On Dynamics and Solutions Expressions of Higher-Order Rational Difference Equations

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Abstract — The principle goal of this paper is to look at some of the qualitative behavior of the critical point of the rational difference equation

$$\Psi_{n+1} = a\Psi_{n-2} + \frac{\beta\Psi_{n-2}\Psi_{n-3}}{\gamma\Psi_{n-3} + \delta\Psi_{n-6}}, \quad n = 0, 1, 2, \ldots,$$

where $a, \beta, \gamma$, and $\delta$ are arbitrary positive real numbers. We also used the proposed equation to get the general solution for particular cases and provided numerical examples to demonstrate our results.

Subject Classification (2020): 39A10.

1. Introduction

One of the most important scientific topics is difference equations, often known as discrete dynamical systems. The study of the qualitative properties of rational difference equations has sparked a lot of attention recently.

Many researchers have opted to utilize difference equations in mathematical models to explain the problems in various sciences, including allowing scientists to introduce their study’s predictions and producing more precise results.

It is particularly fascinating to look into the behavior of the solutions to a system of nonlinear differential equations and examine the local asymptotic stability of their equilibrium points. Numerous studies have been conducted on the technique of identifying the general form of the solution for some special cases of the problem. The systems and behavior of rational difference equations have been the subject of numerous works (can be obtained in the references).

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Article History: Received: 16.06.2022 - Accepted: 23.12.2022 - Published: 01.03.2023
Alayachi et al. [3] studied the qualitative properties of:

\[ y_{n+1} = Ay_{n-1} + \frac{By_{n-1}y_{n-3}}{Cy_{n-3} + Dy_{n-5}}. \]

Almatrafi et al. [6] studied the global behavior of:

\[ \chi_{n+1} = \alpha \chi_n + \frac{\beta \chi_n^2 + \gamma \chi_n \chi_{n-1} + \delta \chi_{n-1}^2}{\lambda \chi_n + \mu \chi_n \chi_{n-1} + \sigma \chi_{n-1}^2}. \]

Alzubaidi and Elsayed [8] examined the dynamics behavior and gave the general form of:

\[ \varphi_{n+1} = \alpha \varphi_{n-2} \pm \frac{\beta \varphi_{n-1} \varphi_{n-2}}{\gamma \varphi_{n-2} \pm \delta \varphi_{n-4}}. \]

Ibrahim et al. [26] investigated the global stability and boundedness of solutions for:

\[ \Upsilon_{n+1} = \alpha + \sum_{i=0}^{k} a_i \Upsilon_{n-i} + \frac{\Upsilon_n \Upsilon_{n-k}}{\beta + \sum_{j=0}^{k} b_j \Upsilon_{n-j}}. \]

Kara and Yazlik [27] found the exact formulas for the solutions of the system:

\[
x_n = \frac{x_{n-2}z_{n-3}}{z_{n-1}(a_n + b_n x_{n-2}z_{n-3})}, \\
y_n = \frac{y_{n-2}x_{n-3}}{x_{n-1}(a_n + \beta_n y_{n-2}x_{n-3})}, \\
z_n = \frac{z_{n-2}y_{n-3}}{y_{n-1}(A_n + B_n z_{n-2}y_{n-3})}.
\]

Karatas et al. [28] investigated the solutions of:

\[ U_{n+1} = \frac{U_{n-5}}{1 + bU_{n-2}U_{n-5}}. \]

Abdul Khaliq et al. [30] investigated the asymptotic behavior of the solutions of:

\[ \omega_{n+1} = \omega_{n-p} \left( \alpha + \frac{\beta \omega_n}{\gamma \omega_n + \delta \omega_{n-r}} \right). \]

In [35] Muna and Mohammad deal with:

\[ V_{n+1} = \frac{(\alpha + \beta V_n)}{(A + BV_n + CV_{n-k})}. \]

The goal of this paper is to find a general solution to some special cases of the fractional recursive equation

\[ \Psi_{n+1} = \alpha \Psi_{n-2} + \frac{\beta \Psi_{n-2} \Psi_{n-3}}{\gamma \Psi_{n-3} + \delta \Psi_{n-6}}, \quad n = 0, 1, 2, \ldots, \]  

where \( \alpha, \beta, \gamma \) and \( \delta \) are arbitrary positive real numbers.
2. Local Stability of the Critical Point

The critical point of Eq. (1), is given by

\[ \bar{\Psi} = \alpha \bar{\Psi} + \beta \bar{\Psi}^2 \gamma + \delta \bar{\Psi}. \]

\[ (1 - \alpha) \bar{\Psi} = \frac{\beta \bar{\Psi}^2}{(\gamma + \delta) \bar{\Psi}} \Rightarrow (1 - \alpha)(\gamma + \delta) \bar{\Psi}^2 = \beta \bar{\Psi}^2. \]

Thus,

\[ [(1 - \alpha)(\gamma + \delta) - \beta] \bar{\Psi}^2 = 0. \]

If \((1 - \alpha)(\gamma + \delta) \neq \beta\) then the unique critical point is \(\bar{\Psi} = 0\).

Assume \(\Phi: (0, \infty)^3 \rightarrow (0, \infty)\) be a \(C^1\) function defined by

\[ \Phi(w_1, w_2, w_3) = \alpha w_1 + \frac{\beta w_1 w_2}{\gamma w_2 + \delta w_3}. \] (2)

In consequence,

\[ \frac{\partial \Phi}{\partial w_1} = \alpha + \frac{\beta w_2}{\gamma w_2 + \delta w_3}, \quad \frac{\partial \Phi}{\partial w_2} = \frac{\beta \delta w_1 w_3}{(\gamma w_2 + \delta w_3)^2}, \quad \frac{\partial \Phi}{\partial w_3} = -\frac{\beta \delta w_1 w_2}{(\gamma w_2 + \delta w_3)^2}. \] (3)

At \(\bar{\Psi} = 0\), we see that

\[ \frac{\partial \Phi}{\partial w_1} (\bar{\Psi}, \bar{\Psi}, \bar{\Psi}) = \alpha + \frac{\beta}{\gamma + \delta} = \gamma_1, \]

\[ \frac{\partial \Phi}{\partial w_2} (\bar{\Psi}, \bar{\Psi}, \bar{\Psi}) = \frac{\beta \delta}{(\gamma + \delta)^2} = \gamma_2, \]

\[ \frac{\partial \Phi}{\partial w_3} (\bar{\Psi}, \bar{\Psi}, \bar{\Psi}) = -\frac{\beta \delta}{(\gamma + \delta)^2} = \gamma_3. \] (4)

Hence,

\[ Z_{n+1} - \left(\alpha + \frac{\beta}{\gamma + \delta}\right) Z_{n-2} - \left(\frac{\beta \delta}{(\gamma + \delta)^2}\right) Z_{n-3} + \left(\frac{\beta \delta}{(\gamma + \delta)^2}\right) Z_{n-6} = 0. \]

**Theorem 2.1.** The critical point \(\bar{\Psi} = 0\) is locally asymptotically stable if

\[ \beta(\gamma + 3\delta) < (1 - \alpha)(\gamma + \delta)^2. \]

**Proof.**

By using the values in the Eq. (4) and by Lemma 1 in [30], ensures that Eq. (1) is asymptotically stable if

\[ \left| \gamma_1 \right| + \left| \gamma_2 \right| + \left| \gamma_3 \right| < 1, \]

\[ \left| \alpha + \frac{\beta}{\gamma + \delta} \right| + \left| \frac{\beta \delta}{(\gamma + \delta)^2} \right| + \left| -\frac{\beta \delta}{(\gamma + \delta)^2} \right| < 1, \]

or

\[ \alpha + \frac{\beta(\gamma + \delta)}{(\gamma + \delta)^2} + \frac{\beta \delta}{(\gamma + \delta)^2} + \frac{\beta \delta}{(\gamma + \delta)^2} < 1, \]
\[ \frac{\beta \gamma + 3 \beta \delta}{(\gamma + \delta)^2} < (1 - \alpha), \]

therefore,

\[ \beta (\gamma + 3 \delta) < (1 - \alpha) (\gamma + \delta)^2. \]

3. Global Attractive of the Critical Point

In this section, we aim to investigate the global asymptotic stability of the positive solutions of Eq.(1).

**Theorem 3.1.** The critical point \( \Psi = 0 \) of Eq.(1) is a global attracting if

\[ \gamma (1 - \alpha) \neq \beta. \]

**Proof.**

From Eq.(3), we note that, the function \( \Phi(w_1, w_2, w_3) \) is increasing in \( w_1 \) and \( w_2 \) and is decreasing in \( w_3 \). Assume that whenever \( (H, h) \) is a solution of the system

\[ H = \Phi(H, H, h), \]
\[ h = \Phi(h, h, H), \]

then, we have

\[ H = \alpha H + \frac{\beta H^2}{\gamma H + \delta h}, \quad \Rightarrow \quad (1 - \alpha) H = \frac{\beta H^2}{\gamma H + \delta h}, \]
\[ \gamma (1 - \alpha) H^2 + \delta (1 - \alpha) h H = \beta H^2. \tag{5} \]

\[ h = \alpha h + \frac{\beta h^2}{\gamma h + \delta H}, \quad \Rightarrow \quad (1 - \alpha) h = \frac{\beta h^2}{\gamma h + \delta H}, \]
\[ \gamma (1 - \alpha) h^2 + \delta (1 - \alpha) h H = \beta h^2. \tag{6} \]

By substrate Eq.(5) from Eq.(6) we obtain

\[ \left[ \gamma (1 - \alpha) - \beta \right] (H^2 - h^2) = 0. \]

In consequence, \( H = h \) if \( \gamma (1 - \alpha) \neq \beta \). It follows by Theorem 1 in [30] the equilibrium point \( \Psi = 0 \) of Eq.(1) is a global attractor.

4. Boundedness of solutions

Here, we demonstrate how the positive solutions to Eq.(1) have boundedness.

**Theorem 4.1.** Every solution of Eq.(1) is bounded if

\[ \left( \alpha + \frac{\beta}{\gamma} \right) < 1. \]
Proof.
Assume that $(\Psi_{n})_{n=-6}^{\infty}$ be a solution of Eq.(1), then

$$
\Psi_{n+1} = a\Psi_{n-2} + \frac{\beta \Psi_{n-2} \Psi_{n-3}}{\gamma \Psi_{n-3} + \delta \Psi_{n-6}}
\leq a\Psi_{n-2} + \frac{\beta \Psi_{n-2} \Psi_{n-3}}{\gamma \Psi_{n-3}}
= \left(\alpha + \frac{\beta}{\gamma}\right)\Psi_{n-2}.
$$

Hence,

$$
\Psi_{n+1} \leq \Psi_{n-2}, \quad \text{for all } n \geq 0.
$$

This implies that the subsequences are bounded from above by

$$
\Psi_{\text{max}} = \max(\Psi_{-6}, \Psi_{-5}, \Psi_{-4}, \Psi_{-3}, \Psi_{-2}, \Psi_{-1}, \Psi_{0}).
$$

5. General Solution for Special Cases

In this section, we will find expressions of solution for some special cases of Eq.(1)

5.1. First Equation

In this subsection, we will find the solution of Eq.(1) when $\alpha = \beta = \delta = \gamma = 1$, so the Eq.(1) become as

$$
\Psi_{n+1} = \Psi_{n-2} + \frac{\Psi_{n-2} \Psi_{n-3}}{\Psi_{n-3} + \Psi_{n-6}}, \quad n = 0, 1, 2, ...
$$

(7)

where the initial conditions $\Psi_{-6}, \Psi_{-5}, \Psi_{-4}, \Psi_{-3}, \Psi_{-2}, \Psi_{-1}$ and $\Psi_{0}$ are arbitrary positive real numbers.

Theorem 5.1. Assume $(\Psi_{n})_{n=-6}^{\infty}$ be a solution of Eq.(7). Thus for $n=0,1,2,...,$

$$
P_{12n-2} = \sigma \prod_{i=0}^{n-1} \left( F_{6i+3} \eta + F_{6i+2} \zeta \right) \left( F_{6i+3} \lambda + F_{6i+4} \mu \right) \left( F_{6i+3} \lambda + F_{6i+6} \tau \right) \left( F_{6i+3} \zeta + F_{6i+2} \kappa \right) |^{\alpha + \frac{\beta}{\gamma}}
$$

$$
P_{12n-1} = \lambda \prod_{i=0}^{n-1} \left( F_{6i+5} \eta + F_{6i+4} \zeta \right) \left( F_{6i+5} \lambda + F_{6i+6} \mu \right) \left( F_{6i+5} \lambda + F_{6i+2} \tau \right) \left( F_{6i+5} \zeta + F_{6i+4} \kappa \right) \left( F_{6i+4} \eta + F_{6i+3} \zeta \right) \left( F_{6i+4} \lambda + F_{6i+3} \mu \right) \left( F_{6i+4} \lambda + F_{6i+6} \tau \right) \left( F_{6i+4} \zeta + F_{6i+3} \kappa \right) |^{\alpha + \frac{\beta}{\gamma}}
$$

$$
P_{12n} = \eta \prod_{i=0}^{n-1} \left( F_{6i+7} \eta + F_{6i+6} \zeta \right) \left( F_{6i+7} \lambda + F_{6i+2} \mu \right) \left( F_{6i+7} \lambda + F_{6i+4} \tau \right) \left( F_{6i+7} \zeta + F_{6i+6} \kappa \right) \left( F_{6i+6} \eta + F_{6i+5} \zeta \right) \left( F_{6i+6} \lambda + F_{6i+5} \mu \right) \left( F_{6i+6} \lambda + F_{6i+2} \tau \right) \left( F_{6i+6} \zeta + F_{6i+5} \kappa \right) |^{\alpha + \frac{\beta}{\gamma}}
$$

$$
P_{12n+1} = \sigma \prod_{i=0}^{n} \left( F_{6i-3} \eta + F_{6i-4} \zeta \right) \left( F_{6i-3} \lambda + F_{6i-2} \mu \right) \left( F_{6i-3} \lambda + F_{6i+1} \tau \right) \left( F_{6i-3} \zeta + F_{6i-2} \kappa \right) |^{\alpha + \frac{\beta}{\gamma}}
$$

$$
P_{12n+2} = \lambda \prod_{i=0}^{n} \left( F_{6i-1} \eta + F_{6i-2} \zeta \right) \left( F_{6i-1} \lambda + F_{6i} \mu \right) \left( F_{6i-1} \lambda + F_{6i+3} \tau \right) \left( F_{6i-1} \zeta + F_{6i-2} \kappa \right) |^{\alpha + \frac{\beta}{\gamma}}
$$

$$
P_{12n+3} = \eta \prod_{i=0}^{n} \left( F_{6i+1} \eta + F_{6i+2} \zeta \right) \left( F_{6i+1} \lambda + F_{6i+2} \mu \right) \left( F_{6i+1} \lambda + F_{6i+1} \tau \right) \left( F_{6i+1} \zeta + F_{6i} \kappa \right) |^{\alpha + \frac{\beta}{\gamma}}
$$
\[
Ψ_{12n+4} = \sigma \prod_{i=0}^{n} (F_{6i+3}\eta + F_{6i+2}\zeta)(F_{6i-1}\lambda + F_{6i-2}\mu)(F_{6i+1}\sigma + F_{6i+1}\tau)(F_{6i+3}\zeta + F_{6i+2}\kappa),
\]
\[
Ψ_{12n+5} = \lambda \prod_{i=0}^{n} (F_{6i-1}\eta + F_{6i-2}\zeta)(F_{6i+1}\lambda + F_{6i+1}\mu)(F_{6i+3}\sigma + F_{6i+4}\kappa)(F_{6i+1}\tau + F_{6i+1}\zeta),
\]
\[
Ψ_{12n+6} = \eta \prod_{i=0}^{n} (F_{6i+1}\eta + F_{6i+2}\zeta)(F_{6i+3}\lambda + F_{6i+3}\mu)(F_{6i+5}\sigma + F_{6i+4}\kappa)(F_{6i+1}\tau + F_{6i+1}\zeta),
\]
\[
Ψ_{12n+7} = \sigma \prod_{i=0}^{n} (F_{6i+2}\eta + F_{6i+1}\zeta)(F_{6i+4}\lambda + F_{6i+4}\mu)(F_{6i+1}\sigma + F_{6i+1}\tau)(F_{6i+1}\zeta + F_{6i+1}\kappa),
\]
\[
Ψ_{12n+8} = \lambda \prod_{i=0}^{n} (F_{6i+4}\eta + F_{6i+3}\zeta)(F_{6i+1}\lambda + F_{6i+1}\mu)(F_{6i+3}\sigma + F_{6i+3}\tau)(F_{6i+1}\zeta + F_{6i+1}\kappa),
\]
\[
Ψ_{12n+9} = \eta \prod_{i=0}^{n} (F_{6i+7}\eta + F_{6i+6}\zeta)(F_{6i+3}\lambda + F_{6i+2}\mu)(F_{6i+5}\sigma + F_{6i+4}\kappa)(F_{6i+7}\zeta + F_{6i+6}\kappa),
\]
where \(Ψ_{-6} = \kappa, Ψ_{-5} = \tau, Ψ_{-4} = \mu, Ψ_{-3} = \zeta, Ψ_{-2} = \sigma, Ψ_{-1} = \lambda, Ψ_{0} = \eta\) and \(\{F_i\}_{i=-5} = \{1, 1, 1, 1, 1, 1, 1, 1, 2, 3, 5, 8, 13, 21, \ldots\}\).

**Proof.**

For \(n = 0\) the result holds. Now suppose that \(n > 0\) and our assumption holds for \(n - 1\), that is

\[
Ψ_{12n-14} = \sigma \prod_{i=0}^{n-2} (F_{6i+3}\eta + F_{6i+2}\zeta)(F_{6i+5}\lambda + F_{6i+4}\mu)(F_{6i+7}\sigma + F_{6i+6}\tau)(F_{6i+3}\zeta + F_{6i+2}\kappa),
\]
\[
Ψ_{12n-13} = \lambda \prod_{i=0}^{n-2} (F_{6i+5}\eta + F_{6i+4}\zeta)(F_{6i+7}\lambda + F_{6i+6}\mu)(F_{6i+3}\sigma + F_{6i+2}\tau)(F_{6i+5}\zeta + F_{6i+4}\kappa),
\]
\[
Ψ_{12n-12} = \eta \prod_{i=0}^{n-2} (F_{6i+7}\eta + F_{6i+6}\zeta)(F_{6i+3}\lambda + F_{6i+2}\mu)(F_{6i+5}\sigma + F_{6i+4}\tau)(F_{6i+7}\zeta + F_{6i+6}\kappa),
\]
\[
Ψ_{12n-11} = \sigma \prod_{i=0}^{n-1} (F_{6i-3}\eta + F_{6i-2}\zeta)(F_{6i-1}\lambda + F_{6i-2}\mu)(F_{6i+1}\sigma + F_{6i+1}\tau)(F_{6i-3}\zeta + F_{6i-2}\kappa),
\]
\[
Ψ_{12n-10} = \lambda \prod_{i=0}^{n-1} (F_{6i-2}\eta + F_{6i-3}\zeta)(F_{6i-1}\lambda + F_{6i-1}\mu)(F_{6i+3}\sigma + F_{6i+2}\tau)(F_{6i-2}\zeta + F_{6i-3}\kappa),
\]
\[
Ψ_{12n-9} = \eta \prod_{i=0}^{n-1} (F_{6i+1}\eta + F_{6i+1}\zeta)(F_{6i+3}\lambda + F_{6i+2}\mu)(F_{6i-1}\sigma + F_{6i-2}\tau)(F_{6i+1}\zeta + F_{6i+1}\kappa),
\]
\[
Ψ_{12n-8} = \sigma \prod_{i=0}^{n-1} (F_{6i+3}\eta + F_{6i+2}\zeta)(F_{6i-1}\lambda + F_{6i-2}\mu)(F_{6i+1}\sigma + F_{6i+1}\tau)(F_{6i+3}\zeta + F_{6i+2}\kappa),
\]
\[
Ψ_{12n-7} = \lambda \prod_{i=0}^{n-1} (F_{6i-1}\eta + F_{6i-2}\zeta)(F_{6i+1}\lambda + F_{6i+1}\mu)(F_{6i+3}\sigma + F_{6i+4}\tau)(F_{6i-1}\zeta + F_{6i-2}\kappa),
\]
\[
Ψ_{12n-6} = \eta \prod_{i=0}^{n-1} (F_{6i+1}\eta + F_{6i+3}\zeta)(F_{6i+3}\lambda + F_{6i+4}\mu)(F_{6i+5}\sigma + F_{6i+4}\tau)(F_{6i+1}\zeta + F_{6i+3}\kappa),
\]
\[
Ψ_{12n-5} = \sigma \prod_{i=0}^{n-1} (F_{6i+3}\eta + F_{6i+2}\zeta)(F_{6i+5}\lambda + F_{6i+4}\mu)(F_{6i+1}\sigma + F_{6i+1}\tau)(F_{6i+3}\zeta + F_{6i+2}\kappa),
\]
\[ \Psi_{12n-4} = \lambda \prod_{i=0}^{n-1} \left[ \frac{F_{6i+5} + \xi}{F_{6i+4} + \eta} \right] \left( \frac{F_{6i+1} \lambda + F_{6i+2} \sigma + F_{6i+3} \tau}{F_{6i+5} \xi + F_{6i+4} \eta + F_{6i+3} \zeta} \right) \]

\[ \Psi_{12n-3} = \eta \prod_{i=0}^{n-1} \left[ \frac{F_{6i+1} \eta + F_{6i+2} \xi}{F_{6i+1} \zeta + F_{6i+2} \zeta} \right] \left( \frac{F_{6i+1} \lambda + F_{6i+2} \sigma + F_{6i+3} \tau}{F_{6i+1} \eta + F_{6i+1} \zeta} \right) \left( \frac{F_{6i+4} \xi + F_{6i+3} \eta + F_{6i+2} \xi}{F_{6i+4} \xi + F_{6i+4} \eta + F_{6i+3} \xi} \right) \]

Now, we prove that the results are holds for \( n \). From Eq.(7), it follows that

\[ \Psi_{12n-2} = \Psi_{12n-5} + \frac{\Psi_{12n-6} - \Psi_{12n-9}}{\Psi_{12n-6} + \Psi_{12n-9}} \]

\[ = \sigma \prod_{i=0}^{n-1} \left[ \frac{F_{6i+3} \eta + F_{6i+2} \xi}{F_{6i+2} \eta + F_{6i+1} \zeta} \right] \left( \frac{F_{6i+5} \lambda + F_{6i+4} \mu}{F_{6i+1} \sigma + F_{6i+1} \tau} \right) \left( \frac{F_{6i+3} \lambda + F_{6i+2} \sigma + F_{6i+1} \tau}{F_{6i+2} \xi + F_{6i+1} \zeta} \right) \left( \frac{F_{6i+3} \xi + F_{6i+2} \eta + F_{6i+1} \zeta}{F_{6i+2} \xi + F_{6i+1} \zeta} \right) \]

\[ = \sigma \prod_{i=0}^{n-1} \left[ \frac{F_{6i+3} \eta + F_{6i+2} \xi}{F_{6i+2} \eta + F_{6i+1} \zeta} \right] \left( \frac{F_{6i+5} \lambda + F_{6i+4} \mu}{F_{6i+1} \sigma + F_{6i+1} \tau} \right) \left( \frac{F_{6i+3} \lambda + F_{6i+2} \sigma + F_{6i+1} \tau}{F_{6i+2} \xi + F_{6i+1} \zeta} \right) \left( \frac{F_{6i+3} \xi + F_{6i+2} \eta + F_{6i+1} \zeta}{F_{6i+2} \xi + F_{6i+1} \zeta} \right) \]

\[ = \sigma \prod_{i=0}^{n-1} \left[ \frac{F_{6i+3} \eta + F_{6i+2} \xi}{F_{6i+2} \eta + F_{6i+1} \zeta} \right] \left( \frac{F_{6i+5} \lambda + F_{6i+4} \mu}{F_{6i+1} \sigma + F_{6i+1} \tau} \right) \left( \frac{F_{6i+3} \lambda + F_{6i+2} \sigma + F_{6i+1} \tau}{F_{6i+2} \xi + F_{6i+1} \zeta} \right) \left( \frac{F_{6i+3} \xi + F_{6i+2} \eta + F_{6i+1} \zeta}{F_{6i+2} \xi + F_{6i+1} \zeta} \right) \]

\[ = \sigma \prod_{i=0}^{n-1} \left[ \frac{F_{6i+3} \eta + F_{6i+2} \xi}{F_{6i+2} \eta + F_{6i+1} \zeta} \right] \left( \frac{F_{6i+5} \lambda + F_{6i+4} \mu}{F_{6i+1} \sigma + F_{6i+1} \tau} \right) \left( \frac{F_{6i+3} \lambda + F_{6i+2} \sigma + F_{6i+1} \tau}{F_{6i+2} \xi + F_{6i+1} \zeta} \right) \left( \frac{F_{6i+3} \xi + F_{6i+2} \eta + F_{6i+1} \zeta}{F_{6i+2} \xi + F_{6i+1} \zeta} \right) \]
\[ \sigma \prod_{i=0}^{n-1} \frac{(F_{6i+3}\eta + F_{6i+2}\zeta)(F_{6i+3}\lambda + F_{6i+4}\mu)(F_{6i+1}\sigma + F_{6i}\tau)(F_{6i+3}\lambda + F_{6i+2}\kappa)}{(F_{6i+1}\eta + F_{6i+1}\tau)(F_{6i+3}\lambda + F_{6i+4}\mu)(F_{6i}\sigma + F_{6i-1}\tau)(F_{6i+2}\zeta + F_{6i+1}\kappa)} \]

Hence, we get
\[ \Psi_{12n-2} = \sigma \prod_{i=0}^{n-1} \frac{(F_{6i+3}\eta + F_{6i+2}\zeta)(F_{6i+5}\lambda + F_{6i+4}\mu)(F_{6i+1}\sigma + F_{6i}\tau)(F_{6i+3}\lambda + F_{6i+2}\kappa)}{(F_{6i+2}\eta + F_{6i+1}\zeta)(F_{6i+4}\lambda + F_{6i+3}\mu)(F_{6i}\sigma + F_{6i-1}\tau)(F_{6i+2}\zeta + F_{6i+1}\kappa)}. \]

Other expressions can be investigated in the same way. The proof has been completed.

### 5.2. Second Equation

In this subsection, we will find the solution of Eq.(1) when \( \alpha = \gamma = \beta = 1 \) and \( \delta = -1 \), so the Eq.(1) become as
\[ \Psi_{n+1} = \Psi_{n-2}\Psi_{n-3} - \Psi_{n-6}, \quad n = 0, 1, 2, \ldots, \] (8)

where the initial conditions \( \Psi_{-6}, \Psi_{-5}, \Psi_{-4}, \Psi_{-3}, \Psi_{-2}, \Psi_{-1} \) and \( \Psi_{0} \) are arbitrary positive real numbers.

**Theorem 5.2.** Assume \((\Psi_{n})_{n=0}^{\infty}\) be a solution of Eq.(8). Thus for \( n=0,1,2,\ldots, \)

\[ \Psi_{12n-2} = \sigma \prod_{i=0}^{n-1} \frac{(F_{3i+3}\eta - F_{3i+1}\zeta)(F_{3i+4}\lambda - F_{3i+2}\mu)(F_{3i+5}\sigma - F_{3i+3}\tau)(F_{3i+3}\lambda - F_{3i+1}\kappa)}{(F_{3i+1}\eta - F_{3i-1}\zeta)(F_{3i+2}\lambda - F_{3i}\mu)(F_{3i+3}\sigma - F_{3i+1}\tau)(F_{3i+1}\zeta - F_{3i-1}\kappa)}, \]

\[ \Psi_{12n-1} = \lambda \prod_{i=0}^{n-1} \frac{(F_{3i+4}\eta - F_{3i+2}\zeta)(F_{3i+5}\lambda - F_{3i+3}\mu)(F_{3i+3}\sigma - F_{3i+1}\tau)(F_{3i+2}\zeta - F_{3i+2}\zeta)}{(F_{3i+2}\eta - F_{3i}\zeta)(F_{3i+3}\lambda - F_{3i+1}\mu)(F_{3i+1}\sigma - F_{3i}\tau)(F_{3i+2}\zeta)}}, \]

\[ \Psi_{12n} = \eta \prod_{i=0}^{n-1} \frac{(F_{3i+5}\eta - F_{3i+3}\zeta)(F_{3i+3}\lambda - F_{3i+1}\mu)(F_{3i+4}\sigma - F_{3i+2}\tau)(F_{3i+5}\lambda - F_{3i+3}\zeta)}{(F_{3i+3}\eta - F_{3i+1}\zeta)(F_{3i+1}\lambda - F_{3i+1}\mu)(F_{3i+2}\sigma - F_{3i+1}\tau)(F_{3i+3}\zeta - F_{3i+1}\kappa)}, \]

\[ \Psi_{12n+1} = \frac{\sigma(2\zeta - \kappa) - (\eta - \zeta)\eta}{(\zeta - \kappa)} \prod_{i=0}^{n-1} \frac{(F_{3i+3}\eta - F_{3i+1}\zeta)(F_{3i+4}\lambda - F_{3i+2}\mu)(F_{3i+5}\sigma - F_{3i+3}\tau)(F_{3i+6}\lambda - F_{3i+4}\kappa)}{(F_{3i+1}\eta - F_{3i+1}\zeta)(F_{3i+2}\lambda - F_{3i+1}\mu)(F_{3i+3}\sigma - F_{3i+1}\tau)(F_{3i+4}\eta - F_{3i+2}\kappa)}, \]

\[ \Psi_{12n+2} = \frac{\lambda(2\sigma + \tau)}{(\sigma - \tau)} \prod_{i=0}^{n-1} \frac{(F_{3i+4}\eta - F_{3i+2}\zeta)(F_{3i+5}\lambda - F_{3i+3}\mu)(F_{3i+6}\sigma - F_{3i+4}\tau)(F_{3i+4}\lambda - F_{3i+2}\zeta)}{(F_{3i+2}\eta - F_{3i}\zeta)(F_{3i+3}\lambda - F_{3i+1}\mu)(F_{3i+4}\sigma - F_{3i+2}\tau)(F_{3i+2}\zeta)}}, \]

\[ \Psi_{12n+3} = \frac{\eta(2\lambda - \mu)}{(\lambda - \mu)} \prod_{i=0}^{n-1} \frac{(F_{3i+5}\eta - F_{3i+3}\zeta)(F_{3i+6}\lambda - F_{3i+4}\mu)(F_{3i+4}\sigma - F_{3i+2}\tau)(F_{3i+5}\lambda - F_{3i+3}\zeta)}{(F_{3i+3}\eta - F_{3i+1}\zeta)(F_{3i+4}\lambda - F_{3i+2}\mu)(F_{3i+2}\sigma - F_{3i+1}\tau)(F_{3i+3}\zeta - F_{3i+1}\kappa)}, \]

\[ \Psi_{12n+4} = \frac{\sigma(2\eta - \zeta)(2\zeta - \kappa)}{(\eta - \zeta)(\zeta - \kappa)} \prod_{i=0}^{n-1} \frac{(F_{3i+6}\eta - F_{3i+4}\zeta)(F_{3i+4}\lambda - F_{3i+2}\mu)}{(F_{3i+4}\eta - F_{3i+2}\zeta)(F_{3i+2}\lambda - F_{3i+1}\mu)(F_{3i+3}\sigma - F_{3i+1}\tau)(F_{3i+3}\zeta - F_{3i+1}\kappa)}.
\[\Psi_{12n+5} = \frac{\lambda(2\sigma - \tau)(3\zeta - \kappa)}{\zeta(\sigma - \tau)} \prod_{i=0}^{n-1} \left( \frac{(\mathcal{F}_{3i+4}\eta - \mathcal{F}_{3i+2}\zeta)(\mathcal{F}_{3i+5}\lambda - \mathcal{F}_{3i+3}\mu)}{(\mathcal{F}_{3i+6}\sigma - \mathcal{F}_{3i+4}\tau)(\mathcal{F}_{3i+7}\zeta - \mathcal{F}_{3i+5}\kappa)} \right),\]

\[\Psi_{12n+6} = \frac{\eta(2\lambda - \mu)(3\sigma - \tau)}{\sigma(\lambda - \mu)} \prod_{i=0}^{n-1} \left( \frac{(\mathcal{F}_{3i+5}\eta - \mathcal{F}_{3i+3}\zeta)(\mathcal{F}_{3i+6}\lambda - \mathcal{F}_{3i+4}\mu)}{(\mathcal{F}_{3i+3}\sigma - \mathcal{F}_{3i+5}\tau)(\mathcal{F}_{3i+5}\zeta - \mathcal{F}_{3i+3}\lambda)} \right),\]

\[\Psi_{12n+7} = \frac{\sigma(2\eta - \zeta)(3\lambda - \mu)(2\zeta - \kappa)}{\lambda(\eta - \zeta)(\zeta - \kappa)} \prod_{i=0}^{n-1} \left( \frac{(\mathcal{F}_{3i+6}\eta - \mathcal{F}_{3i+4}\zeta)(\mathcal{F}_{3i+7}\lambda - \mathcal{F}_{3i+5}\mu)}{(\mathcal{F}_{3i+2}\eta - \mathcal{F}_{3i+1}\zeta)(\mathcal{F}_{3i+5}\lambda - \mathcal{F}_{3i+3}\mu)} \right),\]

\[\Psi_{12n+8} = \frac{\lambda(3\eta - \zeta)(2\sigma - \tau)(3\zeta - \kappa)}{\eta(\sigma - \tau)} \prod_{i=0}^{n-1} \left( \frac{(\mathcal{F}_{3i+5}\eta - \mathcal{F}_{3i+3}\zeta)(\mathcal{F}_{3i+7}(\mathcal{F}_{3i+5}\lambda - \mathcal{F}_{3i+1}\mu)}{(\mathcal{F}_{3i+4}\eta - \mathcal{F}_{3i+2}\zeta)(\mathcal{F}_{3i+5}\lambda - \mathcal{F}_{3i+3}\mu)} \right),\]

\[\Psi_{12n+9} = \frac{\eta(2\lambda - \mu)(3\sigma - \tau)(5\zeta - 2\kappa)}{\sigma(\lambda - \mu)(2\zeta - \kappa)} \prod_{i=0}^{n-1} \left( \frac{(\mathcal{F}_{3i+5}\eta - \mathcal{F}_{3i+3}\zeta)(\mathcal{F}_{3i+4}\lambda - \mathcal{F}_{3i+2}\mu)}{(\mathcal{F}_{3i+3}\eta - \mathcal{F}_{3i+1}\zeta)(\mathcal{F}_{3i+4}\lambda - \mathcal{F}_{3i+2}\mu)} \right),\]

where \(\Psi_{-6} = \kappa, \Psi_{-5} = \tau, \Psi_{-4} = \mu, \Psi_{-3} = \zeta, \Psi_{-2} = \sigma, \Psi_{-1} = \lambda, \Psi_{0} = \eta \) and \(\{\mathcal{F}_{i}\}_{i=-1}^{\infty} = \{1, 0, 1, 2, 3, 5, 8, 13, 21, \ldots\}\).

**Proof.**

For \(n = 0\) the result holds. Now suppose that \(n > 0\) and our assumption holds for \(n - 1\), that is

\[\Psi_{12n-14} = \sigma \prod_{i=0}^{n-2} \left( \frac{(\mathcal{F}_{3i+3}\eta - \mathcal{F}_{3i+1}\zeta)(\mathcal{F}_{3i+4}\lambda - \mathcal{F}_{3i+2}\mu)}{(\mathcal{F}_{3i+2}\eta - \mathcal{F}_{3i+1}\zeta)(\mathcal{F}_{3i+2}\lambda - \mathcal{F}_{3i+1}\mu)} \right),\]

\[\Psi_{12n-13} = \lambda \prod_{i=0}^{n-2} \left( \frac{(\mathcal{F}_{3i+4}\eta - \mathcal{F}_{3i+2}\zeta)(\mathcal{F}_{3i+5}\lambda - \mathcal{F}_{3i+3}\mu)}{(\mathcal{F}_{3i+3}\eta - \mathcal{F}_{3i+1}\zeta)(\mathcal{F}_{3i+4}\lambda - \mathcal{F}_{3i+2}\mu)} \right),\]

\[\Psi_{12n-12} = \eta \prod_{i=0}^{n-2} \left( \frac{(\mathcal{F}_{3i+5}\eta - \mathcal{F}_{3i+3}\zeta)(\mathcal{F}_{3i+3}\lambda - \mathcal{F}_{3i+1}\mu)}{(\mathcal{F}_{3i+2}\eta - \mathcal{F}_{3i+1}\zeta)(\mathcal{F}_{3i+3}\lambda - \mathcal{F}_{3i+1}\mu)} \right),\]

\[\Psi_{12n-11} = \frac{\sigma(2\zeta - \kappa)}{\zeta - \kappa} \prod_{i=0}^{n-2} \left( \frac{(\mathcal{F}_{3i+3}\eta - \mathcal{F}_{3i+1}\zeta)(\mathcal{F}_{3i+4}\lambda - \mathcal{F}_{3i+2}\mu)(\mathcal{F}_{3i+5}\sigma - \mathcal{F}_{3i+3}\tau)}{(\mathcal{F}_{3i+1}\eta - \mathcal{F}_{3i-1}\zeta)(\mathcal{F}_{3i+4}\lambda - \mathcal{F}_{3i+2}\mu)(\mathcal{F}_{3i+3}\sigma - \mathcal{F}_{3i+1}\tau)} \right),\]

\[\Psi_{12n-10} = \frac{\lambda(2\sigma - \tau)}{\sigma - \tau} \prod_{i=0}^{n-2} \left( \frac{(\mathcal{F}_{3i+4}\eta - \mathcal{F}_{3i+2}\zeta)(\mathcal{F}_{3i+5}\lambda - \mathcal{F}_{3i+3}\mu)(\mathcal{F}_{3i+6}\sigma - \mathcal{F}_{3i+4}\tau)}{(\mathcal{F}_{3i+2}\eta - \mathcal{F}_{3i+1}\zeta)(\mathcal{F}_{3i+3}\lambda - \mathcal{F}_{3i+1}\mu)(\mathcal{F}_{3i+4}\sigma - \mathcal{F}_{3i+2}\tau)} \right).\]
\[ \Psi_{12n-9} = \frac{\eta(2\lambda - \mu)}{\lambda - \mu} \prod_{i=0}^{n-2} \frac{(F_{3i+5} - F_{3i+3})(F_{3i+6} - F_{3i+4}\mu)(F_{3i+4}\sigma - F_{3i+2}\tau)(F_{3i+5}\zeta - F_{3i+3}\kappa)}{(F_{3i+3}\eta - F_{3i+1}\zeta)(F_{3i+4}\lambda - F_{3i+2}\mu)(F_{3i+2}\sigma - F_{3i+1}\tau)(F_{3i+3}\zeta - F_{3i+1}\kappa)}, \]

\[ \Psi_{12n-8} = \frac{\sigma(2\eta - \zeta)(2\zeta - \kappa)}{(\eta - \zeta)(\zeta - \kappa)} \prod_{i=0}^{n-2} \frac{(F_{3i+5}\eta - F_{3i+3}\zeta)(F_{3i+4}\lambda - F_{3i+2}\mu)}{(F_{3i+3}\eta - F_{3i+1}\zeta)(F_{3i+4}\lambda - F_{3i+2}\mu)}, \]

\[ \Psi_{12n-7} = \frac{\lambda(2\sigma - \tau)(3\zeta - \kappa)}{\zeta(\sigma - \tau)} \prod_{i=0}^{n-2} \frac{(F_{3i+5}\eta - F_{3i+3}\zeta)(F_{3i+4}\lambda - F_{3i+2}\mu)}{(F_{3i+3}\eta - F_{3i+1}\zeta)(F_{3i+4}\lambda - F_{3i+2}\mu)}, \]

\[ \Psi_{12n-6} = \frac{\eta(2\lambda - \mu)(3\sigma - \tau)}{\sigma(\lambda - \mu)} \prod_{i=0}^{n-2} \frac{(F_{3i+5}\eta - F_{3i+3}\zeta)(F_{3i+4}\lambda - F_{3i+2}\mu)}{(F_{3i+3}\eta - F_{3i+1}\zeta)(F_{3i+4}\lambda - F_{3i+2}\mu)}, \]

\[ \Psi_{12n-5} = \frac{\sigma(2\eta - \zeta)(3\lambda - \mu)(2\zeta - \kappa)}{\lambda(\eta - \zeta)(\zeta - \kappa)} \prod_{i=0}^{n-2} \frac{(F_{3i+5}\eta - F_{3i+3}\zeta)(F_{3i+4}\lambda - F_{3i+2}\mu)}{(F_{3i+3}\eta - F_{3i+1}\zeta)(F_{3i+4}\lambda - F_{3i+2}\mu)}, \]

\[ \Psi_{12n-4} = \frac{\lambda(3\eta - \zeta)(2\sigma - \tau)(3\zeta - \kappa)}{\eta(\sigma - \tau)} \prod_{i=0}^{n-2} \frac{(F_{3i+5}\eta - F_{3i+3}\zeta)(F_{3i+4}\lambda - F_{3i+2}\mu)}{(F_{3i+3}\eta - F_{3i+1}\zeta)(F_{3i+4}\lambda - F_{3i+2}\mu)}, \]

\[ \Psi_{12n-3} = \frac{\eta(2\lambda - \mu)(3\sigma - \tau)(5\zeta - 2\kappa)}{\sigma(\lambda - \mu)(2\zeta - \kappa)} \prod_{i=0}^{n-2} \frac{(F_{3i+5}\eta - F_{3i+3}\zeta)(F_{3i+4}\lambda - F_{3i+2}\mu)}{(F_{3i+3}\eta - F_{3i+1}\zeta)(F_{3i+4}\lambda - F_{3i+2}\mu)} . \]
Now, we prove that the results are holds for \( n \). From Eq.(8), it follows that

\[
\Psi_{12n-2} = \Psi_{12n-5} + \frac{\Psi_{12n-5} \Psi_{12n-6}}{\Psi_{12n-6} - \Psi_{12n-9}}
\]

\[
= \frac{\sigma(2\eta - \zeta)(3\lambda - \mu)(2\zeta - \kappa)}{\lambda(\eta - \zeta)(\zeta - \kappa)} \prod_{i=0}^{n-2} \frac{(\mathcal{F}_{3i+6} \eta - \mathcal{F}_{3i+4} \zeta)(\mathcal{F}_{3i+7} \lambda - \mathcal{F}_{3i+5} \mu)}{(\mathcal{F}_{3i+5} \sigma - \mathcal{F}_{3i+3} \tau)(\mathcal{F}_{3i+6} \zeta - \mathcal{F}_{3i+4} \kappa)}
\]

\[
+ \frac{\eta(2\lambda - \mu)(3\sigma - \tau)}{\sigma(\lambda - \mu)} \prod_{i=0}^{n-2} \frac{(\mathcal{F}_{3i+5} \sigma - \mathcal{F}_{3i+3} \tau)(\mathcal{F}_{3i+6} \lambda - \mathcal{F}_{3i+4} \mu)(\mathcal{F}_{3i+7} \sigma - \mathcal{F}_{3i+5} \tau)(\mathcal{F}_{3i+8} \lambda - \mathcal{F}_{3i+6} \mu)}{(\mathcal{F}_{3i+3} \eta - \mathcal{F}_{3i+1} \zeta)(\mathcal{F}_{3i+4} \lambda - \mathcal{F}_{3i+2} \mu)(\mathcal{F}_{3i+5} \eta - \mathcal{F}_{3i+1} \zeta)(\mathcal{F}_{3i+6} \lambda - \mathcal{F}_{3i+5} \mu)}
\]

\[
= \frac{\sigma(2\eta - \zeta)(3\lambda - \mu)(2\zeta - \kappa)}{\lambda(\eta - \zeta)(\zeta - \kappa)} \prod_{i=0}^{n-2} \frac{(\mathcal{F}_{3i+6} \eta - \mathcal{F}_{3i+4} \zeta)(\mathcal{F}_{3i+7} \lambda - \mathcal{F}_{3i+5} \mu)}{(\mathcal{F}_{3i+5} \sigma - \mathcal{F}_{3i+3} \tau)(\mathcal{F}_{3i+6} \zeta - \mathcal{F}_{3i+4} \kappa)}
\]

\[
+ \frac{\eta(2\lambda - \mu)(3\sigma - \tau)}{\lambda(\eta - \zeta)(\zeta - \kappa)} \prod_{i=0}^{n-2} \frac{\prod_{i=0}^{n-2} \frac{(\mathcal{F}_{3i+7} \sigma - \mathcal{F}_{3i+5} \tau)(\mathcal{F}_{3i+6} \lambda - \mathcal{F}_{3i+4} \mu)(\mathcal{F}_{3i+8} \sigma - \mathcal{F}_{3i+6} \tau)(\mathcal{F}_{3i+9} \lambda - \mathcal{F}_{3i+7} \mu)}{(\mathcal{F}_{3i+3} \eta - \mathcal{F}_{3i+1} \zeta)(\mathcal{F}_{3i+4} \lambda - \mathcal{F}_{3i+2} \mu)(\mathcal{F}_{3i+5} \eta - \mathcal{F}_{3i+1} \zeta)(\mathcal{F}_{3i+6} \lambda - \mathcal{F}_{3i+5} \mu)}}{(\mathcal{F}_{3i+5} \sigma - \mathcal{F}_{3i+3} \tau)(\mathcal{F}_{3i+6} \zeta - \mathcal{F}_{3i+4} \kappa)}
\]
\[ \omega(2\eta - \zeta)(3\lambda - \mu)(2\zeta - \kappa) \]
\[ \frac{\lambda(\eta - \zeta)(\zeta - \kappa)}{\lambda(\eta - \zeta)(\zeta - \kappa)} n \prod_{i=0}^{n-2} \left( \frac{(\mathcal{F}_{3i+6}\eta - \mathcal{F}_{3i+4}\zeta)(\mathcal{F}_{3i+1}\lambda - \mathcal{F}_{3i+3}\mu)}{(\mathcal{F}_{3i+3}\eta - \mathcal{F}_{3i+1}\lambda)(\mathcal{F}_{3i+4}\mu - \mathcal{F}_{3i+2}\kappa)} \right) \]

\[ = \frac{\omega(2\eta - \zeta)(3\lambda - \mu)(2\zeta - \kappa)}{\lambda(\eta - \zeta)(\zeta - \kappa)} \]
\[ \frac{\lambda(\eta - \zeta)(\zeta - \kappa)}{\lambda(\eta - \zeta)(\zeta - \kappa)} n \prod_{i=0}^{n-2} \left( \frac{(\mathcal{F}_{3i+6}\eta - \mathcal{F}_{3i+4}\zeta)(\mathcal{F}_{3i+1}\lambda - \mathcal{F}_{3i+3}\mu)}{(\mathcal{F}_{3i+3}\eta - \mathcal{F}_{3i+1}\lambda)(\mathcal{F}_{3i+4}\mu - \mathcal{F}_{3i+2}\kappa)} \right) \]

\[ = \frac{\sigma(2\eta - \zeta)(3\lambda - \mu)(2\zeta - \kappa)}{\lambda(\eta - \zeta)(\zeta - \kappa)} \]
\[ \frac{\lambda(\eta - \zeta)(\zeta - \kappa)}{\lambda(\eta - \zeta)(\zeta - \kappa)} n \prod_{i=0}^{n-2} \left( \frac{(\mathcal{F}_{3i+6}\eta - \mathcal{F}_{3i+4}\zeta)(\mathcal{F}_{3i+1}\lambda - \mathcal{F}_{3i+3}\mu)}{(\mathcal{F}_{3i+3}\eta - \mathcal{F}_{3i+1}\lambda)(\mathcal{F}_{3i+4}\mu - \mathcal{F}_{3i+2}\kappa)} \right) \]

Therefore,
\[ \Psi_{12n-2} = \sigma n \prod_{i=0}^{n-1} \frac{(\mathcal{F}_{3i+3}\eta - \mathcal{F}_{3i+1}\mu)(\mathcal{F}_{3i+1}\lambda - \mathcal{F}_{3i+3}\mu)}{(\mathcal{F}_{3i+1}\eta - \mathcal{F}_{3i-1}\zeta)(\mathcal{F}_{3i+2}\lambda - \mathcal{F}_{3i+4}\mu)(\mathcal{F}_{3i+3}\sigma - \mathcal{F}_{3i+1}\tau)} \]

The following cases can be proved using a similar technique.

5.3. Third Equation

In this subsection, we will find the solution of Eq.(1) when \( \alpha = \gamma = \delta = 1 \) and \( \beta = -1 \), so the Eq.(1) become as
\[ \Psi_{n+1} = \Psi_{n-1} - \frac{\psi_{n-2}\psi_{n-3}}{\psi_{n-3} + \psi_{n-6}}, \quad n = 0, 1, 2, ..., \]
(9)
where the initial conditions \( \psi_{-6}, \psi_{-5}, \psi_{-4}, \psi_{-3}, \psi_{-2}, \psi_{-1} \) and \( \psi_{0} \) are arbitrary positive real numbers.

Theorem 5.3. Assume \( (\psi_{n})_{n=-6}^{\infty} \) be a solution of Eq.(9). Thus for \( n=0,1,2,..., \)
\[ \Psi_{12n-2} = \sigma n \prod_{i=0}^{n-1} \frac{(\mathcal{F}_{3i+3}\eta - \mathcal{F}_{3i+1}\mu)(\mathcal{F}_{3i+1}\lambda + \mathcal{F}_{3i+3}\mu)(\mathcal{F}_{3i+2}\sigma + \mathcal{F}_{3i+4}\mu)(\mathcal{F}_{3i+1}\zeta + \mathcal{F}_{3i+3}\kappa)}{(\mathcal{F}_{3i+1}\eta + \mathcal{F}_{3i-1}\zeta)(\mathcal{F}_{3i+2}\lambda + \mathcal{F}_{3i+4}\mu)(\mathcal{F}_{3i+3}\sigma + \mathcal{F}_{3i+5}\mu)(\mathcal{F}_{3i+1}\zeta + \mathcal{F}_{3i+3}\kappa)}, \]
\[ \Psi_{12n-1} = \lambda n \prod_{i=0}^{n-1} \frac{(\mathcal{F}_{3i+3}\eta - \mathcal{F}_{3i+1}\mu)(\mathcal{F}_{3i+1}\lambda + \mathcal{F}_{3i+3}\mu)(\mathcal{F}_{3i+2}\sigma + \mathcal{F}_{3i+4}\mu)(\mathcal{F}_{3i+1}\zeta + \mathcal{F}_{3i+3}\kappa)}{(\mathcal{F}_{3i+2}\eta + \mathcal{F}_{3i+4}\mu)(\mathcal{F}_{3i+3}\lambda + \mathcal{F}_{3i+5}\mu)(\mathcal{F}_{3i+4}\sigma + \mathcal{F}_{3i+6}\mu)(\mathcal{F}_{3i+2}\zeta + \mathcal{F}_{3i+4}\kappa)}, \]
\[ \Psi_{12n} = \eta \prod_{i=0}^{n-1} \frac{(F_{3i+2\eta} + F_{3i+3\eta})(F_{3i+1\lambda} + F_{3i+1\mu})(F_{3i+1\sigma} + F_{3i+2\tau})(F_{3i+2\zeta} + F_{3i+3\kappa})}{(F_{3i+3\eta} + F_{3i+4\zeta})(F_{3i+1\lambda} + F_{3i+2\mu})(F_{3i+2\sigma} + F_{3i+3\tau})(F_{3i+3\zeta} + F_{3i+4\kappa})}, \]

\[ \Psi_{12n+1} = \frac{\sigma \kappa}{(\zeta + \kappa)} \prod_{i=0}^{n-1} \frac{(F_{3i+1\eta} + F_{3i+1\xi})(F_{3i+1\lambda} + F_{3i+2\mu})(F_{3i+3\sigma} + F_{3i+4\tau})(F_{3i+3\xi} + F_{3i+4\kappa})}{(F_{3i+1\eta} + F_{3i+2\xi})(F_{3i+2\lambda} + F_{3i+3\mu})(F_{3i+3\sigma} + F_{3i+4\tau})(F_{3i+4\xi} + F_{3i+5\kappa})}, \]

\[ \Psi_{12n+2} = \frac{\lambda \tau}{(\sigma + \tau)} \prod_{i=0}^{n-1} \frac{(F_{3i+1\eta} + F_{3i+2\xi})(F_{3i+2\lambda} + F_{3i+3\mu})(F_{3i+3\sigma} + F_{3i+4\tau})(F_{3i+1\xi} + F_{3i+2\kappa})}{(F_{3i+2\eta} + F_{3i+3\xi})(F_{3i+3\lambda} + F_{3i+4\mu})(F_{3i+4\sigma} + F_{3i+5\tau})(F_{3i+2\xi} + F_{3i+3\kappa})}, \]

\[ \Psi_{12n+3} = \frac{\eta \mu}{(\lambda + \mu)} \prod_{i=0}^{n-1} \frac{(F_{3i+2\eta} + F_{3i+3\xi})(F_{3i+3\lambda} + F_{3i+4\mu})(F_{3i+1\sigma} + F_{3i+2\tau})(F_{3i+2\xi} + F_{3i+3\kappa})}{(F_{3i+3\eta} + F_{3i+4\xi})(F_{3i+1\lambda} + F_{3i+4\mu})(F_{3i+2\sigma} + F_{3i+3\tau})(F_{3i+3\xi} + F_{3i+4\kappa})}, \]

\[ \Psi_{12n+4} = \frac{\sigma \zeta \kappa}{(\eta + \zeta)(\zeta + \kappa)} \prod_{i=0}^{n-1} \frac{(F_{3i+3\eta} + F_{3i+4\xi})(F_{3i+1\lambda} + F_{3i+2\mu})}{(F_{3i+4\eta} + F_{3i+5\xi})(F_{3i+2\lambda} + F_{3i+3\mu})}, \]

\[ \Psi_{12n+5} = \frac{\lambda \tau (\zeta + \kappa)}{(\sigma + \tau)(\zeta + 2\kappa)} \prod_{i=0}^{n-1} \frac{(F_{3i+3\eta} + F_{3i+4\xi})(F_{3i+3\lambda} + F_{3i+4\mu})(F_{3i+4\sigma} + F_{3i+5\tau})(F_{3i+5\xi} + F_{3i+6\kappa})}{(F_{3i+3\sigma} + F_{3i+4\tau})(F_{3i+4\xi} + F_{3i+5\kappa})}, \]

\[ \Psi_{12n+6} = \frac{\eta \mu (\sigma + \tau)}{(\lambda + \mu)(\sigma + 2\tau)} \prod_{i=0}^{n-1} \frac{(F_{3i+4\sigma} + F_{3i+5\tau})(F_{3i+2\xi} + F_{3i+3\kappa})}{(F_{3i+3\eta} + F_{3i+4\xi})(F_{3i+4\lambda} + F_{3i+5\mu})(F_{3i+5\sigma} + F_{3i+6\tau})(F_{3i+3\xi} + F_{3i+4\kappa})}, \]

\[ \Psi_{12n+7} = \frac{\sigma \zeta \kappa (\lambda + \mu)}{(\eta + \zeta)(\lambda + 2\mu)(\zeta + \kappa)} \prod_{i=0}^{n-1} \frac{(F_{3i+2\sigma} + F_{3i+3\tau})(F_{3i+3\xi} + F_{3i+4\kappa})}{(F_{3i+4\eta} + F_{3i+5\xi})(F_{3i+5\lambda} + F_{3i+6\mu})(F_{3i+3\sigma} + F_{3i+4\tau})(F_{3i+4\xi} + F_{3i+5\kappa})}, \]

\[ \Psi_{12n+8} = \frac{\lambda \tau (\eta + \zeta)(\zeta + \kappa)}{(\eta + 2\zeta)(\sigma + \tau)(\zeta + 2\kappa)} \prod_{i=0}^{n-1} \frac{(F_{3i+3\eta} + F_{3i+4\xi})(F_{3i+2\lambda} + F_{3i+3\mu})}{(F_{3i+4\sigma} + F_{3i+5\tau})(F_{3i+2\xi} + F_{3i+3\kappa})(F_{3i+3\eta} + F_{3i+4\xi})(F_{3i+4\lambda} + F_{3i+5\mu})}, \]

\[ \Psi_{12n+9} = \frac{\eta \mu (\sigma + \tau)(\zeta + 2\kappa)}{(\lambda + \mu)(\sigma + 2\tau)(2\zeta + 3\kappa)} \prod_{i=0}^{n-1} \frac{(F_{3i+4\sigma} + F_{3i+5\tau})(F_{3i+5\xi} + F_{3i+6\kappa})}{(F_{3i+3\eta} + F_{3i+4\xi})(F_{3i+3\lambda} + F_{3i+4\mu})(F_{3i+5\sigma} + F_{3i+6\tau})(F_{3i+6\xi} + F_{3i+7\kappa})}, \]

where \( \Psi_{-6} = \kappa, \Psi_{-5} = \tau, \Psi_{-4} = \mu, \Psi_{-3} = \xi, \Psi_{-2} = \sigma, \Psi_{-1} = \lambda, \Psi_0 = \eta \) and \( \{F_i\}_{i=0}^{\infty} = \{0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots\} \).
Proof.
For $n = 0$ the result holds. Now suppose that $n > 0$ and our assumption holds for $n - 1$, that is

$$
\Psi_{12n-14} = \alpha \prod_{i=0}^{n-2} \frac{({F}_{3i+1} + {F}_{3i+3})({F}_{3i+1} + {F}_{3i+2})({F}_{3i+2} + {F}_{3i+3})}{({F}_{3i+1} + {F}_{3i+3})({F}_{3i+2} + {F}_{3i+3})},
$$

$$
\Psi_{12n-13} = \lambda \prod_{i=0}^{n-2} \frac{({F}_{3i+1} + {F}_{3i+2})({F}_{3i+2} + {F}_{3i+3})}{({F}_{3i+1} + {F}_{3i+2})({F}_{3i+2} + {F}_{3i+3})},
$$

$$
\Psi_{12n-12} = \eta \prod_{i=0}^{n-2} \frac{({F}_{3i+1} + {F}_{3i+3})}{({F}_{3i+1} + {F}_{3i+3})},
$$

$$
\Psi_{12n-11} = \frac{\sigma \zeta}{(\eta + \zeta)(\zeta + \kappa)} \prod_{i=0}^{n-2} \frac{({F}_{3i+1} + {F}_{3i+3})}{({F}_{3i+1} + {F}_{3i+3})},
$$

$$
\Psi_{12n-10} = \frac{\lambda \tau}{(\sigma + \tau)(\zeta + \kappa)} \prod_{i=0}^{n-2} \frac{({F}_{3i+1} + {F}_{3i+3})}{({F}_{3i+1} + {F}_{3i+3})},
$$

$$
\Psi_{12n-9} = \frac{\eta \mu}{(\lambda + \mu)(\sigma + \tau)} \prod_{i=0}^{n-2} \frac{({F}_{3i+1} + {F}_{3i+3})}{({F}_{3i+1} + {F}_{3i+3})},
$$

$$
\Psi_{12n-8} = \frac{\alpha \zeta \kappa}{(\eta + \zeta)(\zeta + \kappa)} \prod_{i=0}^{n-2} \frac{({F}_{3i+1} + {F}_{3i+3})}{({F}_{3i+1} + {F}_{3i+3})},
$$

$$
\Psi_{12n-7} = \frac{\lambda \tau}{(\sigma + \tau)(\zeta + \kappa)} \prod_{i=0}^{n-2} \frac{({F}_{3i+1} + {F}_{3i+3})}{({F}_{3i+1} + {F}_{3i+3})},
$$

$$
\Psi_{12n-6} = \frac{\eta \mu}{(\lambda + \mu)(\sigma + \tau)} \prod_{i=0}^{n-2} \frac{({F}_{3i+1} + {F}_{3i+3})}{({F}_{3i+1} + {F}_{3i+3})},
$$

$$
\Psi_{12n-5} = \frac{\sigma \zeta \kappa}{(\eta + \zeta)(\zeta + \kappa)} \prod_{i=0}^{n-2} \frac{({F}_{3i+1} + {F}_{3i+3})}{({F}_{3i+1} + {F}_{3i+3})},
$$
\[
\Psi_{12n-4} = \frac{\lambda \tau(\eta + \zeta)(\zeta + \kappa)}{(\eta + 2\zeta)(\sigma + \tau)(\zeta + 2\kappa)} \prod_{i=0}^{n-2} \left( \frac{F_{3i+4\eta} + F_{3i+4}\zeta(F_{3i+2}\lambda + F_{3i+3}\mu)}{(F_{3i+3\sigma} + F_{3i+4}\tau)(F_{3i+4\zeta} + F_{3i+5}\kappa)} \right),
\]

\[
\Psi_{12n-3} = \frac{\eta \mu(\sigma + \tau)(\zeta + 2\kappa)}{(\lambda + \mu)(\sigma + 2\tau)(2\zeta + 3\kappa)} \prod_{i=0}^{n-2} \left( \frac{F_{3i+3\eta} + F_{3i+3}\zeta(F_{3i+5\lambda} + F_{3i+4}\mu)}{(F_{3i+4\sigma} + F_{3i+5}\tau)(F_{3i+5\zeta} + F_{3i+6}\kappa)} \right).
\]

Now, we prove that the results are holds for \( n \). From Eq.(9), it follows that

\[
\Psi_{12n-2} = \Psi_{12n-5} - \frac{\Psi_{12n-5}\Psi_{12n-6}}{\Psi_{12n-6} + \Psi_{12n-9}}
\]

\[
= \frac{\sigma \zeta \kappa(\lambda + \mu)}{(\eta + \zeta)(\lambda + 2\mu)(\zeta + \kappa)} \prod_{i=0}^{n-2} \left( \frac{F_{3i+3\eta} + F_{3i+3}\zeta(F_{3i+4\lambda} + F_{3i+5}\mu)}{(F_{3i+4\sigma} + F_{3i+5}\tau)(F_{3i+5\zeta} + F_{3i+6}\mu)} \right)
\]

\[
= \frac{\eta \mu(\sigma + \tau)}{(\lambda + \mu)(\sigma + 2\tau)} \prod_{i=0}^{n-2} \left( \frac{F_{3i+2\eta} + F_{3i+2}\zeta(F_{3i+4}\lambda + F_{3i+5}\mu)(F_{3i+4\sigma} + F_{3i+5}\tau)(F_{3i+5}\zeta) + F_{3i+5\kappa}}{(F_{3i+4\sigma} + F_{3i+5}\tau)(F_{3i+5}\zeta) + F_{3i+5\kappa}} \right)
\]

\[
\quad + \frac{\eta \mu(\sigma + \tau)}{(\lambda + \mu)(\sigma + 2\tau)} \prod_{i=0}^{n-2} \left( \frac{F_{3i+2\eta} + F_{3i+2}\zeta(F_{3i+4}\lambda + F_{3i+5}\mu)(F_{3i+4\sigma} + F_{3i+5}\tau)(F_{3i+5}\zeta) + F_{3i+5\kappa}}{(F_{3i+4\sigma} + F_{3i+5}\tau)(F_{3i+5}\zeta) + F_{3i+5\kappa}} \right)
\]

\[
= \frac{\sigma \zeta \kappa(\lambda + \mu)}{(\eta + \zeta)(\lambda + 2\mu)(\zeta + \kappa)} \prod_{i=0}^{n-2} \left( \frac{F_{3i+4\eta} + F_{3i+4}\zeta(F_{3i+5\lambda} + F_{3i+4}\mu)}{(F_{3i+4\sigma} + F_{3i+5}\tau)(F_{3i+4}\zeta) + F_{3i+5\kappa}} \right)
\]

\[
= \frac{\eta \mu(\sigma + \tau)}{(\lambda + \mu)(\sigma + 2\tau)} \prod_{i=0}^{n-2} \left( \frac{F_{3i+4\eta} + F_{3i+4}\zeta(F_{3i+5\lambda} + F_{3i+4}\mu)}{(F_{3i+4\sigma} + F_{3i+5}\tau)(F_{3i+4}\zeta) + F_{3i+5\kappa}} \right)
\]
Thus,

\[
\Psi_{12n-2} = \sigma \prod_{i=0}^{n-1} \left( \frac{(F_{3i+3}\eta + F_{3i+4}\zeta)(F_{3i+4}\lambda + F_{3i+5} \mu)}{(F_{3i+3}\eta + F_{3i+4}\zeta)(F_{3i+4}\lambda + F_{3i+5} \mu) - (F_{3i+3}\sigma + F_{3i+4}\tau)} \right) .
\]

Other relations can be proved in the same way.
5.4. Fourth Equation

In this subsection, we will find the solution of Eq.(1) when $\alpha = \gamma = 1$, and $\beta = \delta = -1$, so the Eq.(1) become as

$$
\Psi_{n+1} = \Psi_{n-2} - \frac{\Psi_{n-2}\Psi_{n-3}}{\Psi_{n-3} - \Psi_{n-6}}, \quad n = 0, 1, 2, ..., \quad (10)
$$

where the initial conditions $\Psi_{-6}, \Psi_{-5}, \Psi_{-4}, \Psi_{-3}, \Psi_{-2}, \Psi_{-1}$ and $\Psi_{0}$ are arbitrary positive real numbers.

**Theorem 5.4.** Assume $(\Psi_{n})_{n=-6}^{\infty}$ be a solution of Eq.(10). Thus for $n=0, 1, 2, ..,$

$$
\begin{align*}
\Psi_{12n-6} &= (-1)^{n} \eta^{n} \mu^{n} (\sigma - \tau)^{n}, \\
\Psi_{12n-5} &= \frac{\sigma^{n} \zeta^{n} K^{n} (\lambda - \mu)^{n}}{\lambda^{n} r^{n} (\eta - \zeta)^{n} (\zeta - \kappa)^{n}}, \\
\Psi_{12n-4} &= \frac{(-1)^{n} \eta^{n} \mu^{n} (\sigma - \tau)^{n}}{\sigma^{n} K^{n} (\lambda - \mu)^{n}}, \\
\Psi_{12n-3} &= \frac{(-1)^{n} \eta^{n} \mu^{n} (\sigma - \tau)^{n}}{\sigma^{n} K^{n} (\lambda - \mu)^{n}}, \\
\Psi_{12n-2} &= \frac{\sigma^{n+1} \zeta^{n} K^{n+1} (\lambda - \mu)^{n+1}}{\lambda^{n} r^{n} (\eta - \zeta)^{n} (\zeta - \kappa)^{n+1}}, \\
\Psi_{12n-1} &= \frac{(-1)^{n+1} \eta^{n+1} \mu^{n+1} (\sigma - \tau)^{n+1}}{\eta^{n} \zeta^{n} K^{n+1} (\zeta - \kappa)^{n+1}}, \\
\Psi_{12n} &= \frac{(-1)^{n} \eta^{n+1} \mu^{n+1} (\sigma - \tau)^{n+1}}{\sigma^{n} K^{n} (\lambda - \mu)^{n}}, \\
\Psi_{12n+1} &= \frac{\sigma^{n+1} \zeta^{n} K^{n+1} (\lambda - \mu)^{n+1}}{\lambda^{n} r^{n} (\eta - \zeta)^{n} (\zeta - \kappa)^{n+1}}, \\
\Psi_{12n+2} &= \frac{(-1)^{n+1} \eta^{n+1} \mu^{n+1} (\sigma - \tau)^{n+1}}{\eta^{n} \zeta^{n} K^{n+1} (\zeta - \kappa)^{n+1}}, \\
\Psi_{12n+3} &= \frac{(-1)^{n+1} \eta^{n+1} \mu^{n+1} (\sigma - \tau)^{n+1}}{\sigma^{n} K^{n} (\lambda - \mu)^{n+1}}, \\
\Psi_{12n+4} &= \frac{\sigma^{n+1} \zeta^{n} K^{n+1} (\lambda - \mu)^{n+1}}{\lambda^{n} r^{n} (\eta - \zeta)^{n+1} (\zeta - \kappa)^{n+1}}, \\
\Psi_{12n+5} &= \frac{(-1)^{n+1} \eta^{n+1} \mu^{n+1} (\sigma - \tau)^{n+1}}{\eta^{n} \zeta^{n} K^{n+1} (\zeta - \kappa)^{n+1}},
\end{align*}
$$

where $\Psi_{-6} = \kappa, \Psi_{-5} = \tau, \Psi_{-4} = \mu, \Psi_{-3} = \zeta, \Psi_{-2} = \sigma, \Psi_{-1} = \lambda, \Psi_{0} = \eta$.

**Proof.**

For $n = 0$ the result holds. Now suppose that $n > 0$ and our assumption holds for $n-1$, that is
Now, we prove that the results are holds for \( n \). From Eq.(10), it follows that

\[
\Psi_{12n-6} = \Psi_{12n-9} - \Psi_{12n-9} \Psi_{12n-10} / \Psi_{12n-10} - \Psi_{12n-13}
\]

\[
= (-1)^{n} \eta^{n} \mu^{n} (\sigma - r)^{n-1} / \sigma^{n-1} \kappa^{n-1} (\lambda - \mu)^{n-1} - (-1)^{n} \lambda^{n} \tau^{n} (\eta - \zeta)^{n-1} (\zeta - \kappa)^{n-1} / \eta^{n-1} \zeta^{n-1} \mu^{n-1} (\sigma - r)^{n-1}
\]

\[
= (-1)^{n} \eta^{n} \mu^{n} (\sigma - r)^{n-1} / \sigma^{n-1} \kappa^{n-1} (\lambda - \mu)^{n-1} - (-1)^{n} \lambda^{n} \tau^{n} (\eta - \zeta)^{n-1} (\zeta - \kappa)^{n-1} / \eta^{n-1} \zeta^{n-1} \mu^{n-1} (\sigma - r)^{n-1}
\]

\[
= (-1)^{n} \eta^{n} \mu^{n} (\sigma - r)^{n-1} / \sigma^{n-1} \kappa^{n-1} (\lambda - \mu)^{n-1} - (-1)^{n} \eta^{n} \mu^{n} \tau^{n} (\sigma - r)^{n-1} / \sigma^{n-1} \kappa^{n-1} (\lambda - \mu)^{n-1}
\]

\[
= (-1)^{n} \eta^{n} \mu^{n} (\sigma - r)^{n-1} / \sigma^{n-1} \kappa^{n-1} (\lambda - \mu)^{n-1}.
\]

So we have

\[
\Psi_{12n-6} = (-1)^{n} \eta^{n} \mu^{n} (\sigma - r)^{n} / \sigma^{n-1} \kappa^{n-1} (\lambda - \mu)^{n}.
\]
Similarly,

\[
\Psi_{12n-5} = \Psi_{12n-8} - \frac{\Psi_{12n-9}}{\Psi_{12n-9} - \Psi_{12n-12}}
= \frac{\sigma_n^\eta \xi_n^\sigma (\eta - \lambda)^n}{\lambda^{-1} n^{-1}} (\eta - \lambda)^n (\zeta - \lambda)^n
- \frac{\sigma_n^\sigma \xi_n^\sigma (\eta - \lambda)^n}{\lambda^{-1} n^{-1}} (\eta - \lambda)^n (\zeta - \lambda)^n
\]

Hence, we obtain

\[
\Psi_{12n-5} = \frac{\sigma_n^\eta \xi_n^\sigma (\eta - \lambda)^n}{\lambda^{-1} n^{-1}} (\eta - \lambda)^n (\zeta - \lambda)^n.
\]

Similarly, by using the same method, we can investigate other relations.

6. Numerical Examples

For our prior results, we present some numerical examples to explain the solution behavior of Eq.(1).

Example 1. In numerical simulation they assumed that for Eq.(7) the initial value are \(\Psi_{-6} = 0.3, \Psi_{-5} = 0.6, \Psi_{-4} = 0.9, \Psi_{-3} = 1.2, \Psi_{-2} = 1.5, \Psi_{-1} = 1.8\) and \(\Psi_0 = 2.1\). Then the solution appear in Figure 1.

![Figure 1. Plotting the solution of \(\Psi_{n+1} = \Psi_{n-2} + \frac{\Psi_{n-2}}{\Psi_{n-3} + \Psi_{n-6}}\).](image)

Example 2. Numerically when the initial value are \(\Psi_{-6} = 4.6, \Psi_{-5} = 2.5, \Psi_{-4} = 1.4, \Psi_{-3} = 3, \Psi_{-2} = 4.5, \Psi_{-1} = 6.3\) and \(\Psi_0 = 3.5\). Figure 2 shows the results of Eq.(8).
Example 3. Figures 3 depict the behavior of Eq.(9), with initial conditions are $\Psi_{-6} = 2.8, \Psi_{-5} = 5.9, \Psi_{-4} = 8.5, \Psi_{-3} = 4.2, \Psi_{-2} = 7.4, \Psi_{-1} = 3.2$ and $\Psi_0 = 6.7$.

Example 4. For Eq.(10) the initial conditions are set as follows: $\Psi_{-6} = 2.2, \Psi_{-5} = 3.9, \Psi_{-4} = 7.5, \Psi_{-3} = 4.2, \Psi_{-2} = 4.8, \Psi_{-1} = 3.2$ and $\Psi_0 = 6.7$, results shows in Figure 4.
7. Conclusions

Studying the dynamics of such equations is a very significant mathematical topic since these equations are strongly related to models in population dynamics and biological sciences. The basic goal of equations dynamics is to predict the global behavior of a equation based on the information of its current state. In this article, we have found general form of the solutions of rational difference equations and we investigated the dynamics of equilibrium point. In sections 2 and 3, we have investigated the existence and uniqueness of equilibrium point and the solutions qualitative behavior is explored, such as local and global stability. Also, we have proven that the solution is bounded in section 4. In section 5, we have obtained expressions of solutions of four special cases of the studied equations 7,8,9 and 10, as applications of Eq.(1). Finally, to support our theoretical discussion some illustrative examples are provided in section 6.

**Author Contributions**

All authors contributed equally to this work. They all read and approved the final version of the manuscript.

**Conflicts of Interest**

The authors declare no conflict of interest.

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