \mathfrak{Img}\mathcal{K}$, we have

$$ \mathcal{K}\mathcal{K}^{-1}_{\psi_1}(v) = {}^{\mathfrak{e}}_{\mathfrak{c}}\mathfrak{D}^{\delta}_{\kappa_1+}\left({}^{\mathfrak{e}}J^{\delta}_{\kappa_1+}v\right) = v. \tag{5} $$

Furthermore, for $\eta \in \mathfrak{Dom}\mathcal{K} \cap \ker\psi_1$ we get

$$ \mathcal{K}^{-1}_{\psi_1}(\mathcal{K}(\eta(\vartheta))) = {}^{\mathfrak{e}}J^{\delta}_{\kappa_1+}\left({}^{\mathfrak{e}}_{\mathfrak{c}}\mathfrak{D}^{\delta}_{\kappa_1+}\eta(\vartheta)\right) = \eta(\vartheta) - \eta(\kappa_1) - {}^{\mathfrak{e}}\mathfrak{D}^{1}_{\kappa_1+}\eta(\kappa_1)(\mathfrak{e}^{\vartheta} - \mathfrak{e}^{\kappa_1}). $$

Since $\eta \in \mathfrak{Dom}\mathcal{K} \cap \ker\psi_1$, then $\eta(\kappa_1) = 0$ and ${}^{\mathfrak{e}}\mathfrak{D}^{1}_{\kappa_1+}\eta(\kappa_1) = 0$. Therefore,

$$ \mathcal{K}^{-1}_{\psi_1}(\mathcal{K}(\eta(\vartheta))) = \eta(\vartheta). \tag{6} $$

Combining (5) with (6), we know that $\mathcal{K}^{-1}_{\psi_1} = (\mathcal{K}|_{\mathfrak{Dom}\mathcal{K}\cap\ker\psi_1})^{-1}$. □

In the sequel we need the following hypotheses:

(T1) There exist constants $\varpi, \overline{\varpi} > 0$ such that

$$ |g(\vartheta, p, q) - g(\vartheta, \bar{p}, \bar{q})| \leq \varpi|p - \bar{p}| + \overline{\varpi}|q - \bar{q}| \text{ for } \vartheta \in \Xi, \text{ and } p, \bar{p}, q, \bar{q} \in \mathbb{R}. $$

$(T2)$ There exist constants $\gamma > 0$ and $\overline{\gamma} \geq 0$ such that

$$|g(\vartheta, \eta_1, \eta_2) - g(\vartheta, \bar{\eta}_1, \bar{\eta}_2)| \geq \gamma|\eta_1 - \bar{\eta}_1| - \overline{\gamma}|\eta_2 - \bar{\eta}_2|,$$

for $\vartheta \in \Xi$ and $\eta_1, \bar{\eta}_1, \eta_2, \bar{\eta}_2 \in \mathbb{R}$.

Lemma 3.3. *Assume that the condition $(T1)$ is verified. Then, the operator Φ is $\mathcal{K}$-compact on any bounded open set $S \subset \Omega$.*

Proof. Let $S = \{\eta \in \Omega : \|\eta\|_\Omega < K\}$ be a bounded open set where $K > 0$.

Claim 1: *$\psi_2\Phi$ is continuous.*

Let $(\eta_\mathfrak{n})_{\mathfrak{n}\in\mathbb{N}}$ be a sequence such that $\eta_\mathfrak{n} \longrightarrow \eta$ in Ω. Then for each $\varrho \in \Xi$, we have

$$\begin{aligned}&|\psi_2\Phi(\eta_\mathfrak{n})(\varrho) - \psi_2\Phi(\eta)(\varrho)| \\ &\quad\leq \frac{\delta - 1}{(\mathfrak{e}^{\kappa_2} - \mathfrak{e}^{\kappa_1})^{\delta-1}} \int_{\kappa_1}^{\kappa_2} (\mathfrak{e}^{\kappa_2} - \mathfrak{e}^{\varrho})^{\delta-2} |\Phi(\eta_\mathfrak{n})(\varrho) - \Phi(\eta)(\varrho)| \mathfrak{e}^{\varrho} d\varrho.\end{aligned}$$

By $(T1)$, we have

$$\begin{aligned}&|\psi_2\Phi(\eta_\mathfrak{n})(\varrho) - \psi_2\Phi(\eta)(\varrho)| \\ &\quad\leq \frac{\varpi(\delta - 1)}{(\mathfrak{e}^{\kappa_2} - \mathfrak{e}^{\kappa_1})^{\delta-1}} \int_{\kappa_1}^{\kappa_2} (\mathfrak{e}^{\kappa_2} - \mathfrak{e}^{\varrho})^{\delta-2} |(\eta_\mathfrak{n})(\varrho) - (\eta)(\varrho)| \mathfrak{e}^{\varrho} d\varrho \\ &\qquad+ \frac{\overline{\varpi}(\delta - 1)}{(\mathfrak{e}^{\kappa_2} - \mathfrak{e}^{\kappa_1})^{\delta-1}} \int_{\kappa_1}^{\kappa_2} (\mathfrak{e}^{\kappa_2} - \mathfrak{e}^{\varrho})^{\delta-2} |{}^{\mathfrak{e}}_{\mathfrak{c}}\mathfrak{D}^{\mu}_{\kappa_1}(\eta_\mathfrak{n})(\varrho) - {}^{\mathfrak{e}}_{\mathfrak{c}}\mathfrak{D}^{\mu}_{\kappa_1}(\eta)(\varrho)| \mathfrak{e}^{\varrho} d\varrho \\ &\quad\leq \frac{(\delta - 1)(\varpi + \overline{\varpi})\|\eta_\mathfrak{n} - \eta\|_\Omega}{(\mathfrak{e}^{\kappa_2} - \mathfrak{e}^{\kappa_1})^{\delta-1}} \int_{\kappa_1}^{\kappa_2} (\mathfrak{e}^{\kappa_2} - \mathfrak{e}^{\varrho})^{\delta-2} \mathfrak{e}^{\varrho} d\varrho \\ &\quad\leq (\varpi + \overline{\varpi})\|\eta_\mathfrak{n} - \eta\|_\Omega.\end{aligned}$$

Thus, for each $\varrho \in \Xi$, we get

$$|\psi_2\Phi(\eta_\mathfrak{n})(\varrho) - \varphi_2\Phi(\eta)(\varrho)| \longrightarrow 0 \text{ as } \mathfrak{n} \longrightarrow +\infty,$$

and hence,

$$\|\psi_2\Phi(\eta_\mathfrak{n})(\varrho) - \psi_2\Phi(\eta)(\varrho)\|_\Theta \longrightarrow 0 \text{ as } \mathfrak{n} \longrightarrow +\infty.$$

Thus, $\psi_2\Phi$ is continuous.

Claim 2: *$\psi_2\Phi(\overline{S})$ is bounded*

For each $\eta \in \overline{S}$ and $\vartheta \in \Xi$, we have

$$\begin{aligned}&|\psi_2\Phi(\eta)(\vartheta)| \\ &\quad\leq \frac{\delta - 1}{(\mathfrak{e}^{\kappa_2} - \mathfrak{e}^{\kappa_1})^{\delta-1}} \int_{\kappa_1}^{\kappa_2} (\mathfrak{e}^{\kappa_2} - \mathfrak{e}^{\varrho})^{\delta-2} \mathfrak{e}^{\varrho} |g(\varrho, \eta(\varrho), {}^{\mathfrak{e}}_{\mathfrak{c}}\mathfrak{D}^{\mu}_{\kappa_1}\eta(\varrho))| d\varrho \\ &\quad\leq \frac{\delta - 1}{(\mathfrak{e}^{\kappa_2} - \mathfrak{e}^{\kappa_1})^{\delta-1}} \int_{\kappa_1}^{\kappa_2} (\mathfrak{e}^{\kappa_2} - \mathfrak{e}^{\varrho})^{\delta-2} \mathfrak{e}^{\varrho} |g(\varrho, \eta(\varrho), {}^{\mathfrak{e}}_{\mathfrak{c}}\mathfrak{D}^{\mu}_{\kappa_1}\eta(\varrho)) - g(\varrho, 0, 0)| d\varrho \\ &\qquad+ \frac{\delta - 1}{(\mathfrak{e}^{\kappa_2} - \mathfrak{e}^{\kappa_1})^{\delta-1}} \int_{\kappa_1}^{\kappa_2} (\mathfrak{e}^{\kappa_2} - \mathfrak{e}^{\varrho})^{\delta-2} \mathfrak{e}^{\varrho} |g(\varrho, 0, 0))| d\varrho \\ &\quad\leq g^* + \frac{\delta - 1}{(\mathfrak{e}^{\kappa_2} - \mathfrak{e}^{\kappa_1})^{\delta-1}} \int_{\kappa_1}^{\kappa_2} (\mathfrak{e}^{\kappa_2} - \mathfrak{e}^{\varrho})^{\delta-2} \mathfrak{e}^{\varrho} \left(\varpi|\eta(\varrho)| + \overline{\varpi}|{}^{\mathfrak{e}}_{\mathfrak{c}}\mathfrak{D}^{\mu}_{\kappa_1}\eta(\varrho)|\right) d\varrho \\ &\quad\leq g^* + K(\varpi + \overline{\varpi}),\end{aligned}$$

where $g^* = \sup_{\vartheta \in J} |g(\vartheta, 0, 0)|$. Thus,

$$\|\psi_2 \Phi(\eta)\|_\Theta \leq g^* + K(\varpi + \overline{\varpi}) := R.$$

This show that $\psi_2\Phi(\overline{S})$ is a bounded set in Θ.

Claim 3: We show that $\mathcal{K}_{\psi_1}^{-1}(id - \psi_2)\Phi : \overline{S} \to \Omega$ is completely continuous.
In view of the Ascoli-Arzela theorem, we need to prove that $\mathcal{K}_{\psi_1}^{-1}(id - \psi_2)\Phi(\overline{S}) \subset \Omega$ is equicontinuous and bounded. First, for each $\eta \in \overline{S}$ and $\vartheta \in \Xi$, we have

$$\begin{aligned}
&\mathcal{K}_{\psi_1}^{-1}(id - \psi_2)\Phi\eta(\vartheta) \\
&\quad = \mathcal{K}_{\psi_1}^{-1}(\Phi\eta(\vartheta) - \psi_2\Phi\eta(\vartheta)) \\
&\quad = {}^{\mathfrak{e}}J^{\delta}_{\kappa_1+}\Bigg[g(\vartheta, \eta(\vartheta), {}^{\mathfrak{e}}_{\mathfrak{c}}\mathfrak{D}^{\mu}_{\kappa_1+}\eta(\vartheta)) \\
&\qquad - \frac{\delta - 1}{(\mathfrak{e}^{\kappa_2} - \mathfrak{e}^{\kappa_1})^{\delta-1}} \int_{\kappa_1}^{\kappa_2} (\mathfrak{e}^{\kappa_2} - \mathfrak{e}^{\varrho})^{\delta-2} g(\varrho, \eta(\varrho), {}^{\mathfrak{e}}_{\mathfrak{c}}\mathfrak{D}^{\mu}_{\kappa_1+}\eta(\varrho))\mathfrak{e}^{\varrho}d\varrho\Bigg] \\
&\quad = \frac{1}{\Gamma(\delta)} \int_{\kappa_1}^{\vartheta} (\mathfrak{e}^{\vartheta} - \mathfrak{e}^{\varrho})^{\delta-1} g(\varrho, \eta(\varrho), {}^{\mathfrak{e}}_{\mathfrak{c}}\mathfrak{D}^{\mu}_{\kappa_1+}\eta(\varrho))\mathfrak{e}^{\varrho}d\varrho \\
&\qquad - \frac{(\delta - 1)(\mathfrak{e}^{\vartheta} - \mathfrak{e}^{\kappa_1})^{\delta}}{\Gamma(\delta + 1)(\mathfrak{e}^{\kappa_2} - \mathfrak{e}^{\kappa_1})^{\delta-1}} \\
&\qquad \times \int_{\kappa_1}^{\kappa_2} (\mathfrak{e}^{\kappa_2} - \mathfrak{e}^{\varrho})^{\delta-2} g(\varrho, \eta(\varrho), {}^{\mathfrak{e}}_{\mathfrak{c}}\mathfrak{D}^{\mu}_{\kappa_1+}\eta(\varrho))\mathfrak{e}^{\varrho}d\varrho.
\end{aligned}$$

In one hand, for each $\eta \in \overline{S}$ and $\vartheta \in \Xi$, we have

$$\begin{aligned}
&|\mathcal{K}_{\psi_1}^{-1}(id - \psi_2)\Phi\eta(\vartheta)| \\
&\quad \leq \frac{1}{\Gamma(\delta)} \int_{\kappa_1}^{\kappa_2} (\mathfrak{e}^{\kappa_2} - \mathfrak{e}^{\varrho})^{\delta-1} |g(\varrho, \eta(\varrho), {}^{\mathfrak{e}}_{\mathfrak{c}}\mathfrak{D}^{\mu}_{\kappa_1+}\eta(\varrho)) - g(\varrho, 0, 0)|\mathfrak{e}^{\varrho}d\varrho \\
&\qquad + \frac{1}{\Gamma(\delta)} \int_{\kappa_1}^{\kappa_2} (\mathfrak{e}^{\kappa_2} - \mathfrak{e}^{\varrho})^{\delta-1} |g(\varrho, 0, 0)|\mathfrak{e}^{\varrho}d\varrho \\
&\qquad + \frac{(\delta - 1)(\mathfrak{e}^{\kappa_2} - \mathfrak{e}^{\kappa_1})^{\delta}}{\Gamma(\delta + 1)(\mathfrak{e}^{\kappa_2} - \mathfrak{e}^{\kappa_1})^{\delta-1}} \\
&\qquad \times \int_{\kappa_1}^{\kappa_2} (\mathfrak{e}^{\kappa_2} - \mathfrak{e}^{\varrho})^{\delta-2} |g(\varrho, \eta(\varrho), {}^{\mathfrak{e}}_{\mathfrak{c}}\mathfrak{D}^{\mu}_{\kappa_1+}\eta(\varrho)) - g(\varrho, 0, 0)|\mathfrak{e}^{\varrho}d\varrho \\
&\qquad + \frac{(\delta - 1)(\mathfrak{e}^{\kappa_2} - \mathfrak{e}^{\kappa_1})^{\delta}}{\Gamma(\delta + 1)(\mathfrak{e}^{\kappa_2} - \mathfrak{e}^{\kappa_1})^{\delta-1}} \int_{\kappa_1}^{\kappa_2} (\mathfrak{e}^{\kappa_2} - \mathfrak{e}^{\varrho})^{\delta-2} |g(\varrho, 0, 0)|\mathfrak{e}^{\varrho}d\varrho \\
&\quad \leq [g^* + K(\varpi + \overline{\varpi})] \frac{2(\mathfrak{e}^{\kappa_2} - \mathfrak{e}^{\kappa_1})^{\delta}}{\Gamma(\delta + 1)} := \beta_1.
\end{aligned}$$

Therefore,

$$\|\mathcal{K}_{\psi_1}^{-1}(id - \psi_2)\Phi\eta\|_\infty \leq \beta_1. \tag{7}$$

On the other hand, we have

$$\begin{aligned}
{}^{\mathfrak{e}}_{\mathfrak{c}}\mathfrak{D}^{\delta}_{\kappa_1+}\left(\mathcal{K}_{\psi_1}^{-1}(id - \psi_2)\Phi\eta(\vartheta)\right) &= g(\vartheta, \eta(\vartheta), {}^{\mathfrak{e}}_{\mathfrak{c}}\mathfrak{D}^{\mu}_{\kappa_1+}\eta(\vartheta)) - \frac{(\delta - 1)}{(\mathfrak{e}^{\kappa_2} - \mathfrak{e}^{\kappa_1})^{(\delta-1)}} \\
&\quad \times \int_{\kappa_1}^{\kappa_2} (\mathfrak{e}^{\kappa_2} - \mathfrak{e}^{\varrho})^{\delta-2} g(\varrho, \eta(\varrho), {}^{\mathfrak{e}}_{\mathfrak{c}}\mathfrak{D}^{\mu}_{\kappa_1+}\eta(\varrho))\mathfrak{e}^{\varrho}d\varrho,
\end{aligned}$$

which implies that for each $\eta \in \overline{S}$ and $\vartheta \in \Xi$, we have

$$\left|{}^{\mathfrak{e}}_{\mathfrak{c}}\mathfrak{D}^{\delta}_{\kappa_1+}\left(\mathcal{K}^{-1}_{\psi_1}(id-\psi_2)\Phi\eta(\vartheta)\right)\right| \leq 2g^* + 2K(\varpi+\overline{\varpi}) := \beta_2.$$

Thus,

$$\left\|{}^{\mathfrak{e}}_{\mathfrak{c}}\mathfrak{D}^{\delta}_{\kappa_1+}\left(\mathcal{K}^{-1}_{\psi_1}(id-\psi_2)\Phi\eta\right)\right\|_{\infty} \leq \beta_2. \tag{8}$$

Also, we have

$$\begin{aligned}
&{}^{\mathfrak{e}}_{\mathfrak{c}}\mathfrak{D}^{\mu}_{\kappa_1+}\left(\mathcal{K}^{-1}_{\psi_1}(id-\psi_2)\Phi\eta(\vartheta)\right)\\
&= {}^{\mathfrak{e}}J^{(\delta-\mu)}_{\kappa_1+}[g(\vartheta,\eta(\vartheta),{}^{\mathfrak{e}}_{\mathfrak{c}}\mathfrak{D}^{\mu}_{\kappa_1+}\eta(\vartheta))\\
&\quad - \frac{(\delta-1)}{(\mathfrak{e}^{\kappa_2}-\mathfrak{e}^{\kappa_1})^{(\delta-1)}}\int_{\kappa_1}^{\kappa_2}(\mathfrak{e}^{\kappa_2}-\mathfrak{e}^{\varrho})^{\delta-2}g(\varrho,\eta(\varrho),{}^{\mathfrak{e}}_{\mathfrak{c}}\mathfrak{D}^{\mu}_{\kappa_1+}\eta(\varrho))\mathfrak{e}^{\varrho}d\varrho]\\
&= \frac{1}{\Gamma(\delta-\mu)}\int_{\kappa_1}^{\vartheta}(\mathfrak{e}^{\vartheta}-\mathfrak{e}^{\varrho})^{(\delta-\mu-1)}g(\varrho,\eta(\varrho),{}^{\mathfrak{e}}_{\mathfrak{c}}\mathfrak{D}^{\mu}_{\kappa_1+}\eta(\varrho))\mathfrak{e}^{\varrho}d\varrho\\
&\quad - \frac{(\mathfrak{e}^{\vartheta}-\mathfrak{e}^{\kappa_1})^{(\delta-\mu)}(\delta-1)}{\Gamma(\delta-\mu+1)(\mathfrak{e}^{\kappa_2}-\mathfrak{e}^{\kappa_1})^{(\delta-1)}}\\
&\quad \times \int_{\kappa_1}^{\kappa_2}(\mathfrak{e}^{\kappa_2}-\mathfrak{e}^{\varrho})^{\delta-2}g(\varrho,\eta(\varrho),{}^{\mathfrak{e}}_{\mathfrak{c}}\mathfrak{D}^{\mu}_{\kappa_1+}\eta(\varrho))\mathfrak{e}^{\varrho}d\varrho,
\end{aligned}$$

which implies that for each $\eta \in \overline{S}$ and $\vartheta \in \Xi$, we have

$$\begin{aligned}
&\left|{}^{\mathfrak{e}}_{\mathfrak{c}}\mathfrak{D}^{\mu}_{\kappa_1+}\left(\mathcal{K}^{-1}_{\psi_1}(id-\psi_2)\Phi\eta(\vartheta)\right)\right|\\
&\leq \frac{1}{\Gamma(\delta-\mu)}\int_{\kappa_1}^{\kappa_2}(\mathfrak{e}^{\vartheta}-\mathfrak{e}^{\varrho})^{(\delta-\mu-1)}\left|g(\varrho,\eta(\varrho),{}^{\mathfrak{e}}_{\mathfrak{c}}\mathfrak{D}^{\mu}_{\kappa_1+}\eta(\varrho))-g(\varrho,0,0)\right|\mathfrak{e}^{\varrho}d\varrho\\
&\quad + \frac{1}{\Gamma(\delta-\mu)}\int_{\kappa_1}^{\kappa_2}(\mathfrak{e}^{\vartheta}-\mathfrak{e}^{\varrho})^{(\delta-\mu-1)}\left|g(\varrho,0,0)\right|\mathfrak{e}^{\varrho}d\varrho\\
&\quad + \frac{(\mathfrak{e}^{\kappa_2}-\mathfrak{e}^{\kappa_1})^{(\delta-\mu)}(\delta-1)}{\Gamma(\delta-\mu+1)(\mathfrak{e}^{\kappa_2}-\mathfrak{e}^{\kappa_1})^{(\delta-1)}}\\
&\quad \times \int_{\kappa_1}^{\kappa_2}(\mathfrak{e}^{\kappa_2}-\mathfrak{e}^{\varrho})^{\delta-2}\left|g(\varrho,\eta(\varrho),{}^{\mathfrak{e}}_{\mathfrak{c}}\mathfrak{D}^{\mu}_{\kappa_1+}\eta(\varrho))-g(\varrho,0,0)\right|\mathfrak{e}^{\varrho}d\varrho\\
&\quad + \frac{(\mathfrak{e}^{\kappa_2}-\mathfrak{e}^{\kappa_1})^{(\delta-\mu)}(\delta-1)}{\Gamma(\delta-\mu+1)(\mathfrak{e}^{\kappa_2}-\mathfrak{e}^{\kappa_1})^{(\delta-1)}}\int_{\kappa_1}^{\kappa_2}(\mathfrak{e}^{\kappa_2}-\mathfrak{e}^{\varrho})^{\delta-2}\left|g(\varrho,0,0)\right|\mathfrak{e}^{\varrho}d\varrho\\
&\leq [g^*+K(\varpi+\overline{\varpi})]\frac{2(\mathfrak{e}^{\kappa_2}-\mathfrak{e}^{\kappa_1})^{(\delta-\mu)}}{\Gamma(\delta-\mu+1)} := \beta_3.
\end{aligned}$$

Thus,

$$\left\|{}^{\mathfrak{e}}_{\mathfrak{c}}\mathfrak{D}^{\mu}_{\kappa_1+}\left(\mathcal{K}^{-1}_{\psi_1}(id-\psi_2)\Phi\eta\right)\right\|_{\infty} \leq \beta_3. \tag{9}$$

By inequalities (7),(8) and (9), we have

$$\|\mathcal{K}^{-1}_{\psi_1}(id-\psi_2)\Phi\eta\|_{\Omega} \leq \max\{\beta_1,\beta_2,\beta_3\},$$

which shows that $\mathcal{K}^{-1}_{\psi_1}(id-\psi_2)\Phi(\overline{S})$ is uniformly bounded in Ω.

Now, we need to prove that $\mathcal{K}^{-1}_{\psi_1}(id-\psi_2)\Phi(\overline{S})$ is equicontinuous. Furthermore, for $\kappa_1 < \vartheta_1 < \vartheta_2 \leq \kappa_2$ and $\eta \in \overline{S}$, we have firstly

$$|\mathcal{K}^{-1}_{\psi_1}(id-\psi_2)\Phi\eta(\vartheta_2) - \mathcal{K}^{-1}_{\psi_1}(id-\psi_2)\Phi\eta(\vartheta_1)|$$

$$
\begin{aligned}
&\leq \frac{g^* + K(\varpi + \overline{\varpi})}{\Gamma(\delta)} \left[\int_{\vartheta_1}^{\vartheta_2} (\mathfrak{e}^{\vartheta_2} - \mathfrak{e}^{\varrho})^{\delta-1} \mathfrak{e}^{\varrho} d\varrho \right. \\
&\quad \left. + \int_{\mathfrak{a}}^{\vartheta_1} \left| (\mathfrak{e}^{\vartheta_2} - \mathfrak{e}^{\varrho})^{\delta-1} - (\mathfrak{e}^{\vartheta_1} - \mathfrak{e}^{\varrho})^{\delta-1} \right| \mathfrak{e}^{\varrho} d\varrho \right] \\
&\quad + \frac{K(\varpi + \overline{\varpi}) + g^*}{\Gamma(\delta+1)} \left[(\mathfrak{e}^{\vartheta_2} - \mathfrak{e}^{\mathfrak{a}})^{\delta} - (\mathfrak{e}^{\vartheta_1} - \mathfrak{e}^{\mathfrak{a}})^{\delta} \right].
\end{aligned}
$$

Secondly,

$$
\begin{aligned}
&\left| {}^{\mathfrak{c}}_{\mathfrak{c}}\mathfrak{D}^{\delta}_{\kappa_1+} \left(\mathcal{K}^{-1}_{\psi_1} (id - \psi_2) \Phi \eta(t_2) \right) - {}^{\mathfrak{c}}_{\mathfrak{c}}\mathfrak{D}^{\delta}_{\kappa_1+} \left(\mathcal{K}^{-1}_{\psi_1} (id - \psi_2) \Phi \eta(t_1) \right) \right| \\
&\quad \leq \left| g(t_2, \eta(t_2), {}^{\mathfrak{c}}_{\mathfrak{c}}\mathfrak{D}^{\mu}_{\kappa_1+} \eta(t_2)) - g(t_1, \eta(t_1), {}^{\mathfrak{c}}_{\mathfrak{c}}\mathfrak{D}^{\mu}_{\kappa_1+} \eta(t_1)) \right|.
\end{aligned}
$$

Finally, we have

$$
\begin{aligned}
&{}^{\mathfrak{c}}_{\mathfrak{c}}\mathfrak{D}^{\mu}_{\kappa_1+} \left(\mathcal{K}^{-1}_{\psi_1} (id - \psi_2) \Phi \eta(\vartheta) \right) \\
&\quad = {}^{\mathfrak{c}}J^{(\delta-\mu)}_{\kappa_1+} \left| g(t_2, \eta(t_2), {}^{\mathfrak{c}}_{\mathfrak{c}}\mathfrak{D}^{\mu}_{\kappa_1+} \eta(t_2)) - g(t_1, \eta(t_1), {}^{\mathfrak{c}}_{\mathfrak{c}}\mathfrak{D}^{\mu}_{\kappa_1+} \eta(t_1)) \right|.
\end{aligned}
$$

We conclude that as $\vartheta_1 \to \vartheta_2$, the right-hand side of the above three inequalities tends to zero. Thus, $\mathcal{K}^{-1}_{\psi_1}(id - \psi_2)\Phi(\overline{S})$ is equicontinuous in Ω. By the Ascoli-Arzela theorem, $\mathcal{K}^{-1}_{\psi_1}(id - \psi_2)\Phi(\overline{S})$ is relatively compact. Consequently, Φ is $\mathcal{K}$-compact in $\overline{S}$. □

Lemma 3.4. *Suppose that* $(T1)$ *is satisfied. If*

$$
\varpi + \overline{\varpi} \leq \min \left\{ 1, \frac{\Gamma(\delta+1)}{(\mathfrak{e}^{\kappa_2} - \mathfrak{e}^{\kappa_1})^{\delta}}, \frac{\Gamma(\delta-\mu+1)}{(\mathfrak{e}^{\kappa_2} - \mathfrak{e}^{\kappa_1})^{\delta-\mu}} \right\}, \tag{10}
$$

then there exists $\mathcal{H} > 0$ *independent of* ε *such that,*

$$
\mathcal{K}(\eta) - \Phi(\eta) = -\varepsilon \left[\mathcal{K}(\eta) + \Phi(-\eta) \right] \Longrightarrow \|\eta\|_{\Omega} \leq \mathcal{H}, \quad \varepsilon \in (0, 1]. \tag{11}
$$

Proof. Let $\eta \in \Omega$ satisfies (11). Then,

$$
\mathcal{K}(\eta) - \Phi(\eta) = -\varepsilon \mathcal{K}(\eta) - \varepsilon \Phi(-\eta).
$$

Thus,

$$
\mathcal{K}(\eta) = \frac{1}{1+\varepsilon} \Phi(\eta) - \frac{\varepsilon}{1+\varepsilon} \Phi(-\eta). \tag{12}
$$

By employing the definition of $\mathcal{K}$ and Φ, we get for each $\vartheta \in \Xi$

$$
\begin{aligned}
|\mathcal{K}\eta(\vartheta)| &= \left| {}^{\mathfrak{c}}_{\mathfrak{c}}\mathfrak{D}^{\delta}_{\kappa_1+} \eta(\vartheta) \right| \\
&\leq \frac{1}{1+\varepsilon} \left| g(\vartheta, \eta(\vartheta), {}^{\mathfrak{c}}_{\mathfrak{c}}\mathfrak{D}^{\mu}_{\kappa_1+} \eta(\vartheta)) \right| + \frac{\varepsilon}{1+\varepsilon} \left| g(\vartheta, -\eta(\vartheta), -{}^{\mathfrak{c}}_{\mathfrak{c}}\mathfrak{D}^{\mu}_{\kappa_1+} \eta(\vartheta)) \right| \\
&\leq \frac{1}{1+\varepsilon} \left[\left| g(\vartheta, \eta(\vartheta), {}^{\mathfrak{c}}_{\mathfrak{c}}\mathfrak{D}^{\mu}_{\kappa_1+} \eta(\vartheta)) - g(\vartheta, 0, 0) \right| + g^* \right] \\
&\quad + \frac{\varepsilon}{1+\varepsilon} \left[\left| g(\vartheta, -\eta(\vartheta), -{}^{\mathfrak{c}}_{\mathfrak{c}}\mathfrak{D}^{\mu}_{\kappa_1+} \eta(\vartheta)) - g(\vartheta, 0, 0) \right| + g^* \right] \\
&\leq \left(\frac{1}{1+\varepsilon} + \frac{\varepsilon}{1+\varepsilon} \right) g^* + \frac{1}{1+\varepsilon} \left[\varpi |\eta(\vartheta)| + \overline{\varpi} |{}^{\mathfrak{c}}_{\mathfrak{c}}\mathfrak{D}^{\mu}_{\kappa_1+} \eta(\vartheta)| \right] \\
&\quad + \frac{\varepsilon}{1+\varepsilon} \left[\varpi |-\eta(\vartheta)| + \overline{\varpi} |-{}^{\mathfrak{c}}_{\mathfrak{c}}\mathfrak{D}^{\mu}_{\kappa_1+} \eta(\vartheta)| \right]
\end{aligned}
$$

$$\leq g^* + \left(\frac{1}{1+\varepsilon} + \frac{\varepsilon}{1+\varepsilon}\right)\left[\varpi|\eta(\vartheta)| + \overline{\varpi}|{}^{\mathfrak{c}}_{\mathfrak{c}}\mathfrak{D}^{\mu}_{\kappa_1+}\eta(\vartheta)|\right]$$
$$\leq g^* + \left[\varpi|\eta(\vartheta)| + \overline{\varpi}|{}^{\mathfrak{c}}_{\mathfrak{c}}\mathfrak{D}^{\mu}_{\kappa_1+}\eta(\vartheta)|\right]$$
$$\leq g^* + \varpi\|\eta\|_\infty + \overline{\varpi}\|{}^{\mathfrak{c}}_{\mathfrak{c}}\mathfrak{D}^{\mu}_{\kappa_1+}\eta\|_\infty,$$

which implies that

$$\|{}^{\mathfrak{c}}_{\mathfrak{c}}\mathfrak{D}^{\delta}_{\kappa_1+}\eta\|_\infty \leq g^* + \varpi\|\eta\|_\infty + \overline{\varpi}\|{}^{\mathfrak{c}}_{\mathfrak{c}}\mathfrak{D}^{\mu}_{\mathfrak{a}+}\eta\|_\infty. \tag{13}$$

By (12), for each $\vartheta \in \Xi$, we have

$$\eta(\vartheta) = \frac{1}{1+\varepsilon}\mathcal{K}^{-1}_{\psi_1}\Phi(\eta)(\vartheta) - \frac{\varepsilon}{1+\varepsilon}\mathcal{K}^{-1}_{\psi_1}\Phi(-\eta)(\vartheta).$$

Then,

$$\begin{aligned}
|\eta(\vartheta)| &\leq \frac{\int_{\kappa_1}^{\vartheta}(\mathfrak{e}^{\vartheta} - \mathfrak{e}^{\varrho})^{\delta-1}|g(\varrho,\eta(\varrho),{}^{\mathfrak{c}}_{\mathfrak{c}}\mathfrak{D}^{\mu}_{\kappa_1+}\eta(\varrho)) - g(\varrho,0,0)|\mathfrak{e}^{\varrho}d\varrho}{(1+\varepsilon)\Gamma(\delta)} \\
&+ \frac{\varepsilon\int_{\kappa_1}^{\vartheta}(\mathfrak{e}^{\vartheta} - \mathfrak{e}^{\varrho})^{\delta-1}|g(\varrho,-\eta(\varrho),-{}^{\mathfrak{c}}_{\mathfrak{c}}\mathfrak{D}^{\mu}_{\kappa_1+}\eta(\varrho)) - g(\varrho,0,0)|\mathfrak{e}^{\varrho}d\varrho}{(1+\varepsilon)\Gamma(\delta)} \\
&+ \frac{g^*(\mathfrak{e}^{\kappa_2} - \mathfrak{e}^{\kappa_1})^{\delta}}{(1+\varepsilon)\Gamma(\delta+1)} + \frac{\varepsilon g^*(\mathfrak{e}^{\kappa_2} - \mathfrak{e}^{\kappa_1})^{\delta}}{(1+\varepsilon)\Gamma(\delta+1)} \\
&\leq \left(\frac{\varepsilon}{1+\varepsilon} + \frac{1}{1+\varepsilon}\right)\frac{(\mathfrak{e}^{\kappa_2} - \mathfrak{e}^{\kappa_1})^{\delta}}{\Gamma(\delta+1)}\left(\varpi\|\eta\|_\infty + \overline{\varpi}\|{}^{\mathfrak{c}}_{\mathfrak{c}}\mathfrak{D}^{\mu}_{\kappa_1+}\eta\|_\infty\right) \\
&+ \left(\frac{\varepsilon}{1+\varepsilon} + \frac{1}{1+\varepsilon}\right)\frac{g^*(\mathfrak{e}^{\kappa_2} - \mathfrak{e}^{\kappa_1})^{\delta}}{\Gamma(\delta+1)} \\
&= \frac{(\mathfrak{e}^{\kappa_2} - \mathfrak{e}^{\kappa_1})^{\delta}}{\Gamma(\delta+1)}\left(\varpi\|\eta\|_\infty + \overline{\varpi}\|{}^{\mathfrak{c}}_{\mathfrak{c}}\mathfrak{D}^{\mu}_{\kappa_1+}\eta\|_\infty\right) + \frac{g^*(\mathfrak{e}^{\kappa_2} - \mathfrak{e}^{\kappa_1})^{\delta}}{\Gamma(\delta+1)}.
\end{aligned}$$

Hence,

$$\|\eta\|_\infty \leq \left[g^* + \varpi\|\eta\|_\infty + \overline{\varpi}\|{}^{\mathfrak{c}}_{\mathfrak{c}}\mathfrak{D}^{\mu}_{\kappa_1+}\eta\|_\infty\right]\frac{(\mathfrak{e}^{\kappa_2} - \mathfrak{e}^{\kappa_1})^{\delta}}{\Gamma(\delta+1)}. \tag{14}$$

On one hand, for each $\vartheta \in \Xi$, we have

$$\eta(\vartheta) = \frac{1}{1+\varepsilon}\mathcal{K}^{-1}_{\psi_1}\Phi(\eta)(\vartheta) - \frac{\varepsilon}{1+\varepsilon}\mathcal{K}^{-1}_{\psi_1}\Phi(-\eta)(\vartheta).$$

Then,

$${}^{\mathfrak{c}}_{\mathfrak{c}}\mathfrak{D}^{\mu}_{\kappa_1+}\eta(\vartheta) = \frac{1}{1+\varepsilon}{}^{\mathfrak{e}}J^{(\delta-\mu)}_{\kappa_1+}\Phi(\eta)(\vartheta) - \frac{\varepsilon}{1+\varepsilon}{}^{\mathfrak{e}}J^{(\delta-\mu)}_{\kappa_1+}\Phi(-\eta)(\vartheta).$$

Hence,

$$\begin{aligned}
&|{}^{\mathfrak{c}}_{\mathfrak{c}}\mathfrak{D}^{\mu}_{\kappa_1+}\eta(\vartheta)| \\
&\leq \frac{\int_{\kappa_1}^{\vartheta}(\mathfrak{e}^{\vartheta} - \mathfrak{e}^{\varrho})^{(\delta-\mu-1)}|g(\varrho,\eta(\varrho),{}^{\mathfrak{c}}_{\mathfrak{c}}\mathfrak{D}^{\mu}_{\kappa_1+}\eta(\varrho)) - g(\varrho,0,0)|\mathfrak{e}^{\varrho}d\varrho}{(1+\varepsilon)\Gamma(\delta-\mu)}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\varepsilon\int_{\kappa_1}^{\vartheta}(\mathfrak{e}^{\vartheta}-\mathfrak{e}^{\varrho})^{(\delta-\mu-1)}|g(\varrho,-\eta(\varrho),-{}_{\mathfrak{c}}^{\mathfrak{e}}\mathfrak{D}_{\kappa_1^+}^{\mu}\eta(\varrho))-g(\varrho,0,0)|\mathfrak{e}^{\varrho}d\varrho}{(1+\varepsilon)\Gamma(\delta-\mu)}\\
&+\frac{g^*(\mathfrak{e}^{\kappa_2}-\mathfrak{e}^{\kappa_1})^{(\delta-\mu)}}{(1+\varepsilon)\Gamma(\delta-\mu+1)}+\frac{\varepsilon g^*(\mathfrak{e}^{\kappa_2}-\mathfrak{e}^{\kappa_1})^{(\delta-\mu)}}{(1+\varepsilon)\Gamma(\delta-\mu+1)}\\
\le&\left(\frac{\varepsilon}{1+\varepsilon}+\frac{1}{1+\varepsilon}\right)\frac{(\mathfrak{e}^{\kappa_2}-\mathfrak{e}^{\kappa_1})^{(\delta-\mu)}}{\Gamma(\delta-\mu+1)}\left(\varpi\|\eta\|_\infty+\overline{\varpi}\|{}_{\mathfrak{c}}^{\mathfrak{e}}\mathfrak{D}_{\kappa_1^+}^{\mu}\eta\|_\infty\right)\\
&+\left(\frac{\varepsilon}{1+\varepsilon}+\frac{1}{1+\varepsilon}\right)\frac{g^*(\mathfrak{e}^{\kappa_2}-\mathfrak{e}^{\kappa_1})^{(\delta-\mu)}}{\Gamma(\delta-\mu+1)}\\
=&\frac{(\mathfrak{e}^{\kappa_2}-\mathfrak{e}^{\kappa_1})^{(\delta-\mu)}}{\Gamma(\delta-\mu+1)}\left(g^*+\varpi\|\eta\|_\infty+\overline{\varpi}\|{}_{\mathfrak{c}}^{\mathfrak{e}}\mathfrak{D}_{\kappa_1^+}^{\mu}\eta\|_\infty\right).
\end{aligned}
$$

Therefore,

$$
\|{}_{\mathfrak{c}}^{\mathfrak{e}}\mathfrak{D}_{\kappa_1^+}^{\mu}\eta(\vartheta)\|_\infty\le\left[g^*+\varpi\|\eta\|_\infty+\overline{\varpi}\|{}_{\mathfrak{c}}^{\mathfrak{e}}\mathfrak{D}_{\kappa_1^+}^{\mu}\eta\|_\infty\right]\frac{(\mathfrak{e}^{\kappa_2}-\mathfrak{e}^{\kappa_1})^{\delta-\mu}}{\Gamma(\delta-\mu+1)}. \tag{15}
$$

Using the definition of the norm $\|\cdot\|_\Omega$ and the inequalities (13), we see that if $\|\eta\|_\Omega=\|{}_{\mathfrak{c}}^{\mathfrak{e}}\mathfrak{D}_{\kappa_1^+}^{\delta}\eta\|_\infty$, then

$$
\begin{aligned}
\|\eta\|_\Omega &\le g^*+\varpi\|\eta\|_\infty+\overline{\varpi}\|{}_{\mathfrak{c}}^{\mathfrak{e}}\mathfrak{D}_{\kappa_1^+}^{\mu}\eta\|_\infty\\
&\le g^*+(\varpi+\overline{\varpi})\|\eta\|_\Omega,
\end{aligned}
$$

which implies that

$$
\|\eta\|_\Omega\le\frac{g^*}{1-(\varpi+\overline{\varpi})}:=\mathcal{H}_1.
$$

On the other hand, if $\|\eta\|_\Omega=\|\eta\|_\infty$, then (14) implies

$$
\begin{aligned}
\|\eta\|_\Omega &\le [g^*+\varpi\|\eta\|_\infty+\overline{\varpi}\|{}_{\mathfrak{c}}^{\mathfrak{e}}\mathfrak{D}_{\kappa_1^+}^{\mu}\eta\|_\infty]\frac{(\mathfrak{e}^{\kappa_2}-\mathfrak{e}^{\kappa_1})^{\delta}}{\Gamma(\delta+1)}\\
&\le [g^*+\varpi\|\eta\|_\Omega+\overline{\varpi}\|\eta\|_\Omega]\frac{(\mathfrak{e}^{\kappa_2}-\mathfrak{e}^{\kappa_1})^{\delta}}{\Gamma(\delta+1)}\\
&\le [g^*+(\varpi+\overline{\varpi})\|\eta\|_\Omega]\frac{(\mathfrak{e}^{\kappa_2}-\mathfrak{e}^{\kappa_1})^{\delta}}{\Gamma(\delta+1)},
\end{aligned}
$$

and so

$$
\|\eta\|_\Omega \le \frac{g^*}{\frac{\Gamma(\delta+1)}{(\mathfrak{e}^{\kappa_2}-\mathfrak{e}^{\kappa_1})^{\delta}}-(\varpi+\overline{\varpi})}:=\mathcal{H}_2.
$$

And, if $\|\eta\|_\Omega=\|{}_{\mathfrak{c}}^{\mathfrak{e}}\mathfrak{D}_{\kappa_1^+}^{\mu}\eta\|_\infty$, then by inequalities (15) we have

$$
\begin{aligned}
\|\eta\|_X &\le \left[g^*+\varpi\|\eta\|_\infty+\overline{\varpi}\|{}_{\mathfrak{c}}^{\mathfrak{e}}\mathfrak{D}_{\kappa_1^+}^{\mu}\eta\|_\infty\right]\frac{(\mathfrak{e}^{\kappa_2}-\mathfrak{e}^{\kappa_1})^{\delta-\mu}}{\Gamma(\delta-\mu+1)}\\
&\le \left[g^*+\varpi\|\eta\|_X+\overline{\varpi}\|\eta\|_X\right]\frac{(\mathfrak{e}^{\kappa_2}-\mathfrak{e}^{\kappa_1})^{\delta-\mu}}{\Gamma(\delta-\mu+1)}\\
&\le \left[g^*+(\varpi+\overline{\varpi})\|\eta\|_X\right]\frac{(\mathfrak{e}^{\kappa_2}-\mathfrak{e}^{\kappa_1})^{\delta-\mu}}{\Gamma(\delta-\mu+1)},
\end{aligned}
$$

and so

$$
\|\eta\|_\Omega \le \frac{g*}{\frac{\Gamma(\delta-\mu+1)}{(\mathfrak{e}^{\kappa_2}-\mathfrak{e}^{\kappa_1})^{\delta-\mu}}-(\varpi+\overline{\varpi})}:=\mathcal{H}_3.
$$

Therefore,

$$\|\eta\|_\Omega \le \max\{\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3\} := \mathcal{H}.$$

□

Lemma 3.5. *If the conditions* $(T1)$ *and (10) are satisfied, then there is a bounded open set* $S \subset \Omega$ *such that*

$$\mathcal{K}(\eta) - \Phi(\eta) \neq -\varepsilon[\mathcal{K}(\eta) + \Phi(-\eta)],$$

for all $\eta \in \partial S$ *and all* $\varepsilon \in (0,1]$.

Proof. By Lemma 3.4, there exists $\mathcal{H} > 0$ independent of ε where, if η verifies $\mathcal{K}(\eta) - \Phi(\eta) = -\varepsilon[\mathcal{K}(\eta) + \Phi(-\eta)], \varepsilon \in (0,1]$, then $\|\eta\|_\Omega \le \mathcal{H}$. Thus, if

$$S = \{\eta \in \Omega; \|\eta\|_\Omega < \mathcal{F}\}, \tag{16}$$

where $\mathcal{F} > \mathcal{H}$, we conclude that

$$\mathcal{K}(\eta) - \Phi(\eta) \neq -\varepsilon[\mathcal{K}(\eta) - \Phi(-\eta)],$$

for every $\eta \in \partial S = \{\eta \in \Omega; \|\eta\|_\Omega = \mathcal{F}\}$ and $\varepsilon \in (0,1]$. □

Theorem 3.6. *If the conditions* $(T1)$ *and (10) hold, then the problem* (1)-(2) *has at least one solution.*

Proof. The set S given in (16) is symmetric, $0 \in S$ and $\Omega \cap \overline{S} = \overline{S} \neq \emptyset$. Moreover, by Lemma 3.5 and if $(T1)$ is verified, then

$$\mathcal{K}(\eta) - \Phi(\eta) \neq -\varepsilon[\mathcal{K}(\eta) - \Phi(-\eta)],$$

for all $\eta \in \Omega \cap \partial S = \partial S$ and all $\varepsilon \in (0,1]$. And, by Lemma 2.9, we conclude that (1)-(2) has at least one solution. □

4. Uniqueness of Solution

Theorem 4.1. *Consider that the hypotheses* $(T1)$*-*$(T2)$ *are satisfied. If*

$$\max\left\{\left(\frac{\varpi(\mathfrak{e}^{\kappa_2}-\mathfrak{e}^{\kappa_1})^\delta}{\Gamma(\delta+1)} + \frac{\overline{\varpi}(\mathfrak{e}^{\kappa_2}-\mathfrak{e}^{\kappa_1})^{\delta-\mu}}{\Gamma(\delta-\mu+1)}\right); \left(\frac{\overline{\gamma}}{\gamma} + \frac{2(\varpi+\overline{\varpi})(\mathfrak{e}^{\kappa_2}-\mathfrak{e}^{\kappa_1})^\delta}{\Gamma(\delta+1)}\right)\right\} < 1, \tag{17}$$

then the problem (1)-(2) *has a unique solution in* $\mathfrak{Dom}\mathcal{K} \cap \overline{S}$.

Proof. Note that condition (17) is stronger than condition (10). Then, by Theorem 3.6 we obtain that the problem (1)-(2) has at least one solution in $\mathfrak{Dom}\mathcal{K} \cap \overline{S}$.
Now, we prove the uniqueness result. Suppose that the problem (1)-(2) has two different solutions $\eta_1, \eta_2 \in \mathfrak{Dom}\mathcal{K} \cap \overline{S}$. Then, we have for each $\vartheta \in \Xi$

$${}^{\mathfrak{c}}_{\mathfrak{c}}\mathfrak{D}^\delta_{\kappa_1+}\eta_1(\vartheta) = g(\vartheta, \eta_1(\vartheta), {}^{\mathfrak{c}}_{\mathfrak{c}}\mathfrak{D}^\mu_{\kappa_1+}\eta_1(\vartheta)),$$

and

$${}^{\mathfrak{c}}_{\mathfrak{c}}\mathfrak{D}^\delta_{\kappa_1+}\eta_2(\vartheta) = g(\vartheta, \eta_2(\vartheta), {}^{\mathfrak{c}}_{\mathfrak{c}}\mathfrak{D}^\mu_{\kappa_1+}\eta_2(\vartheta)),$$

where

$$\eta_1(\kappa_1) = \eta_1(\kappa_2),\ \eta_2(\kappa_1) = \eta_2(\kappa_2),$$

and

$${}^{\mathfrak{c}}_{\mathfrak{c}}\mathfrak{D}^1_{\kappa_1+}\eta_1(\vartheta)(\kappa_1) = {}^{\mathfrak{c}}_{\mathfrak{c}}\mathfrak{D}^1_{\kappa_1+}\eta_1(\vartheta)(\kappa_2),\ {}^{\mathfrak{c}}_{\mathfrak{c}}\mathfrak{D}^1_{\kappa_1+}\eta_2(\vartheta)(\kappa_1) = {}^{\mathfrak{c}}_{\mathfrak{c}}\mathfrak{D}^1_{\kappa_1+}\eta_2(\vartheta)(\kappa_2).$$

Let $U(\vartheta) = \eta_1(\vartheta) - \eta_2(\vartheta)$, for all $\vartheta \in \Xi$. Then,

$$LU(\vartheta) = {}^{\mathfrak{c}}_{\mathfrak{c}}\mathfrak{D}^\delta_{\kappa_1+}U(\vartheta)$$

$$\begin{aligned} &= {}^{\mathfrak{e}}_{\mathfrak{c}}\mathfrak{D}^{\delta}_{\kappa_1+}\eta_1(\vartheta) - {}^{\mathfrak{e}}_{\mathfrak{c}}\mathfrak{D}^{\delta}_{\kappa_1+}\eta_2(\vartheta) \\ &= g(\vartheta,\eta_1(\vartheta), {}^{\mathfrak{e}}_{\mathfrak{c}}\mathfrak{D}^{\mu}_{\kappa_1+}\eta_1(\vartheta)) - g(\vartheta,\eta_2(\vartheta), {}^{\mathfrak{e}}_{\mathfrak{c}}\mathfrak{D}^{\mu}_{\kappa_1+}\eta_2(\vartheta)). \end{aligned} \tag{18}$$

Using the fact that $\mathfrak{Img}\mathcal{K} = \ker \psi_2$, we have

$$\int_{\kappa_1}^{\kappa_2} (\mathfrak{e}^{\kappa_2} - \mathfrak{e}^{\varrho})^{\delta-2} \left[g(\varrho,\eta_1(\varrho), {}^{\mathfrak{e}}_{\mathfrak{c}}\mathfrak{D}^{\mu}_{\kappa_1+}\eta_1(\varrho)) - g(\varrho,\eta_2(\varrho), {}^{\mathfrak{e}}_{\mathfrak{c}}\mathfrak{D}^{\mu}_{\kappa_1+}\eta_2(\varrho)) \right] \mathfrak{e}^{\varrho} d\varrho$$
$$\times \frac{(\delta-1)}{(\mathfrak{e}^{\kappa_2} - \mathfrak{e}^{\kappa_1})^{(\delta-1)}} = 0.$$

Since g is continuous function, there exist $\vartheta_0 \in \Xi$ such that

$$g(\vartheta_0,\eta_1(\vartheta_0), {}^{\mathfrak{e}}_{\mathfrak{c}}\mathfrak{D}^{\mu}_{\kappa_1+}\eta_1(\vartheta_0)) - g(\vartheta_0,\eta_2(\vartheta_0), {}^{\mathfrak{e}}_{\mathfrak{c}}\mathfrak{D}^{\mu}_{\kappa_1+}\eta_2(\vartheta_0)) = 0.$$

In view of $(T2)$, we have

$$\left|\eta_1(\vartheta_0) - \eta_2(\vartheta_0)\right| \le \frac{\overline{\gamma}}{\gamma}\left| {}^{\mathfrak{e}}_{\mathfrak{c}}\mathfrak{D}^{\mu}_{\kappa_1+}\eta_1(\vartheta_0) - {}^{\mathfrak{e}}_{\mathfrak{c}}\mathfrak{D}^{\mu}_{\kappa_1+}\eta_2(\vartheta_0)\right| \le \frac{\overline{\gamma}}{\gamma}\|\eta_1 - \eta_2\|_{\Omega}.$$

Then,

$$|U(\vartheta_0)| \le \frac{\overline{\gamma}}{\gamma}\|U\|_{\Omega}. \tag{19}$$

On the other hand, by Lemma 2.5, we have

$${}^{\mathfrak{e}}J^{\delta}_{\kappa_1+}({}^{\mathfrak{e}}_{\mathfrak{c}}\mathfrak{D}^{\delta}_{\kappa_1+}U(\vartheta)) = U(\vartheta) - U(\kappa_1) - {}^{\mathfrak{e}}_{\mathfrak{c}}\mathfrak{D}^{1}_{\kappa_1+}U(\kappa_1)(\mathfrak{e}^{\vartheta} - \mathfrak{e}^{\kappa_1}),$$

which implies that

$$U(\kappa_1) + {}^{\mathfrak{e}}_{\mathfrak{c}}\mathfrak{D}^{1}_{\kappa_1+}U(\kappa_1)(\mathfrak{e}^{\vartheta} - \mathfrak{e}^{\kappa_1}) = U(\vartheta_0) - {}^{\mathfrak{e}}J^{\delta}_{\kappa_1+}({}^{\mathfrak{e}}_{\mathfrak{c}}\mathfrak{D}^{\delta}_{\kappa_1+}U(\vartheta_0)),$$

and therefore

$$U(\vartheta) = {}^{\mathfrak{e}}J^{\delta}_{\kappa_1+}({}^{\mathfrak{e}}_{\mathfrak{c}}\mathfrak{D}^{\delta}_{\kappa_1+}U(\vartheta)) - U(\vartheta_0) + {}^{\mathfrak{e}}J^{\delta}_{\kappa_1+}({}^{\mathfrak{e}}_{\mathfrak{c}}\mathfrak{D}^{\delta}_{\kappa_1+}U(\vartheta_0)).$$

Using (19), for every $\vartheta \in \Xi$, we obtain

$$\begin{aligned} |U(\vartheta)| &\le |{}^{\mathfrak{e}}J^{\delta}_{\kappa_1+}({}^{\mathfrak{e}}_{\mathfrak{c}}\mathfrak{D}^{\delta}_{\kappa_1+}U(\vartheta))| + |U(\vartheta_0)| + |{}^{\mathfrak{e}}J^{\delta}_{\kappa_1+}({}^{\mathfrak{e}}_{\mathfrak{c}}\mathfrak{D}^{\delta}_{\kappa_1+}U(\vartheta_0))| \\ &\le \frac{\overline{\gamma}}{\gamma}\|U\|_{\Omega} + \frac{2(\mathfrak{e}^{\kappa_2} - \mathfrak{e}^{\kappa_1})^{\delta}}{\Gamma(\delta+1)}\|{}^{\mathfrak{e}}_{\mathfrak{c}}\mathfrak{D}^{\delta}_{\kappa_1+}U\|_{\infty}. \end{aligned}$$

Then,

$$\|U\|_{\infty} \le \frac{\overline{\gamma}}{\gamma}\|U\|_{\Omega} + \frac{2(\mathfrak{e}^{\kappa_2} - \mathfrak{e}^{\kappa_1})^{\delta}}{\Gamma(\delta+1)}\|{}^{\mathfrak{e}}_{\mathfrak{c}}\mathfrak{D}^{\delta}_{\kappa_1+}U\|_{\infty}. \tag{20}$$

By (18) and $(T1)$, we find that

$$\begin{aligned} \left|{}^{\mathfrak{e}}_{\mathfrak{c}}\mathfrak{D}^{\delta}_{\kappa_1+}U(\vartheta)\right| &= \left|{}^{\mathfrak{e}}_{\mathfrak{c}}\mathfrak{D}^{\delta}_{\kappa_1+}\eta_1(\vartheta) - {}^{\mathfrak{e}}_{\mathfrak{c}}\mathfrak{D}^{\delta}_{\kappa_1+}\eta_2(\vartheta)\right| \\ &= \left|g(\vartheta,\eta_1(\vartheta), {}^{\mathfrak{e}}_{\mathfrak{c}}\mathfrak{D}^{\mu}_{\kappa_1+}\eta_1(\vartheta)) - g(\vartheta,\eta_2(\vartheta), {}^{\mathfrak{e}}_{\mathfrak{c}}\mathfrak{D}^{\mu}_{\kappa_1+}\eta_2(\vartheta)))\right| \\ &\le (\varpi + \overline{\varpi})\, \|W\|_{\Omega}. \end{aligned}$$

Then,

$$\left\|{}^{\mathfrak{e}}_{\mathfrak{c}}\mathfrak{D}^{\delta}_{\kappa_1+}U\right\|_{\infty} \le (\varpi + \overline{\varpi})\, \|U\|_{\Omega}. \tag{21}$$

By (21) and (20) we get,

$$\|U\|_{\infty} \le \left(\frac{\overline{\gamma}}{\gamma} + \frac{2(\varpi + \overline{\varpi})(\mathfrak{e}^{\kappa_2} - \mathfrak{e}^{\kappa_1})^{\delta}}{\Gamma(\delta+1)}\right)\|U\|_{\Omega}.$$

On the other hand, we have

$$\begin{aligned}\left|{}_{\mathfrak{c}}^{\mathfrak{c}}\mathfrak{D}^{\mu}_{\kappa_1^+}U(\vartheta)\right| &= \left|{}_{\mathfrak{c}}^{\mathfrak{c}}\mathfrak{D}^{\delta}_{\kappa_1^+}[{}^{\mathfrak{c}}J^{\delta}_{\kappa_1^+}({}_{\mathfrak{c}}^{\mathfrak{c}}\mathfrak{D}^{\mu}_{\kappa_1^+}U(\vartheta))]\right| \\ &= \left|{}_{\mathfrak{c}}^{\mathfrak{c}}\mathfrak{D}^{\delta}_{\kappa_1^+}[{}^{\mathfrak{c}}J^{\delta}_{\kappa_1^+}({}_{\mathfrak{c}}^{\mathfrak{c}}\mathfrak{D}^{\mu}_{\kappa_1^+}\eta_1(\vartheta))] - {}_{\mathfrak{c}}^{\mathfrak{c}}\mathfrak{D}^{\delta}_{\kappa_1^+}[{}^{\mathfrak{c}}J^{\delta}_{\kappa_1^+}({}_{\mathfrak{c}}^{\mathfrak{c}}\mathfrak{D}^{\mu}_{\kappa_1^+}\eta_2(\vartheta))]\right| \\ &= \Big|g(\vartheta, {}^{\mathfrak{c}}J^{\delta}_{\kappa_1^+}({}_{\mathfrak{c}}^{\mathfrak{c}}\mathfrak{D}^{\mu}_{\kappa_1^+}\eta_1(\vartheta)), {}^{\mathfrak{c}}J^{\delta-\mu}_{\kappa_1^+}({}_{\mathfrak{c}}^{\mathfrak{c}}\mathfrak{D}^{\mu}_{\kappa_1^+}\eta_1(\vartheta))) \\ &\quad - g(\vartheta, {}^{\mathfrak{c}}J^{\delta}_{\kappa_1^+}({}_{\mathfrak{c}}^{\mathfrak{c}}\mathfrak{D}^{\mu}_{\kappa_1^+}\eta_2(\vartheta)), {}^{\mathfrak{c}}J^{\delta-\mu}_{\kappa_1^+}({}_{\mathfrak{c}}^{\mathfrak{c}}\mathfrak{D}^{\mu}_{\kappa_1^+}\eta_2(\vartheta)))\Big| \\ &\le \left(\frac{\varpi(\mathfrak{e}^{\kappa_2}-\mathfrak{e}^{\kappa_1})^{\delta}}{\Gamma(\delta+1)} + \frac{\overline{\varpi}(\mathfrak{e}^{\kappa_2}-\mathfrak{e}^{\kappa_1})^{\delta-\mu}}{\Gamma(\delta-\mu+1)}\right)\|{}_{\mathfrak{c}}^{\mathfrak{c}}\mathfrak{D}^{\mu}_{\kappa_1^+}U\|_{\infty} \\ &\le \left(\frac{\varpi(\mathfrak{e}^{\kappa_2}-\mathfrak{e}^{\kappa_1})^{\delta}}{\Gamma(\delta+1)} + \frac{\overline{\varpi}(\mathfrak{e}^{\kappa_2}-\mathfrak{e}^{\kappa_1})^{\delta-\mu}}{\Gamma(\delta-\mu+1)}\right)\|U\|_{\Omega}.\end{aligned}$$

Then,

$$\|{}_{\mathfrak{c}}^{\mathfrak{c}}\mathfrak{D}^{\mu}_{\kappa_1^+}U\|_{\infty} \le \left(\frac{\varpi(\mathfrak{e}^{\kappa_2}-\mathfrak{e}^{\kappa_1})^{\delta}}{\Gamma(\delta+1)} + \frac{\overline{\varpi}(\mathfrak{e}^{\kappa_2}-\mathfrak{e}^{\kappa_1})^{\delta-\mu}}{\Gamma(\delta-\mu+1)}\right)\|U\|_{\Omega}. \tag{22}$$

Hence, by (17), we conclude that

$$\|U\|_{\Omega} = 0.$$

As a result, for any $\vartheta \in \Xi$ we get

$$U(\vartheta) = 0 \Longrightarrow \eta_1(\vartheta) = \eta_2(\vartheta).$$

□

5. Examples

Example 5.1. *Consider the following problem for non-linear implicit fractional differential equations*

$${}_{\mathfrak{c}}^{\mathfrak{c}}\mathfrak{D}^{\frac{3}{2}}_{0^+}\varphi(\vartheta) = \frac{\mathfrak{e}^{-\vartheta}}{(11+\mathfrak{e}^{\vartheta})}\left[\frac{|\varphi(\vartheta)|}{1+|\varphi(\vartheta)|} - \frac{|{}_{\mathfrak{c}}^{\mathfrak{c}}\mathfrak{D}^{\frac{1}{2}}_{0^+}\varphi(\vartheta)|}{1+|{}_{\mathfrak{c}}^{\mathfrak{c}}\mathfrak{D}^{\frac{1}{2}}_{0^+}\varphi(\vartheta)|}\right] + \mathfrak{e}^{\vartheta},\ \ \vartheta \in \left[0, \frac{\pi}{2}\right], \tag{23}$$

$$\varphi(0) = \varphi\left(\frac{\pi}{2}\right) \quad \textit{and} \quad {}^{\mathfrak{c}}\mathfrak{D}^{1}_{0^+}\varphi(0) = {}^{\mathfrak{c}}\mathfrak{D}^{1}_{0^+}\varphi\left(\frac{\pi}{2}\right). \tag{24}$$

Here we have

$$g(\vartheta, \varphi_1, \varphi_2) = \frac{\mathfrak{e}^{-\vartheta}}{(11+\mathfrak{e}^{\vartheta})}\left(\frac{\varphi_1}{1+\varphi_1} - \frac{\varphi_2}{1+\varphi_2}\right) + \mathfrak{e}^{\vartheta},$$

where $\vartheta \in [0, \pi]$, $\varphi_1, \varphi_2 \in [0, +\infty)$.
Clearly, the function g *is jointly continuous.*
For each $\varphi_1, \bar{\varphi_1}, \varphi_2, \bar{\varphi_2} \in [0, +\infty)$ *and* $\vartheta \in [0, \frac{\pi}{2}]$, *we have*

$$\begin{aligned}|g(\vartheta, \varphi_1, \varphi_2) - g(\vartheta, \bar{\varphi_1}, \bar{\varphi_2})| &\le \frac{\mathfrak{e}^{-\vartheta}}{(11+\mathfrak{e}^{\vartheta})}[|\varphi_1 - \bar{\varphi_1}| + |\varphi_2 - \bar{\varphi_2}|] \\ &\le \frac{1}{12}(|\varphi_1 - \bar{\varphi_1}| + |\varphi_2 - \bar{\varphi_2}|).\end{aligned}$$

Hence, the condition $(T1)$ *is satisfied with* $\varpi = \overline{\varpi} = \dfrac{1}{12}$.
And on the other hand, we have

$$\varpi + \overline{\varpi} = \frac{1}{6}$$

$$< \min\left\{1, \frac{\Gamma\left(\frac{5}{2}\right)}{(e^{\frac{\pi}{2}}-e^{0})^{\frac{3}{2}}}, \frac{\Gamma(2)}{(e^{\frac{\pi}{2}}-e^{0})}\right\}$$
$$= \min\left\{1, \frac{3\sqrt{\pi}}{4(e^{\frac{\pi}{2}}-1)^{\frac{3}{2}}}, \frac{1}{(e^{\frac{\pi}{2}}-e^{0})}\right\}$$
$$= \frac{3\sqrt{\pi}}{4(e^{\frac{\pi}{2}}-1)^{\frac{3}{2}}},$$

which implies that the condition (10) is satisfied. It follows from Theorem 3.6 that the problem (23)–(24) has at least one solution.

Example 5.2. *Consider the following problem:*

$$ {}^{\mathfrak{c}}_{\mathfrak{c}}\mathfrak{D}^{\frac{6}{5}}_{0^+}\varphi(\vartheta) = g\left(\vartheta, \varphi(\vartheta), {}^{\mathfrak{c}}_{\mathfrak{c}}\mathfrak{D}^{\frac{2}{3}}_{0^+}\varphi(\vartheta)\right), \ \vartheta \in J = \left[0, \frac{\pi}{3}\right], \tag{25}$$

$$\varphi(0) = \varphi\left(\frac{\pi}{3}\right) \quad \text{and} \quad {}^{\mathfrak{c}}\mathfrak{D}^{1}_{0^+}\varphi(0) = {}^{\mathfrak{c}}\mathfrak{D}^{1}_{0^+}\varphi\left(\frac{\pi}{3}\right), \tag{26}$$

where

$$g(\vartheta,\varphi_1,\varphi_2) = \frac{\ln(\vartheta+2)}{5\sqrt{\pi}}\varphi_1 + \frac{1}{5}e^{-\vartheta-\pi}\left(\sin\varphi_1 + \frac{1}{\varphi_2+1}\right) + e^{\vartheta}, \quad \vartheta \in \left[0, \frac{\pi}{3}\right],$$

such that $\varphi_1, \varphi_2 \in \mathbb{R}^+$.
Clearly, the function g is jointly continuous.
For any $\varphi_1, \bar{\varphi_1}, \varphi_2, \bar{\varphi_2} \in \mathbb{R}^+$ *and* $\vartheta \in [0, \frac{\pi}{3}]$, *we have*

$$\begin{aligned}|g(\vartheta,\varphi_1,\varphi_2) - g(\vartheta,\bar{\varphi_1},\bar{\varphi_2})| &\le \frac{\ln(\frac{\pi}{3}+2)}{5\sqrt{\pi}}|\varphi_1-\bar{\varphi_1}| + \frac{1}{5\mathfrak{e}^{\pi}}|\sin\varphi_1 - \sin\bar{\varphi_1}| \\ &\quad + \frac{1}{5\mathfrak{e}^{\pi}}|\varphi_2-\bar{\varphi_2}| \\ &\le \frac{\mathfrak{e}^{\pi}\ln(\frac{\pi}{3}+2)+\sqrt{\pi}}{5\sqrt{\pi}\mathfrak{e}^{\pi}}|\varphi_1-\bar{\varphi_1}| + \frac{1}{5\mathfrak{e}^{\pi}}|\varphi_2-\bar{\varphi_2}|.\end{aligned}$$

Hence, the condition $(T1)$ *is satisfied with*

$$\varpi = \frac{\mathfrak{e}^{\pi}\ln(\frac{\pi}{3}+2)+\sqrt{\pi}}{5\sqrt{\pi}\mathfrak{e}^{\pi}}$$

and

$$\overline{\varpi} = \frac{1}{5\mathfrak{e}^{\pi}}.$$

For any $\varphi_1, \bar{\varphi_1}, \varphi_2, \bar{\varphi_2} \in \mathbb{R}$ *and* $\varphi \in [0, \frac{\pi}{3}]$, *we get*

$$\begin{aligned}|g(\vartheta,\varphi_1,\varphi_2) - g(\vartheta,\bar{\varphi_1},\bar{\varphi_2})| &\ge \frac{\ln(2)}{5\sqrt{\pi}}|\varphi_1-\bar{\varphi_1}| - \frac{1}{5\mathfrak{e}^{\pi}}|\sin\varphi_1 - \sin\bar{\varphi_1}| \\ &\quad - \frac{1}{5\mathfrak{e}^{\pi}}|\varphi_2-\bar{\varphi_2}| \\ &\ge \frac{\mathfrak{e}^{\pi}\ln(2)-\sqrt{\pi}}{5\sqrt{\pi}\mathfrak{e}^{\pi}}|\varphi_1-\bar{\varphi_1}| - \frac{1}{5\mathfrak{e}^{\pi}}|\varphi_2-\bar{\varphi_2}|.\end{aligned}$$

Hence, the condition $(T2)$ *is satisfied with* $\gamma = \dfrac{\mathfrak{e}^{\pi}\ln(2)-\sqrt{\pi}}{5\sqrt{\pi}\mathfrak{e}^{\pi}}, \overline{\gamma} = \dfrac{1}{5\mathfrak{e}^{\pi}}$. *Also, we have*

$$\left[\frac{\varpi(\mathfrak{e}^{\frac{\pi}{3}}-1)^{\frac{6}{5}}}{\Gamma(\frac{11}{5})} + \frac{\overline{\varpi}(\mathfrak{e}^{\frac{\pi}{3}}-1)^{\frac{8}{15}}}{\Gamma(\frac{23}{15})}\right] \approx 0.2685 < 1,$$

and

$$\begin{aligned}\left[\frac{\overline{\gamma}}{\gamma} + 2(\varpi + \overline{\varpi})\left(\frac{(\mathfrak{e}^{\frac{\pi}{3}} - 1)^{\frac{6}{5}}}{\Gamma(\frac{6}{5}+1)}\right)\right] &= \frac{\sqrt{\pi}}{\mathfrak{e}^{\pi}\ln(2) - \sqrt{\pi}} \\ &\quad + 2\frac{\mathfrak{e}^{\pi}\ln(\frac{\pi}{3}+2) + 2\sqrt{\pi}}{5\sqrt{\pi}\mathfrak{e}^{\pi}}\left[\frac{(\mathfrak{e}^{\frac{\pi}{3}} - 1)^{\frac{6}{5}}}{\Gamma\left(\frac{11}{5}\right)}\right] \\ &\approx 0.6672 \\ &< 1.\end{aligned}$$

Consequently, Theorem 4.1 implies that the problem (25)–(26) has a unique solution.

Conclusion

In the present research, we have investigated existence and uniqueness criteria for the solutions of a boundary value problem for a class of problems involving non-linear implicit fractional differential equations with the exponentially fractional derivative of Caputo. To achieve the desired results for the given problem, we employ Mawhin's theory of degree of coincidence. Two examples are provided to demonstrate how the major results can be applied. Our results in the given configuration are novel and substantially contribute to the literature on this new field of study. We feel that there are multiple potential study avenues such as coupled systems, problems with integro-differential equations, and many more due to the limited number of publications on implicit differential equations with periodic conditions.

References

[1] S. Abbas, M. Benchohra and G.M. N'Guérékata, Topics in Fractional Differential Equations, Springer-Verlag, New York, 2012.

[2] S. Abbas, M. Benchohra and G M. N'Guérékata, Advanced Fractional Differential and Integral Equations, Nova Science Publishers, New York, 2014.

[3] R.S. Adiguzel, U. Aksoy, E. Karapınar, I.M. Erhan, On the solution of a boundary value problem associated with a fractional differential equation, Math. Meth. Appl. Sci. (2020), 1-12.

[4] R.S. Adiguzel, U. Aksoy, E. Karapınar, I.M. Erhan, Uniqueness of solution for higher-order nonlinear fractional differential equations with multi-point and integral boundary conditions, Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. RACSAM, (2021), 115-155.

[5] R.S. Adiguzel, U. Aksoy, E. Karapınar, I.M. Erhan, On the solutions of fractional differential equations via Geraghty type hybrid contractions, Appl. Comput. Math, **20** (2021), 313-333.

[6] H. Afshari and E. Karapinar, A solution of the fractional differential equations in the setting of b-metric space. *Carpathian Math. Publ.* **13** (2021), 764-774. https://doi.org/10.15330/cmp.13.3.764-774

[7] H. Afshari and E. Karapınar, A discussion on the existence of positive solutions of the boundary value problems via ψ-Hilfer fractional derivative on b-metric spaces, Adv. Difference Equ., **2020** (2020), 616. https://doi.org/10.1186/s13662-020-03076-z

[8] G.A. Anastassiou, Advances on Fractional Inequalities, Springer, New York, 2011.

[9] D. Baleanu, K. Diethelm, E. Scalas, and J.J. Trujillo, Fractional Calculs Models and Numerical Methods, World Scientific Publishing, New York, 2012.

[10] D. Baleanu, Z.B. Güvenç and J.A.T. Machado, New Trends in Nanotechnology and Fractional Calculus Applications, Springer, New York, 2010.

[11] D. Baleanu, J.A.T. Machado and A.C.-J. Luo, Fractional Dynamics and Control, Springer, 2012.

[12] M. Benchohra and S. Bouriah, Existence and stability results for nonlinear boundary value problem for implicit differential equations of fractional order, Moroccan J. Pure. Appl. Anal., **1** (1), (2015), 22-36.

[13] M. Benchohra, S. Bouriah and J.J. Nieto, Existence of periodic solutions for nonlinear implicit Hadamard fractional differential equations, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM, **112** (1) (2018), 25–35.

[14] M. Benchohra, S. Bouriah and J.R. Graef, Nonlinear implicit differential equation of fractional order at resonance, Electron. J. Differential Equations Vol. 2016 (2016), No. 324, pp. 1-10.

[15] M. Benchohra and J.E. Lazreg, Nonlinear fractional implicit differential equations. Commun. Appl. Anal., 17 (2013), 471-482.

[16] S. Bouriah, D. Foukrach, M. Benchohra and J. Graef, Existence and uniqueness of periodic solutions for some nonlinear fractional pantograph differential equations with ψ-Caputo derivative, Arab. J. Math., **10** (2021), 575-587. https://doi.org/10.1007/s40065-021-00343-z

[17] C. Derbazi, H. Hammouche, A. Salim and M. Benchohra, Measure of noncompactness and fractional hybrid differential equations with Hybrid conditions, Differ. Equ. Appl., **14** (2022), 145-161. http://dx.doi.org/10.7153/dea-2022-14-09

[18] Y. Feng and Z. Bai, Solvability of some nonlocal fractional boundary value problems at resonance in R^n, Fractal Fract., **6** (2022), 16pages. https://doi.org/10.3390/fractalfract6010025

[19] D. Foukrach, S. Bouriah, M. Benchohra and E. Karapınar, Some new results for ψ-Hilfer fractional pantograph-type differential equation depending on ψ-Riemann-Liouville integral, J. Anal (2021). https://doi.org/10.1007/s41478-021-00339-0

[20] D. Foukrach, S. Bouriah, S. Abbas and M. Benchohra, Periodic solutions of nonlinear fractional pantograph integro-differential equations with Ψ-Caputo derivative, Ann. Univ. Ferrara (2022). https://doi.org/10.1007/s11565-022-00396-8

[21] R. Hermann, Fractional Calculus: An Introduction For Physicists, World Scientific Publishing Co. Pte. Ltd. 2011.

[22] A. Heris, A. Salim, M. Benchohra and E. Karapınar, Fractional partial random differential equations with infinite delay, Results in Physics (2022). https://doi.org/10.1016/j.rinp.2022.105557

[23] R. Hilfer, Applications of Fractional Calculus in Physics, World Scientific, Singapore, 2000.

[24] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, Theory and Applications of Fractional Differenatial Equations, North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam, 2006.

[25] N. Laledj, A. Salim, J.E. Lazreg, S. Abbas, B. Ahmad and M. Benchohra, On implicit fractional q-difference equations: Analysis and stability, Math. Meth. Appl. Sci., **2** (2022), 1-23. https://doi.org/10.1002/mma.8417

[26] J.E. Lazreg, M. Benchohra and A. Salim, Existence and Ulam stability of k-Generalized ψ-Hilfer Fractional Problem, J. Innov. Appl. Math. Comput. Sci., **2** (2022), 01-13.

[27] A.J. Luo and V. Afraimovich, Long-range Interactions, Stochasticity and Fractional Dynamics, Springer, New York, Dordrecht, Heidelberg, London, 2010.

[28] J. Mawhin, NSFCBMS Regional Conference Series in Mathematics, American Mathematical Society, Providence, RI, 1979.

[29] S.K. Ntouyas, J. Tariboon, C. Sawaddee, Nonlocal initial and boundary value problems via fractional calculus with exponential singular kernel, J. Nonlinear Sci. Appl., **11** (2018), 1015-1030.

[30] D. O'Regan, Y.J. Chao, Y.Q. Chen, Topological Degree Theory and Application, Taylor and Francis Group, Boca Raton, London, NewYork, 2006.

[31] M.D. Otigueira, Fractional Calculus for Scientists and Engineers, Lecture Notes in Electrical Engineering, 84. Springer, Dordrecht, 2011.

[32] I. Petras, Fractional-Order Nonlinear Systems: Modeling, Analysis and Simulation, Springer Heidelberg Dordrecht London New York, 2011.

[33] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, 1999.

[34] P. Sahoo, T. Barman and J.P. Davim, Fractal Analysis in Machining, Springer, New York, Dordrecht, Heidelberg, London, 2011.

[35] A. Salim, M. Benchohra, J.R. Graef and J.E. Lazreg, Initial value problem for hybrid ψ-Hilfer fractional implicit differential equations, J. Fixed Point Theory Appl., **24** (2022), 14 pp. https://doi.org/10.1007/s11784-021-00920-x

[36] A. Salim, M. Benchohra, J.E. Lazreg and J. Henderson, On k-generalized ψ-Hilfer boundary value problems with retardation and anticipation, Adv. Theor. Nonl. Anal. Appl., **6** (2022), 173-190. https://doi.org/10.31197/atnaa.973992

[37] A. Salim, M. Benchohra, J.E. Lazreg and E. Karapınar, On k-generalized ψ-Hilfer impulsive boundary value problem with retarded and advanced arguments, J. Math. Ext., **15** (2021), 1-39. https://doi.org/10.30495/JME.SI.2021.2187

[38] A. Salim, J.E. Lazreg, B. Ahmad, M. Benchohra and J.J. Nieto, A Study on k-Generalized ψ-Hilfer Derivative Operator, Vietnam J. Math., (2022). https://doi.org/10.1007/s10013-022-00561-8