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Research Article

Coefficient estimates for starlike and convex functions associated with cosine function

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Abstract

This paper deals with the new classes $\mathcal{S}^*_{\cos}$ and $\mathcal{S}^{\mathbf{c}}_{\cos}$ of starlike and convex functions, respectively, associated with the cosine function. We give initial coefficient bounds for the first seven coefficients of the functions that belong to these classes, and we evaluate the upper bounds for the Hankel determinant of order three and four. We found the upper bound of Zalcman functional for the above mentioned classes for the cases $n = 3$ and $n = 4$, showing that the Zalcman conjecture holds for these values. Moreover, we determined lower and upper bounds for the difference $|a_4| - |a_3|$ of the coefficients for the functions that belong to these classes.

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1. Introduction and preliminaries

Let denote by $\mathcal{A}$ the class of all analytic and normalized functions f of the form

$$f(z) = z + a_2z^2 + a_3z^3 + \ldots,\ z \in \mathbb{D}, \tag{1.1}$$

where $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ is the open unit disc, and let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting in the univalent functions in $\mathbb{D}$.

Let F and G be the two analytic functions in $\mathbb{D}$. The function F is said to be *subordinated* to G, written symbolically as $F(z) \prec G(z)$, if there exists a function η analytic in $\mathbb{D}$, with $\eta(0) = 0$ and $|\eta(z)| < 1$ for all $z \in \mathbb{D}$, such that $F(z) = G(\eta(z))$, $z \in \mathbb{D}$. In the case if G is univalent in $\mathbb{D}$ the next equivalence holds:

$$F(z) \prec G(z) \Leftrightarrow F(0) = G(0) \text{ and } F(\mathbb{D}) \subset G(\mathbb{D}).$$

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Let us define by $\mathcal{P}$ the well-known Carathéodory class, that is the family of holomorphic functions p in $\mathbb{D}$ that satisfies the condition $\operatorname{Re} p(z) > 0$, $z \in \mathbb{D}$, and of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} t_n z^n, \ z \in \mathbb{D}. \tag{1.2}$$

The subclass of $\mathcal{S}$ defined by

$$\mathcal{S}^* := \left\{ f \in \mathcal{A} : \operatorname{Re} \frac{zf'(z)}{f(z)} > 0, \ z \in \mathbb{D} \right\}$$

is called the class of starlike (univalent) functions in $\mathbb{D}$. Based on the geometric properties of the image of the open unit disc by some analytic functions, the functions can be categorized into different families. Thus, in 1992 Ma and Minda [25] extended various subclasses of starlike functions for which the quantity $\dfrac{zf'(z)}{f(z)}$ is subordinated to a more general function. They considered an analytic function ϕ with positive real part in the unit disc $\mathbb{D}$, with $\phi(0) = 1$, $\phi'(0) > 0$, and ϕ maps $\mathbb{D}$ onto a starlike domain with respect to 1 and symmetric with respect to the real axis. The class of Ma-Minda starlike functions consists of functions $f \in \mathcal{A}$ satisfying the subordination $\dfrac{zf'(z)}{f(z)} \prec \phi(z)$.

Remark 1.1. By varying the function ϕ we can obtain several familiar subclasses as follows:

(i) For $\phi(z) = \dfrac{1+Az}{1+Bz}$, $-1 \le B < A \le 1$, we obtain the class $\mathcal{S}^*(A, B)$ defined and studied in [17];
(ii) For $\phi(z) = \sqrt{1+z}$ we get the class $\mathcal{S}^*_L$ defined and studied in [40];
(iii) For $\phi(z) = z + \sqrt{1+z^2}$ we obtain the class $\mathcal{S}^*_l$ introduced and investigated in [34];
(iv) For $\phi(z) = 1 + \dfrac{2}{\pi^2}\left(\log \dfrac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2$ we get the class defined and studied in [36];
(v) For $\phi(z) = \cosh z$ we get the class $\mathcal{S}^*_{\cosh}$ defined and investigated in [2];
(vi) For $\phi(z) = 1 + \dfrac{4}{3}z + \dfrac{2}{3}z^2$ we obtain the class $\mathcal{S}^*_c$ introduced and studied in [37];
(vii) For $\phi(z) = e^z$ we have the class $\mathcal{S}^*_e$ defined in [29];
(viii) For $\phi(z) = 1 + \sin z$ we get the class $\mathcal{S}^*_{\sin}$ (for details see [4, 13, 42]);
(ix) For $\phi(z) = \dfrac{2}{1+e^{-z}}$ we obtain the class $\mathcal{S}^*_{SG}$ defined in [15] and extensively studied in [28].

Recently, in [7] the authors introduced and studied the class $\mathcal{S}^*_{\cos}$ defined by

$$\mathcal{S}^*_{\cos} := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \cos z =: \phi(z) \right\},$$

and let define the class

$$\mathcal{S}^{\mathbf{c}}_{\cos} := \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \cos z =: \phi(z) \right\}.$$

It is noteworthy that the functions cosine and cosine hyperbolic functions have the same image of the open unit disc $\mathbb{D}$, hence $\mathcal{S}^*_{\cos} \equiv \mathcal{S}^*_{\widehat{\phi}}$ and $\mathcal{S}^{\mathbf{c}}_{\cos} \equiv \mathcal{S}^{\mathbf{c}}_{\widehat{\phi}}$, where $\widehat{\phi} = \cosh$.

In recent years, finding upper bounds for the modules of Hankel determinants for different subclasses of analytic functions become an active area of research in the Geometric Function Theory. The Hankel determinant $H_{j,k}(f)$ has been introduced by Pommerenke

[32] as follows

$$H_{j,k}(f)=\begin{vmatrix} a_k & a_{k+1} & \cdots & a_{k+j-1} \\ a_{k+1} & a_{k+2} & \cdots & a_{k+j} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k+j-1} & a_{k+j} & \cdots & a_{k+2j-2} \end{vmatrix},$$

where $f \in \mathcal{A}$ and $j, k \in \mathbb{N}$.

By specializing the different values for j and k we can obtain different Hankel determinants. Thus, for $j = 2$ and $k = 1$ we get

$$H_{2,1}(f)=\begin{vmatrix} 1 & a_2 \\ a_2 & a_3 \end{vmatrix}=a_3-a_2^2,$$

and note that $H_{2,1}(f)$ is the classical Fekete-Szegő functional. For different subclasses of $\mathcal{A}$, the maximum value of $|H_{2,1}(f)|$ has been obtained by different authors, like [16, 22, 26, 27, 38, 41].

Furthermore, for $j = 2$ and $k = 2$ we have

$$H_{2,2}(f)=\begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix}=a_2a_4-a_3^2,$$

and the upper bound for $|H_{2,2}(f)|$ has been investigated by several authors (see also [1, 30, 31]).

For the third order Hankel determinant

$$H_{3,1}(f)=\begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}=a_5\left(a_3-a_2^2\right)-a_4(a_4-a_2a_3)+a_3\left(a_2a_4-a_3^2\right), \tag{1.3}$$

Babalola [6] was the first that studied this determinant for subclasses of $\mathcal{S}$ (also refer to the articles [12, 19, 21–23, 41]).

Recently, the upper bound for $|H_{4,1}(f)|$ has been studied by several authors like [3, 5, 21, 28]. The fourth order Hankel determinant is obtained as follows,

$$H_{4,1}(f)=a_7H_{3,1}(f)-a_6\delta_1+a_5\delta_2-a_4\delta_3. \tag{1.4}$$

where $H_{3,1}(f)$ is given by (1.3) and

$$\delta_1 := (a_3a_6-a_4a_5)-a_2(a_2a_6-a_3a_5)+a_4\left(a_2a_4-a_3^2\right), \tag{1.5}$$

$$\delta_2 := (a_4a_6-a_5^2)-a_2(a_3a_6-a_4a_5)+a_3(a_3a_5-a_4^2), \tag{1.6}$$

$$\delta_3 := a_2(a_4a_6-a_5^2)-a_3(a_3a_6-a_4a_5)+a_4(a_3a_5-a_4^2). \tag{1.7}$$

Very recently, Khan et al. [20] obtained the upper bound for $|H_{3,1}(f)|$ for the class $\mathcal{S}_s^*(\phi)$ of starlike functions with respect to symmetric points related with sine function (see also [4]). Inspired by the above work we have determined the bound for the initial seven coefficients, the upper bounds for the fourth order Hankel determinant and for the Zalcman functional for the classes $\mathcal{S}^*_{\cos}$ and $\mathcal{S}^{\mathbf{c}}_{\cos}$ associated with cosine function.

The proofs of our first main results will use the following lemmas.

Lemma 1.2. *If $p \in \mathcal{P}$ has the form* (1.2)*, then*

$$|t_n| \le 2, \text{ for } n \ge 1, \tag{1.8}$$

$$|t_{i+j}-\mu t_i t_j| \le 2\max\{1; |1-2\mu|\}, \text{ for } \mu \in \mathbb{C}, \tag{1.9}$$

and for any complex number ζ we have

$$|t_2-\zeta t_1^2| \le 2\max\{1; |2\zeta-1|\}. \tag{1.10}$$

We mention that the first inequality is the well-known Carathéodory's result (see [10, 11]), the second one was obtained in [14], and the third in [18] (see also [33]).

Lemma 1.3 ([4, Lemma 2.2]). *If $p \in \mathcal{P}$ has the form* (1.2), *then*

$$|\alpha t_1^3 - \beta t_1 t_2 + \gamma t_3| \leq 2|\alpha| + 2|\beta - 2\alpha| + 2|\alpha - \beta + \gamma|. \tag{1.11}$$

Lemma 1.4 ([35, Lemma 2.1]). *Let ℓ, j, k, and r be real numbers such that $0 < \ell < 1$, $0 < r < 1$, and*

$$8r(1-r)\left[(\ell j - 2k)^2 + (\ell(r+\ell) - j)^2\right] + \ell(1-\ell)(j - 2r\ell)^2 \leq 4\ell^2(1-\ell)^2 r(1-r).$$

If $p \in \mathcal{P}$ has the form (1.2), *then*

$$\left|k t_1^4 + r t_2^2 + 2\ell t_1 t_3 - \frac{3}{2} j t_1^2 t_2 - t_4\right| \leq 2. \tag{1.12}$$

2. Initial coefficients estimates for the classes $\mathcal{S}^*_{\cos}$ and $\mathcal{S}^{\mathbf{c}}_{\cos}$

In this section we analyse the coefficients of the functions of the class $\mathcal{S}^*_{\cos}$ and $\mathcal{S}^{\mathbf{c}}_{\cos}$, and we find the upper bounds for the first seven coefficients.

Theorem 2.1. *If $f \in \mathcal{S}^*_{\cos}$ has the form* (1.1), *then*

$$a_2 = 0, \quad |a_3| \leq \frac{1}{4}, \quad |a_4| \leq \frac{1}{3}, \quad |a_5| \leq \frac{11}{24}, \quad |a_6| \leq \frac{4}{5}, \quad |a_7| \leq \frac{179}{252}.$$

Proof. If $f \in \mathcal{S}^*_{\cos}$, then there exists a Schwartz function η, that is η is analytic in $\mathbb{D}$ and satisfy the conditions $\eta(0) = 0$ and $|\eta(z)| < 1$ for all $z \in \mathbb{D}$, such that

$$\frac{zf'(z)}{f(z)} = \phi(\eta(z)) = \cos \eta(z), \ z \in \mathbb{D}.$$

Since the function f has the form (1.1), it follows that

$$\begin{aligned}\frac{zf'(z)}{f(z)} =& 1 + a_2 z + \left(2a_3 - a_2^2\right) z^2 + \left(3a_4 - 3a_2 a_3 + a_2^3\right) z^3 \\ &+ \left(4a_5 - 2a_3^2 + 4a_2^2 a_3 - a_2^4 - 4a_2 a_4\right) z^4 \\ &+ \left(5a_6 - 5a_2 a_5 - 5a_3 a_4 + 5a_2^2 a_4 + 5a_2 a_3^2 - 5a_3 a_2^3 + a_2^5\right) z^5 \\ &+ \Big(6a_7 - 6a_2 a_6 - 6a_3 a_5 - 3a_4^2 + 6a_2^2 a_5 + 12 a_2 a_3 a_4 + 2a_3^3 - 6a_2^3 a_4 - 9a_2^2 a_3^2 \\ &+ 6a_2^4 a_3 - a_2^6\Big) z^6 + \ldots, \ z \in \mathbb{D}.\end{aligned} \tag{2.1}$$

From the fact that $\eta(0) = 0$ and $|\eta(z)| < 1$ for all $z \in \mathbb{D}$, if we define the function p by

$$p(z) := \frac{1 + \eta(z)}{1 - \eta(z)} = 1 + t_1 z + t_2 z^2 + \ldots, \ z \in \mathbb{D},$$

we obtain that $p \in \mathcal{P}$ and

$$\eta(z) = \frac{p(z) - 1}{p(z) + 1} = \frac{t_1 z + t_2 z^2 + \ldots}{2 + t_1 z + t_2 z^2 + \ldots}, \ z \in \mathbb{D}.$$

According to the above relation we get

$$\begin{aligned}\cos \eta(z) =& 1 - \frac{t_1^2}{8} z^2 + \left(-\frac{t_1 t_2}{4} + \frac{t_1^3}{8}\right) z^3 + \left(-\frac{35\, t_1^4}{384} - \frac{t_2^2}{8} + \frac{3\, t_2 t_1^2}{8} - \frac{t_1 t_3}{4}\right) z^4 \\ &+ \left(-\frac{t_3 t_2}{4} + \frac{3\, t_3 t_1^2}{8} + \frac{3\, t_1 t_2^2}{8} - \frac{35\, t_2 t_1^3}{96} + \frac{11\, t_1^5}{192} - \frac{t_1 t_4}{4}\right) z^5 \\ &+ \Bigg(\frac{3\, t_3 t_1 t_2}{4} + \frac{t_2^3}{8} - \frac{1501\, t_1^6}{46080} - \frac{t_1 t_5}{4} - \frac{t_4 t_2}{4} + \frac{3\, t_4 t_1^2}{8} - \frac{t_3^2}{8} - \frac{35\, t_3 t_1^3}{96} \\ &- \frac{35\, t_2^2 t_1^2}{64} + \frac{55\, t_2 t_1^4}{192}\Bigg) z^6 + \ldots, \ z \in \mathbb{D},\end{aligned} \tag{2.2}$$

and equating the corresponding coefficients of (2.1) and (2.2) we obtain

$$a_2 =0, \tag{2.3}$$
$$a_3 =-\frac{1}{16}t_1^2, \tag{2.4}$$
$$a_4 =\frac{1}{3}\left(\frac{1}{8}t_1^3-\frac{1}{4}t_1t_2\right), \tag{2.5}$$
$$a_5 =-\frac{1}{8}\left(\frac{1}{6}t_1^4+\frac{1}{4}t_2^2-\frac{3}{4}t_1^2t_2+\frac{1}{2}t_3t_1\right), \tag{2.6}$$
$$a_6 =\frac{1}{5}\left(\frac{17}{384}t_1^5-\frac{65}{192}t_1^3t_2+\frac{3}{8}t_1^2t_3+\frac{3}{8}t_1t_2^2-\frac{1}{4}t_1t_4-\frac{1}{4}t_2t_3\right), \tag{2.7}$$
$$a_7 =\frac{1}{6}\left(-\frac{1757}{92160}t_1^6-\frac{395}{768}t_1^2t_2^2-\frac{131}{384}t_1^3t_3+\frac{177}{768}t_1^4t_2+\frac{3}{8}t_1^2t_4+\frac{1}{8}t_2^3-\frac{1}{4}t_1t_5\right.$$
$$\left.+\frac{3}{4}t_1t_2t_3-\frac{1}{4}t_2t_4-\frac{1}{8}t_3^2\right). \tag{2.8}$$

Using (2.4) we get

$$|a_3|=\frac{1}{16}|t_1|^2,$$

and from (1.8) we have $|t_1|\leq 2$, hence

$$|a_3|\leq\frac{1}{4}.$$

The relation (2.5) leads to

$$|a_4|=\left|\frac{t_1}{12}\right|\left|t_2-\frac{t_1^2}{2}\right|,$$

and according to (1.8) and (1.10), we obtain

$$|a_4|\leq\frac{2}{12}\cdot 2=\frac{1}{3}.$$

We can write the equality (2.6) like

$$a_5=-\frac{t_1}{8}\left(\frac{1}{6}t_1^3-\frac{3}{4}t_1t_2+\frac{1}{2}t_3\right)-\frac{t_2^2}{32},$$

and using triangle inequality this implies

$$|a_5|\leq\frac{|t_1|}{8}\left|\frac{1}{6}t_1^3-\frac{3}{4}t_1t_2+\frac{1}{2}t_3\right|+\frac{1}{32}|t_2|^2.$$

From (1.8) and Lemma 1.3 for the appropriate values $\alpha=\frac{1}{6}$, $\beta=\frac{3}{4}$, and $\gamma=\frac{1}{2}$, the above inequality implies that

$$|a_5|\leq\frac{11}{24}.$$

If we add and subtract $\frac{t_1t_4}{20}$ from the righthand side of (2.7), and using then the triangle inequality we have

$$|a_6|=\left|\frac{1}{10}t_1\left(\frac{17}{192}t_1^4-\frac{65}{96}t_1^2t_2+\frac{3}{4}t_1t_3+\frac{3}{4}t_2^2-t_4\right)+\frac{1}{20}t_1t_4-\frac{1}{20}t_2t_3\right|$$
$$\leq\left|\frac{t_1}{10}\right|\left|\frac{17}{192}t_1^4-\frac{65}{96}t_1^2t_2+\frac{3}{4}t_1t_3+\frac{3}{4}t_2^2-t_4\right|+\left|\frac{1}{20}t_1t_4\right|+\left|\frac{1}{20}t_2t_3\right|.$$

From (1.8) and Lemma 1.4 for the values $k=\frac{17}{192}$, $r=\frac{3}{4}$, $\ell=\frac{3}{8}$, and $j=\frac{65}{144}$, from the inequality (1.12) we conclude that

$$|a_6| \leq \frac{4}{10}+\frac{4}{20}+\frac{4}{20}=\frac{4}{5}.$$

From (2.8) we have

$$\begin{aligned}a_7 =&\frac{1}{6}\left(-\frac{1757}{92160}t_1^6-\frac{1185}{2304}t_1^2t_2^2-\frac{131}{384}t_1^3t_3+\frac{531}{2304}t_1^4t_2+\frac{3}{8}t_1^2t_4+\frac{1}{8}t_2^3-\frac{1}{4}t_1t_5\right.\\ &\left.+\frac{3}{4}t_1t_2t_3-\frac{1}{4}t_2t_4-\frac{1}{8}t_3^2\right),\end{aligned}$$

and rearranging the terms of the above equality we get

$$\begin{aligned}6a_7 =&\frac{1}{8}\left(t_2-t_1^2\right)\left(\frac{1757}{11520}t_1^4+\frac{64}{96}t_2^2+\frac{59}{48}t_1t_3-\frac{110}{96}t_1^2t_2-t_4\right)\\ &+\frac{t_2}{12}\left(\frac{4084}{7680}t_1^4+\frac{32}{64}t_2^2+\frac{56}{32}t_1t_3-\frac{115}{64}t_1^2t_2-t_4\right)\\ &+\frac{t_2}{12}\left(\frac{2199}{7680}t_1^4+\frac{1}{4}t_2^2+\frac{48}{32}t_1t_3-\frac{71}{64}t_1^2t_2-t_4\right)-\frac{t_3}{2}\left(\frac{72}{192}t_1^3-\frac{125}{192}t_1t_2+\frac{1}{4}t_3\right)\\ &+\frac{t_1t_2}{21}\left(t_3-\frac{735}{768}t_1t_2\right)-\frac{t_1}{4}\left(t_5-t_1t_4\right)-\frac{1}{21}t_1t_2t_3-\frac{1}{48}t_2^3+\frac{1}{24}t_2t_4,\end{aligned}$$

and from the triangle inequality it follows that

$$\begin{aligned}6|a_7| \leq&\frac{1}{8}\left|t_2-t_1^2\right|\left|\frac{1757}{11520}t_1^4+\frac{64}{96}t_2^2+\frac{59}{48}t_1t_3-\frac{110}{96}t_1^2t_2-t_4\right|\\ &+\frac{|t_2|}{12}\left|\frac{4084}{7680}t_1^4+\frac{32}{64}t_2^2+\frac{56}{32}t_1t_3-\frac{115}{64}t_1^2t_2-t_4\right|\\ &+\frac{|t_2|}{12}\left|\frac{2199}{7680}t_1^4+\frac{1}{4}t_2^2+\frac{48}{32}t_1t_3-\frac{71}{64}t_1^2t_2-t_4\right|\\ &+\frac{|t_3|}{2}\left|\frac{72}{192}t_1^3-\frac{125}{192}t_1t_2+\frac{1}{4}t_3\right|+\frac{|t_1||t_2|}{21}\left|t_3-\frac{735}{768}t_1t_2\right|+\frac{|t_1|}{4}\left|t_5-t_1t_4\right|\\ &+\frac{1}{21}|t_1||t_2||t_3|+\frac{1}{48}|t_2|^3+\frac{1}{24}|t_2||t_4|.\end{aligned}$$

Now we will use the inequalities (1.8), (1.9), (1.10) of Lemma 1.2, together with the inequality (1.11) of Lemma 1.3, and (1.12) of Lemma 1.4. Since it is easy to check that the assumption of Lemma 1.4 holds in each of the four above cases, the previous inequality leads to

$$6|a_7| \leq \frac{1}{8}\cdot 2\cdot 2+\frac{2}{12}\cdot 2+\frac{2}{12}\cdot 2+\frac{2}{2}+\frac{4}{21}\cdot 2+\frac{2}{4}\cdot 2+\frac{1}{21}\cdot 8+\frac{1}{48}\cdot 8+\frac{1}{24}\cdot 4=\frac{179}{42},$$

and all the estimations of the theorem are proved. □

Remark 2.2. The upper bounds given by Theorem 2.1 are not the best possible, excepting those for the first two coefficients. Thus, if we consider the function

$$f_*(z):=z\exp\left(\int_0^z\frac{\cos(c\,t)-1}{t}\,\mathrm{d}t\right)=z-\frac{c^2}{4}z^3+\frac{c^4}{24}z^5-\frac{47\,c^6}{8640}z^7+\cdots,\ z\in\mathbb{D},\ \text{with } |c|=1,$$

it is easy to check that $f_*\in\mathcal{S}^*_{\cos}$. For this function we have

$$|a_4|=0,\quad |a_5|=\frac{1}{24}<\frac{11}{120},\quad |a_6|=0,\quad |a_7|=\frac{47}{8640}<\frac{179}{252},$$

hence the estimations given by Theorem 2.1 are not sharp.

Theorem 2.3. *If* $f \in \mathcal{S}^{\mathbf{c}}_{\cos}$ *has the form* (1.1), *then*

$$a_2 = 0, \quad |a_3| \le \frac{1}{12}, \quad |a_4| \le \frac{1}{12}, \quad |a_5| \le \frac{11}{120}, \quad |a_6| \le \frac{2}{15}, \quad |a_7| \le \frac{179}{1764}.$$

Proof. From the definitions of the classes $\mathcal{S}^*_{\cos}$ and $\mathcal{S}^*_{\cos}$ it follows that

$$f \in \mathcal{S}^{\mathbf{c}}_{\cos} \Leftrightarrow zf'(z) = \sum_{n=1}^{\infty} na_n z^n \in \mathcal{S}^*_{\cos}.$$

Therefore, if $f \in \mathcal{S}^{\mathbf{c}}_{\cos}$, from the above equivalence and the inequalities of Theorem 2.1 it follows that

$$2a_2 = 0, \quad 3|a_3| \le \frac{1}{4}, \quad 4|a_4| \le \frac{1}{3}, \quad 5|a_5| \le \frac{11}{24}, \quad 6|a_6| \le \frac{4}{5}, \quad 7|a_7| \le \frac{179}{252},$$

and all our estimations are proved. □

Remark 2.4. (i) Like it was shown in the Remark 2.2, the upper bounds given by Theorem 2.1 are not the best possible, therefore the estimations given in the above theorem not sharp.

(ii) From the above equivalence, by using the relations (2.2)–(2.7) it follows that if $f \in \mathcal{S}^{\mathbf{c}}_{\cos}$ has the form (1.1), then

$$a_2 = 0, \tag{2.9}$$

$$a_3 = -\frac{1}{48}t_1^2, \tag{2.10}$$

$$a_4 = \frac{1}{12}\left(\frac{1}{8}t_1^3 - \frac{1}{4}t_1t_2\right), \tag{2.11}$$

$$a_5 = -\frac{1}{20}\left(\frac{1}{12}t_1^4 + \frac{1}{8}t_2^2 - \frac{3}{8}t_1^2t_2 + \frac{1}{4}t_3t_1\right), \tag{2.12}$$

$$a_6 = \frac{17}{11520}t_1^5 - \frac{13}{1152}t_2t_1^3 - \frac{1}{120}t_3t_2 + \frac{1}{80}t_3t_1^2 + \frac{1}{80}t_1t_2^2 - \frac{1}{120}t_1t_4, \tag{2.13}$$

$$a_7 = -\frac{251}{552960}t_1^6 - \frac{395}{32256}t_2^2t_1^2 + \frac{59}{10752}t_2t_1^4 - \frac{131}{16128}t_3t_1^3 + \frac{1}{56}t_3t_1t_2 - \frac{1}{168}t_1t_5 - \frac{1}{168}t_4t_2 + \frac{1}{112}t_4t_1^2 - \frac{1}{336}t_3^2 + \frac{1}{336}t_2^3. \tag{2.14}$$

3. Hankel determinants upper bound for the classes $\mathcal{S}^*_{\cos}$ and $\mathcal{S}^{\mathbf{c}}_{\cos}$

In this section we determine the upper bounds of the modules for the second, third, and fourth order Hankel determinant for the functions that belong to the classes $\mathcal{S}^*_{\cos}$ and $\mathcal{S}^{\mathbf{c}}_{\cos}$. For $H_{2,1}(f)$, $H_{2,2}(f)$, and $H_{3,1}(f)$ the results are immediately, as follows, respectively:

Theorem 3.1. *If* $f \in \mathcal{S}^*_{\cos}$ *has the form* (1.1), *then*

$$(i) \quad |H_{2,1}(f)| \le \frac{1}{4}, \qquad (ii) \quad |H_{2,2}(f)| \le \frac{1}{16}, \qquad (iii) \quad |H_{3,1}(f)| \le \frac{139}{576}.$$

Proof. (i) If $f \in \mathcal{S}^*_{\cos}$, then from (2.3) and (2.4) we get

$$H_{2,1}(f) = a_3 - a_2^2 = -\frac{1}{16}t_1^2,$$

and using (1.8) it follows that

$$|H_{2,1}(f)| = \left|a_3 - a_2^2\right| = \frac{1}{16}|t_1|^2 \le \frac{1}{4}.$$

(ii) From (2.3), (2.4), and (2.5) we have

$$H_{2,2}(f) = a_2a_4 - a_3^2 = -\frac{1}{256}t_1^4,$$

and according to (1.8) we conclude that

$$|H_{2,2}(f)| = \left|a_2a_4 - a_3^2\right| = \frac{1}{256}|t_1|^4 \le \frac{1}{16}.$$

(iii) From the relation (1.3) the third order Hankel determinant written as

$$H_{3,1}(f) = a_5\left(a_3 - a_2^2\right) - a_4(a_4 - a_2a_3) + a_3\left(a_2a_4 - a_3^2\right),$$

which implies

$$|H_{3,1}(f)| \le |a_5|\left|a_3 - a_2^2\right| + |a_4|\,|a_4 - a_2a_3| + |a_3|\left|a_2a_4 - a_3^2\right|.$$

Since (2.3) shows that $a_2 = 0$, then

$$|H_{3,1}(f)| \le |a_5|\,|a_3| + |a_4|\,|a_4| + |a_3|^3,$$

and using the estimations obtained in Theorem 2.1 the above inequality implies

$$|H_{3,1}(f)| \le |a_5|\,|a_3| + |a_4|^2 + |a_3|^3 \le \frac{11}{24}\cdot\frac{1}{4} + \frac{1}{9} + \frac{1}{64} = \frac{139}{576}.$$

□

Theorem 3.2. *If $f \in \mathcal{S}^{\mathbf{c}}_{\cos}$ has the form* (1.1)*, then*

$$(i)\quad |H_{2,1}(f)| \le \frac{1}{12}, \qquad (ii)\quad |H_{2,2}(f)| \le \frac{1}{144}, \qquad (iii)\quad |H_{3,1}(f)| \le \frac{131}{8640}.$$

Proof. (i) If $f \in \mathcal{S}^{\mathbf{c}}_{\cos}$, then from (2.9) and (2.10) we get

$$a_3 - a_2^2 = -\frac{1}{48}t_1^2,$$

and using (1.8) it follows that

$$\left|a_3 - a_2^2\right| = \frac{1}{48}|t_1|^2 \le \frac{1}{12}.$$

(ii) From (2.9), (2.10), and (2.11) we have

$$a_2a_4 - a_3^2 = -\frac{1}{2304}t_1^4,$$

and (1.8) leads to

$$\left|a_2a_4 - a_3^2\right| = \frac{1}{2304}|t_1|^4 \le \frac{1}{144}.$$

(iii) Using the relation (1.3) the third order Hankel determinant has the form

$$H_{3,1}(f) = a_5\left(a_3 - a_2^2\right) - a_4(a_4 - a_2a_3) + a_3\left(a_2a_4 - a_3^2\right),$$

hence

$$|H_{3,1}(f)| \le |a_5|\left|a_3 - a_2^2\right| + |a_4|\,|a_4 - a_2a_3| + |a_3|\left|a_2a_4 - a_3^2\right|.$$

From (2.9) we have $a_2 = 0$, thus

$$|H_{3,1}(f)| \le |a_5|\,|a_3| + |a_4|^2 + |a_3|^3,$$

and using the estimations given by Theorem 2.3 it follows

$$|H_{3,1}(f)| \le |a_5|\,|a_3| + |a_4|^2 + |a_3|^3 \le \frac{11}{120}\cdot\frac{1}{12} + \frac{1}{144} + \frac{1}{1728} = \frac{131}{8640}.$$

□

To find the upper bound for $|H_{4,1}(f)|$ for the class $\mathcal{S}^*_{\cos}$ we will use the following preparing lemma.

Lemma 3.3. *If $f \in \mathcal{S}^*_{\cos}$ has the form* (1.1)*, then*

$$(i)\quad |a_3a_6 - a_4a_5| \le \frac{19}{24}, \qquad (ii)\quad \left|a_3a_5 - a_4^2\right| \le \frac{1}{6}, \qquad (iii)\quad |a_4a_6 - a_5^2| \le \frac{1261}{2880}.$$

Proof. (i) If $f \in \mathcal{S}^*_{\cos}$, using the relations (2.4)–(2.7) we get

$$\begin{aligned} a_3a_6 - a_4a_5 =& \frac{29}{92160}t_1^7 - \frac{13}{9216}t_1^5t_2 - \frac{1}{480}t_1^4t_3 + \frac{17}{3840}t_1^3t_2^2 + \frac{1}{320}t_1^3t_4 \\ & - \frac{1}{384}t_1t_2^3 - \frac{1}{480}t_1^2t_2t_3 \\ =& \frac{t_1^3}{64}\left(\frac{29}{1440}t_1^4 + \frac{17}{60}t_2^2 + \frac{3}{15}t_1t_3 - \frac{13}{144}t_1^2t_2 - t_4\right) \\ & - \frac{1}{480}t_1^2t_2t_3 - \frac{1}{384}t_1t_2^3 - \frac{1}{192}t_1^4t_3 + \frac{3}{160}t_1^3t_4, \end{aligned}$$

and from the triangle inequality it follows that

$$\begin{aligned} |a_3a_6 - a_4a_5| \leq& \frac{1}{64}|t_1|^3\left|\frac{29}{1440}t_1^4 + \frac{17}{60}t_2^2 + \frac{3}{15}t_1t_3 - \frac{13}{144}t_1^2t_2 - t_4\right| \\ & + \frac{1}{480}|t_1|^2|t_2||t_3| + \frac{1}{384}|t_1||t_2|^3 + \frac{1}{192}|t_1|^4|t_3| + \frac{3}{160}|t_1|^3|t_4|. \end{aligned}$$

Using the inequality (1.8) of Lemma 1.2 together with the inequality (1.12) of Lemma 1.4, since the assumption of the Lemma 1.4 holds we obtain

$$|a_3a_6 - a_4a_5| \leq \frac{1}{64}\cdot 16 + \frac{1}{480}\cdot 16 + \frac{1}{384}\cdot 16 + \frac{1}{192}\cdot 32 + \frac{3}{160}\cdot 16 = \frac{19}{24}.$$

(ii) Using the relations (2.4)–(2.6) we have

$$\begin{aligned} a_3a_5 - a_4^2 =& -\frac{1}{2304}t_1^6 - \frac{23}{4608}t_1^2t_2^2 + \frac{1}{256}t_1^3t_3 + \frac{5}{4608}t_1^4t_2 \\ =& -\frac{t_1^3}{256}\left(\frac{1}{9}t_1^3 - \frac{5}{18}t_1t_2 - t_3\right) + \frac{t_2^2}{192}\left(t_2 - \frac{23}{24}t_1^2\right) - \frac{t_2^3}{192}. \end{aligned}$$

From the above relation, using the triangle inequality, the inequalities (1.8) and (1.10) of Lemma 1.2, combined with the inequality of (1.11) of Lemma 1.3 for the values $\alpha = \frac{1}{9}$, $\beta = \frac{5}{18}$, and $\gamma = -1$, we deduce that

$$\left|a_3a_5 - a_4^2\right| \leq \frac{|t_1|^3}{256}\left|\frac{1}{9}t_1^3 - \frac{5}{18}t_1t_2 - t_3\right| + \frac{|t_2|^2}{192}\left|t_2 - \frac{23}{24}t_1^2\right| + \frac{|t_2|^3}{192} \leq \frac{1}{6}.$$

(iii) According to the relations (2.5)–(2.7) we get

$$\begin{aligned} a_4a_6 - a_5^2 =& -\frac{1}{15360}t_1^8 + \frac{1}{2880}t_1^6t_2 + \frac{1}{1920}t_1^5t_3 - \frac{61}{46080}t_1^4t_2^2 - \frac{1}{480}t_1^4t_4 \\ & + \frac{13}{3840}t_1^3t_2t_3 - \frac{1}{2560}t_1^2t_2^3 + \frac{1}{3840}t_1t_2^2t_3 - \frac{1}{1024}t_2^4 - \frac{1}{256}t_1^2t_3^2 + \frac{1}{240}t_1^2t_2t_4 \\ =& \frac{t_1^2}{480}\left(t_2 - t_1^2\right)\left(\frac{1}{32}t_1^4 + \frac{30326}{57888}t_2^2 + \frac{3}{4}t_1t_3 - \frac{2}{6}t_1^2t_2 - t_4\right) \\ & - \frac{t_1^2t_2}{480}\left(\frac{1}{48}t_1^4 + \frac{41180}{57888}t_2^2 + \frac{3}{8}t_1t_3 - \frac{12839}{57888}t_1^2t_2 - t_4\right) \\ & + \frac{1}{240}t_1^2t_4\left(t_2 - t_1^2\right) - \frac{1}{256}t_1^2t_3\left(t_3 - \frac{256}{384}t_1t_2\right) + \frac{1}{480}t_1^5\left(t_3 - \frac{49}{96}t_1t_2\right) \\ & + \frac{t_2}{2}\left(\frac{1}{1920}t_1t_2t_3 - \frac{t_2^3}{512}\right) + \frac{1}{1440}t_1^6t_2, \end{aligned}$$

and from the triangle inequality it follows that

$$\begin{aligned}|a_4a_6-a_5^2|\leq&\frac{|t_1|^2}{480}\left|t_2-t_1^2\right|\left|\frac{1}{32}t_1^4+\frac{30326}{57888}t_2^2+\frac{3}{4}t_1t_3-\frac{2}{6}t_1^2t_2-t_4\right|\\&+\frac{|t_1|^2\,|t_2|}{480}\left|\frac{1}{48}t_1^4+\frac{41180}{57888}t_2^2+\frac{3}{8}t_1t_3-\frac{12839}{57888}t_1^2t_2-t_4\right|\\&+\frac{1}{240}|t_1|^2\,|t_4|\left|t_2-t_1^2\right|+\frac{1}{256}|t_1|^2\,|t_3|\left|t_3-\frac{256}{384}t_1t_2\right|+\frac{1}{480}|t_1|^5\left|t_3-\frac{49}{96}t_1t_2\right|\\&+\frac{|t_2|}{2}\frac{1}{1920}|t_1|\,|t_2|\,|t_3|+\frac{1}{1024}|t_2|^4+\frac{1}{1440}|t_1|^6\,|t_2|.\end{aligned}$$

We will use now the inequalities (1.8), (1.9), (1.10) of Lemma 1.2, together with the inequality (1.11) of Lemma 1.3, and (1.12) of Lemma 1.4. It is easy to check that the assumption of Lemma 1.4 holds in each of the two above cases, hence the above inequality implies

$$|a_4a_6-a_5^2|\leq\frac{16}{480}+\frac{16}{480}+\frac{16}{240}+\frac{16}{256}+\frac{64}{480}+\frac{8}{1920}+\frac{8}{512}+\frac{128}{1440}=\frac{1261}{2880}.$$

□

Theorem 3.4. *If $f\in\mathcal{S}^*_{\cos}$ has the form* (1.1)*, then*

$$|H_{4,1}(f)|\leq\frac{6533521}{5806080}\simeq 1.125289524.$$

Proof. If $f\in\mathcal{S}^*_{\cos}$, from the relation (1.5) we have

$$|\delta_1|\leq|a_3a_6-a_4a_5|+|a_2||a_2a_6-a_3a_5|+|a_4||a_2a_4-a_3^2|,$$

and using the estimations of the Lemma 3.3(i), Theorem 2.1 and Theorem 3.1(ii) we obtain

$$|\delta_1|\leq\frac{19}{24}+\frac{1}{3}\cdot\frac{1}{16}=\frac{13}{16}. \tag{3.1}$$

From (1.6) it follows

$$|\delta_2|\leq|a_4a_6-a_5^2|+|a_2|\,|a_3a_6-a_4a_5|+|a_3|\left|a_3a_5-a_4^2\right|$$

and making use of Theorem 2.1 and Lemma 3.3(i), (ii), (iii) we get

$$|\delta_2|\leq\frac{1261}{2880}+\frac{1}{4}\cdot\frac{95}{576}=\frac{5519}{11520}. \tag{3.2}$$

Using the relation (1.7) it follows

$$|\delta_3|\leq|a_2||a_4a_6-a_5^2|+|a_3|\,|a_3a_6-a_4a_5|+|a_4|\left|a_3a_5-a_4^2\right|,$$

and from the results of the Theorem 2.1 and Lemma 3.3(i), (ii) we obtain

$$|\delta_3|\leq\frac{1}{4}\cdot\frac{19}{24}+\frac{1}{3}\cdot\frac{95}{576}=\frac{437}{1728}. \tag{3.3}$$

Finally, the equality (1.4) leads to

$$|H_{4,1}(f)|\leq|a_7||H_{3,1}(f)|+|a_6||\delta_1|+|a_5||\delta_2|+|a_4||\delta_3|,$$

according to the estimations given by the Theorem 3.1(iii), Lemma 3.3, and the inequalities (3.1)–(3.3), we conclude that

$$|H_{4,1}(f)|\leq\frac{179}{252}\cdot\frac{139}{576}+\frac{4}{5}\cdot\frac{13}{16}+\frac{11}{24}\cdot\frac{5519}{11520}+\frac{1}{3}\cdot\frac{437}{1728}=\frac{6533521}{5806080}.$$

□

The next lemma will be used to determine the upper bound for $|H_{4,1}(f)|$ for the class $\mathcal{S}^{\mathbf{c}}_{\cos}$.

Lemma 3.5. *If $f \in \mathcal{S}^{\mathbf{c}}_{\cos}$ has the form* (1.1), *then*

$$(i)\quad |a_3a_6 - a_4a_5| \le \frac{13}{480}, \qquad (ii)\quad \left|a_3a_5 - a_4^2\right| \le \frac{7}{720}, \qquad (iii)\quad |a_4a_6 - a_5^2| \le \frac{817}{20800}.$$

Proof. (i) If $f \in \mathcal{S}^{\mathbf{c}}_{\cos}$, using the relations (2.9)–(2.13) we get

$$\begin{aligned} a_3a_6 - a_4a_5 =& \frac{7}{552960}t_1^7 - \frac{13}{276480}t_1^5t_2 - \frac{1}{7680}t_1^4t_3 + \frac{1}{5120}t_1^3t_2^2 + \frac{1}{5760}t_1^3t_4 \\ &- \frac{1}{7680}t_1t_2^3 - \frac{1}{11520}t_1^2t_2t_3 \\ =& \frac{t_1^3}{2560}\left(\frac{7}{216}t_1^4 + \frac{1}{2}t_2^2 + \frac{1}{3}t_1t_3 - \frac{13}{108}t_1^2t_2 - t_4\right) \\ &- \frac{1}{11520}t_1^2t_2t_3 - \frac{1}{7680}t_1t_2^3 - \frac{1}{3840}t_1^4t_3 + \frac{13}{23040}t_1^3t_4, \end{aligned}$$

and from the triangle inequality it follows that

$$\begin{aligned} |a_3a_6 - a_4a_5| \le& \frac{|t_1|^3}{2560}\left|\frac{7}{216}t_1^4 + \frac{1}{2}t_2^2 + \frac{1}{3}t_1t_3 - \frac{13}{108}t_1^2t_2 - t_4\right| \\ &+ \frac{1}{11520}|t_1|^2|t_2||t_3| + \frac{1}{7680}|t_1||t_2|^3 + \frac{1}{3840}|t_1|^4|t_3| + \frac{13}{23040}|t_1|^3|t_4|. \end{aligned}$$

Using (1.8) and the inequality (1.12) of Lemma 1.4 we obtain

$$|a_3a_6 - a_4a_5| \le \frac{8}{2560}\cdot 2 + \frac{1}{11520}\cdot 16 + \frac{1}{7680}\cdot 16 + \frac{1}{3840}\cdot 32 + \frac{13}{23040}\cdot 16 = \frac{13}{480}.$$

(ii) From the relations (2.10)–(2.12) we have

$$\begin{aligned} a_3a_5 - a_4^2 =& -\frac{1}{46080}t_1^6 - \frac{7}{23040}t_1^2t_2^2 + \frac{1}{3840}t_1^3t_3 + \frac{1}{23040}t_1^4t_2 \\ =& -\frac{t_1^3}{3840}\left(\frac{1}{12}t_1^3 - \frac{1}{6}t_1t_2 - t_3\right) - \frac{7}{23040}t_1^2t_2^2. \end{aligned}$$

Then, using the triangle inequality, the inequalities (1.8) and (1.10) of Lemma 1.2, combined with the inequality of (1.11) of Lemma 1.3 for the values $\alpha = \frac{1}{12}$, $\beta = \frac{1}{6}$, and $\gamma = -1$, we deduce that

$$\left|a_3a_5 - a_4^2\right| \le \frac{|t_1|^3}{3840}\left|\frac{1}{12}t_1^3 - \frac{1}{6}t_1t_2 - t_3\right| + \frac{7}{23040}|t_1|^2|t_2|^2 \le \frac{7}{720}.$$

(iii) Using the relations (2.10)–(2.12) we get

$$\begin{aligned} a_4a_6 - a_5^2 =& -\frac{11}{5529600}t_1^8 + \frac{11}{1382400}t_1^6t_2 + \frac{1}{38400}t_1^5t_3 - \frac{53}{1382400}t_1^4t_2^2 - \frac{1}{11520}t_1^4t_4 \\ &+ \frac{7}{57600}t_1^3t_2t_3 - \frac{1}{38400}t_1^2t_2^3 + \frac{1}{57600}t_1t_2^2t_3 - \frac{1}{25600}t_2^4 - \frac{1}{6400}t_1^2t_3^2 + \frac{1}{5760}t_1^2t_2t_4 \\ =& -\frac{t_1^4}{2880}\left(\frac{11}{1920}t_1^4 + \frac{53}{480}t_2^2 + \frac{1}{13}t_1t_3 - \frac{11}{480}t_1^2t_2 - t_4\right) \\ &+ \frac{7}{57600}t_1^3t_2t_3 - \frac{1}{38400}t_1^2t_2^3 + \frac{1}{57600}t_1t_2^2t_3 - \frac{1}{25600}t_2^4 - \frac{1}{6400}t_1^2t_3^2 + \frac{1}{5760}t_1^2t_2t_4 \\ &+ \frac{79}{1497600}t_1^5t_3 - \frac{1}{2304}t_1^4t_4, \end{aligned}$$

and from the triangle inequality it follows that

$$\begin{aligned}|a_4a_6-a_5^2|\leq&\frac{|t_1|^4}{2880}\left|\frac{11}{1920}t_1^4+\frac{53}{480}t_2^2+\frac{1}{13}t_1t_3-\frac{11}{480}t_1^2t_2-t_4\right|\\&+\frac{7}{57600}|t_1|^3|t_2||t_3|+\frac{1}{38400}|t_1|^2|t_2|^3+\frac{1}{57600}|t_1||t_2|^2|t_3|+\frac{1}{25600}|t_2|^4\\&+\frac{1}{6400}|t_1|^2|t_3|^2+\frac{1}{5760}|t_1|^2|t_2||t_4|+\frac{79}{1497600}|t_1|^5|t_3|+\frac{1}{2304}|t_1|^4|t_4|.\end{aligned}$$

We will use now the inequalities (1.8) and (1.12) of Lemma 1.4. It is easy to check that the assumption of Lemma 1.4 holds in the above case, hence the above inequality implies

$$\begin{aligned}|a_4a_6-a_5^2|\leq&\frac{32}{2880}+\frac{7\cdot 32}{57600}+\frac{32}{38400}+\frac{16}{57600}+\frac{16}{25600}\\&+\frac{16}{6400}+\frac{16}{5760}+\frac{79\cdot 64}{1497600}+\frac{32}{2304}=\frac{817}{20800}.\end{aligned}$$

□

Theorem 3.6. *If $f\in\mathcal{S}^{\mathbf{c}}_{\cos}$ has the form* (1.1)*, then*

$$|H_{4,1}(f)|\leq\frac{90717383}{9906624000}\simeq 0.009157244991.$$

Proof. If $f\in\mathcal{S}^{\mathbf{c}}_{\cos}$, from the relation (1.5) we have

$$|\delta_1|\leq|a_3a_6-a_4a_5|+|a_2||a_2a_6-a_3a_5|+|a_4||a_2a_4-a_3^2|,$$

and using the estimations of the Lemma 3.5(i), Theorem 2.3, and Theorem 3.2(ii) we obtain

$$|\delta_1|\leq\frac{13}{480}+\frac{1}{12}\cdot\frac{1}{144}=\frac{239}{8640}. \tag{3.4}$$

From (1.6) it follows

$$|\delta_2|\leq|a_4a_6-a_5^2|+|a_2|\,|a_3a_6-a_4a_5|+|a_3|\left|a_3a_5-a_4^2\right|$$

and making use of Lemma 3.5(i), (ii), (iii) we get

$$|\delta_2|\leq\frac{817}{20800}+\frac{1}{12}\cdot\frac{7}{720}=\frac{11257}{280800}. \tag{3.5}$$

Using the relation (1.7) it follows

$$|\delta_3|\leq|a_2||a_4a_6-a_5^2|+|a_3|\,|a_3a_6-a_4a_5|+|a_4|\left|a_3a_5-a_4^2\right|,$$

and from the results of the Theorem 2.3, Lemma 3.5(i), (ii) we obtain

$$|\delta_3|\leq 0+\frac{1}{12}\cdot\frac{13}{480}+\frac{1}{12}\cdot\frac{7}{720}=\frac{53}{17280}. \tag{3.6}$$

Finally, the equality (1.4) leads to

$$|H_{4,1}(f)|\leq|a_7||H_{3,1}(f)|+|a_6||\delta_1|+|a_5||\delta_2|+|a_4||\delta_3|,$$

according to the estimations given by the Theorem 2.3, Theorem 3.2(iii), and the inequalities (3.4)–(3.6), we conclude that

$$|H_{4,1}(f)|\leq\frac{179}{1764}\cdot\frac{131}{8640}+\frac{2}{15}\cdot\frac{239}{8640}+\frac{11}{120}\cdot\frac{11257}{280800}+\frac{1}{12}\cdot\frac{53}{17280}=\frac{90717383}{9906624000}.$$

□

Remark 3.7. From the proofs of Theorem 3.4 and Theorem 3.6 that use the estimations of Theorem 2.1 and Theorem 2.3 which are not sharp, it follows that the upper bounds given by these theorems are not the best possible. To find the best estimations of the modules of these Hankel determinant remain an interesting open problem.

4. The Zalcman functional estimate for the classes $\mathcal{S}^*_{\cos}$ and $\mathcal{S}^{\mathbf{c}}_{\cos}$

In the 1960's, Lawrence Zalcman conjectured that the coefficients of the functions $f \in \mathcal{S}$ having the form (1.1) satisfy the inequality

$$\left|a_n^2 - a_{2n-1}\right| \leq (n-1)^2,\ n \geq 2,$$

and the equality holds only for the Koebe function $k(z) = \dfrac{z}{(1-z)^2}$ and its rotations. Like it was shown in [9,43] it implies the Bieberbach conjecture, that is $|a_n| \leq n$, $n \geq 2$. Remark that for $n = 2$ the above inequality is a well-known consequence of the *Area Theorem*, and could be found in [33, Theorem 1.5]. In the literature the Zalcman functional has been studied by many researchers (see, for example, [8], [24], [20]).

The Theorem 3.1(i) shows that the above conjecture holds for the classes $\mathcal{S}^*_{\cos}$ and $\mathcal{S}^{\mathbf{c}}_{\cos}$ if $n = 2$. Our next four results prove that the Zalcman inequality holds for the class $\mathcal{S}^*_{\cos}$ and $\mathcal{S}^{\mathbf{c}}_{\cos}$ if $n = 3$ and $n = 4$, respectively.

For $n = 3$, the Zalcman functional upper bounds are given in the next two theorems.

Theorem 4.1. *If $f \in \mathcal{S}^*_{\cos}$ has the form* (1.1)*, then*

$$\left|a_3^2 - a_5\right| \leq \frac{41}{96}.$$

Proof. For $f \in \mathcal{S}^*_{\cos}$, using the equalities (2.4) and (2.6) we obtain

$$a_3^2 - a_5 = \frac{19}{768}t_1^4 + \frac{1}{32}t_2^2 - \frac{3}{32}t_1^2 t_2 + \frac{1}{16}t_3 t_1,$$

and from the triangle inequality

$$\left|a_3^2 - a_5\right| \leq \frac{|t_1|}{8}\left|\frac{19}{96}t_1^3 - \frac{3}{4}t_1 t_3 + \frac{t_3}{2}\right| + \frac{1}{32}|t_2|^2.$$

Using the inequalities (1.8) of Lemma 1.2, and (1.11) of Lemma 1.3, the above relation leads to

$$\left|a_3^2 - a_5\right| \leq \frac{2}{8} \cdot \frac{58}{48} + \frac{4}{32} = \frac{41}{96}.$$

□

Theorem 4.2. *If $f \in \mathcal{S}^{\mathbf{c}}_{\cos}$ has the form* (1.1)*, then*

$$\left|a_3^2 - a_5\right| \leq \frac{127}{1440}.$$

Proof. From the relations (2.10) and (2.12) it follows

$$a_3^2 - a_5 = \frac{53}{11520}t_1^4 + \frac{1}{160}t_2^2 - \frac{3}{160}t_1^2 t_2 + \frac{1}{80}t_3 t_1,$$

and using the triangle inequality we get

$$\left|a_3^2 - a_5\right| \leq \frac{|t_1|}{40}\left|\frac{53}{288}t_1^3 - \frac{3}{4}t_1 t_2 + \frac{t_3}{2}\right| + \frac{1}{160}|t_2|^2.$$

The inequalities (1.8) of Lemma 1.2, and (1.11) of Lemma 1.3, leads to

$$\left|a_3^2 - a_5\right| \leq \frac{2}{40} \cdot \frac{91}{72} + \frac{4}{160} = \frac{127}{1440}.$$

□

For $n = 4$, the Zalcman functional estimations are obtained in the next two results.

Theorem 4.3. *If $f \in \mathcal{S}^*_{\cos}$ has the form* (1.1)*, then*

$$\left|a_4^2 - a_7\right| \leq \frac{25}{12}.$$

Proof. For $f \in \mathcal{S}^*_{\cos}$, using the equalities (2.5) and (2.8) we obtain

$$\begin{aligned} a_4^2 - a_7 =& \frac{2717\,t_1^6}{552960} - \frac{209\,t_1^4 t_2}{4608} + \frac{427\,t_1^2 t_2^2}{4608} + \frac{t_2 t_4}{24} - \frac{t_2^3}{48} - \frac{t_1 t_2 t_3}{8} \\ &+ \frac{131\,t_1^3 t_3}{2304} - \frac{t_1^2 t_4}{16} + \frac{t_3^2}{48} + \frac{t_1 t_5}{24} \\ =& \frac{t_1^2}{16}\left(\frac{2717}{34560} t_1^4 + \frac{239}{288} t_2^2 + \frac{131}{144} t_1 t_3 - \frac{209}{288} t_1^2 t_2 - t_4\right) \\ &- \frac{1}{8} t_1 t_2 \left(t_3 - \frac{188}{576} t_1 t_2\right) + \frac{t_3^2}{48} + \frac{t_2 t_4}{24} + \frac{t_1 t_5}{24} - \frac{t_2^3}{48}, \end{aligned}$$

and from the triangle inequality

$$\begin{aligned} \left|a_4^2 - a_7\right| \le& \frac{|t_1|^2}{16}\left|\frac{2717}{34560} t_1^4 + \frac{239}{288} t_2^2 + \frac{131}{144} t_1 t_3 - \frac{209}{288} t_1^2 t_2 - t_4\right| \\ &+ \frac{1}{8}|t_1||t_2|\left|t_3 - \frac{188}{576} t_1 t_2\right| + \frac{|t_3|^2}{48} + \frac{|t_2||t_4|}{24} + \frac{|t_1||t_5|}{24} + \frac{|t_2|^3}{48}. \end{aligned}$$

Using the inequalities (1.8) and (1.9) of Lemma 1.2, and (1.12) of Lemma 1.4, the above relation leads to

$$\left|a_4^2 - a_7\right| \le \frac{4}{16} \cdot 2 + \frac{1}{8} \cdot 2 \cdot 2 \cdot 2 + \frac{4}{48} + \frac{4}{24} + \frac{4}{24} + \frac{8}{48} = \frac{25}{12}.$$

□

Theorem 4.4. *If $f \in \mathcal{S}^{\mathbf{c}}_{\cos}$ has the form* (1.1)*, then*

$$\left|a_4^2 - a_7\right| \le \frac{25}{84}.$$

Proof. If $f \in \mathcal{S}^{\mathbf{c}}_{\cos}$, from the relations (2.11) and (2.14) it follows

$$\begin{aligned} a_4^2 - a_7 =& \frac{311\,t_1^6}{552960} - \frac{191\,t_1^4 t_2}{32256} + \frac{409\,t_1^2 t_2^2}{32256} + \frac{t_2 t_4}{168} - \frac{t_2^3}{336} - \frac{t_1 t_2 t_3}{56} \\ &+ \frac{131\,t_1^3 t_3}{16128} - \frac{t_1^2 t_4}{112} + \frac{t_3^2}{336} + \frac{t_1 t_5}{168} \\ =& \frac{t_1^2}{112}\left(\frac{34832}{552960} t_1^4 + \frac{222}{288} t_2^2 + \frac{131}{144} t_1 t_3 - \frac{191}{288} t_1^2 t_2 - t_4\right) \\ &- \frac{1}{56} t_1 t_2 \left(t_3 - \frac{187}{576} t_1 t_2\right) + \frac{t_3^2}{336} + \frac{t_2 t_4}{168} + \frac{t_1 t_5}{168} - \frac{t_2^3}{336}, \end{aligned}$$

and using the triangle inequality we have

$$\begin{aligned} \left|a_4^2 - a_7\right| \le& \frac{|t_1|^2}{112}\left|\frac{34832}{552960} t_1^4 + \frac{222}{288} t_2^2 + \frac{131}{144} t_1 t_3 - \frac{191}{288} t_1^2 t_2 - t_4\right| \\ &+ \frac{1}{56}|t_1||t_2|\left|t_3 - \frac{187}{576} t_1 t_2\right| + \frac{|t_3|^2}{336} + \frac{|t_2||t_4|}{168} + \frac{|t_1||t_5|}{168} + \frac{|t_2|^3}{336}. \end{aligned}$$

From here, the inequalities (1.8) and (1.9) of Lemma 1.2, and (1.12) of Lemma 1.4, leads to

$$\left|a_3^2 - a_5\right| \le \frac{4}{112} \cdot 2 + \frac{1}{56} \cdot 2 \cdot 2 \cdot 2 + \frac{4}{336} + \frac{4}{168} + \frac{4}{168} + \frac{8}{336} = \frac{25}{84}.$$

□

5. Logarithmic coefficient bounds for the classes $\mathcal{S}^{\mathbf{c}}_{\cos}$ and $\mathcal{S}^{*}_{\cos}$

It is well known that the *logarithmic coefficients* $\beta_n := \beta_n(f)$, $n \in \mathbb{N}$, for function $f \in \mathcal{S}$ are defined by

$$\log \frac{f(z)}{z} = 2\sum_{n=1}^{\infty} \beta_n z^n, \ z \in \mathbb{D}. \tag{5.1}$$

Since the function $\Phi(z) := \cos z$ has real positive part in $\mathbb{D}$, and moreover (see Figure 1 made with MAPLE™ computer software)

$$\operatorname{Re} \Phi(z) > \frac{1}{2}, \ z \in \mathbb{D},$$

it follows that the classes $\mathcal{S}^{\mathbf{c}}_{\cos}$ and $\mathcal{S}^{*}_{\cos}$ are subsets of the class of convex and starlike (normalized) functions, respectively, therefore $\mathcal{S}^{\mathbf{c}}_{\cos} \subset \mathcal{S}$ and $\mathcal{S}^{*}_{\cos} \subset \mathcal{S}$.

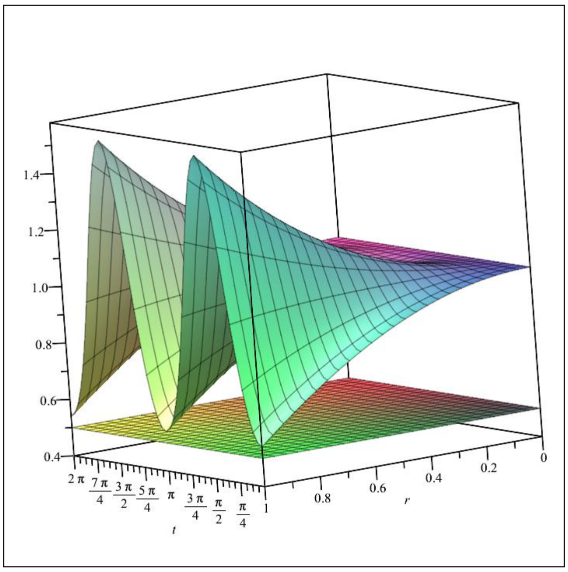

Figure 1. The image of $\operatorname{Re} \Phi\left(re^{it}\right)$, $r \in [0,1]$, $t \in [0, 2\pi]$

In this section we will give the upper bounds estimates for the first six coefficients of the functions that belong to the classes $\mathcal{S}^{\mathbf{c}}_{\cos}$ and $\mathcal{S}^{*}_{\cos}$, respectively.

Theorem 5.1. *If* $f \in \mathcal{S}^{\mathbf{c}}_{\cos}$ *is given by* (1.1), *then*

$$\beta_1 = 0, \quad |\beta_2| \le \frac{1}{24}, \quad |\beta_3| \le \frac{1}{24}, \quad |\beta_4| \le \frac{259}{5760}, \quad |\beta_5| \le \frac{5}{48}, \quad |\beta_6| \le \frac{149}{840}.$$

Proof. If $f \in \mathcal{S}^{\mathbf{c}}_{\cos}$ has the form (1.1), it follows that

$$\begin{aligned}\log \frac{f(z)}{z} =& a_2 z + \left(-\frac{a_2^2}{2} + a_3\right) z^2 + \left(-a_2 a_3 + a_4 + \frac{a_2^3}{3}\right) z^3 \\ &+ \left(-a_2 a_4 + a_5 + a_2^2 a_3 - \frac{{a_3}^2}{2} - \frac{a_2^4}{4}\right) z^4 \\ &+ \left(-a_2 a_5 + a_6 + a_2^2 a_4 - a_3 a_4 - a_2^3 a_3 + a_2 {a_3}^2 + \frac{a_2^5}{5}\right) z^5 \\ &+ \Bigg(- a_2 a_6 + a_7 + a_2^2 a_5 - a_3 a_5 - {a_2}^3 a_4 + 2\, a_2 a_3 a_4 - \frac{a_4^2}{2} \\ &\qquad + a_2^4 a_3 - \frac{3\, a_2^2 {a_3}^2}{2} + \frac{{a_3}^3}{3} - \frac{a_2^6}{6}\Bigg) z^6 + \cdots, \ z \in \mathbb{D},\end{aligned}$$

and equating the first six coefficients of the relation (5.1) we get

$$\beta_1 = \frac{a_2}{2}, \tag{5.2}$$

$$\beta_2 = \frac{1}{4}\left(2a_3 - a_2^2\right), \tag{5.3}$$

$$\beta_3 = \frac{1}{6}\left(a_2^3 - 3a_2a_3 + 3a_4\right), \tag{5.4}$$

$$\beta_4 = \frac{1}{8}\left(-a_2^4 + 4a_2^2a_3 - 4a_2a_4 - 2a_3^2 + 4a_5\right), \tag{5.5}$$

$$\beta_5 = \frac{1}{10}\left(a_2^5 - 5a_2^3a_3 + 5a_2^2a_4 + 5a_2a_3^2 - 5a_2a_5 - 5a_3a_4 + 5a_6\right), \tag{5.6}$$

$$\beta_6 = \frac{1}{12}\Big(-a_2^6 + 6a_2^4a_3 - 6a_2^3a_4 - 9a_2^2a_3^2 + 6a_2^2a_5 + 12a_2a_3a_4 + 2a_3^3 - 6a_2a_6 - 6a_3a_5 - 3a_4^2 + 6a_7\Big). \tag{5.7}$$

Substituting (2.9)–(2.14) into (5.2)–(5.7) it follows that

$$\beta_1 = 0,$$

$$\beta_2 = -\frac{1}{96}t_1^2, \tag{5.8}$$

$$\beta_3 = \frac{1}{192}t_1^3 - \frac{1}{96}t_1t_2, \tag{5.9}$$

$$\beta_4 = -\frac{101}{46080}t_1^4 - \frac{1}{320}t_2^2 + \frac{3}{320}t_2t_1^2 - \frac{1}{160}t_1t_3, \tag{5.10}$$

$$\beta_5 = \frac{13\,t_1^5}{15360} - \frac{3\,t_2t_1^3}{512} - \frac{t_3t_2}{240} + \frac{t_3t_1^2}{160} + \frac{t_1{t_2}^2}{160} - \frac{t_1t_4}{240}, \tag{5.11}$$

$$\beta_6 = -\frac{31\,t_1^6}{103680} - \frac{677\,{t_2}^2t_1^2}{107520} + \frac{983\,t_2t_1^4}{322560} - \frac{169\,t_3t_1^3}{40320} + \frac{t_3t_1t_2}{112} - \frac{t_1t_5}{336} - \frac{t_4t_2}{336} + \frac{t_4t_1^2}{224} - \frac{t_3^2}{672} + \frac{{t_2}^3}{672}. \tag{5.12}$$

Using (5.8), since (1.8) shows that $|t_1| \le 2$, we get

$$|\beta_2| \le \frac{1}{24}.$$

The relation (5.9) leads to

$$|\beta_3| = \left|\frac{t_1}{96}\right|\left|t_2 - \frac{t_1^2}{2}\right|,$$

and according to (1.8) and (1.10), we obtain

$$|\beta_3| \le \frac{2}{96}\cdot 2 = \frac{1}{24}.$$

From (5.10), by using triangle inequality it follows that

$$|\beta_4| \le \frac{|t_1|}{160}\left|\frac{101}{288}t_1^3 - \frac{3}{2}t_1t_2 + t_3\right| + \frac{1}{320}|t_2|^2,$$

and using (1.8) and Lemma 1.3 for the appropriate values $\alpha = \frac{101}{288}$, $\beta = \frac{3}{2}$, and $\gamma = 1$, the above inequality implies that

$$|\beta_4| \le \frac{2}{160}\cdot\frac{187}{72} + \frac{4}{320} = \frac{259}{5760}.$$

If we add and subtract $\frac{t_1t_4}{80}$ from the righthand side of (5.11), and using the triangle inequality we have

$$\begin{aligned}|\beta_5| &= \left|\frac{t_1}{80}\left(\frac{11}{192}t_1^4-\frac{240}{512}t_1^2t_2+\frac{1}{2}t_1t_3+\frac{1}{2}t_2^2-t_4\right)+\frac{1}{7680}t_1^5+\frac{1}{120}t_1t_4-\frac{1}{240}t_2t_3\right|\\ &\leq \left|\frac{t_1}{80}\right|\left|\frac{11}{192}t_1^4-\frac{240}{512}t_1^2t_2+\frac{1}{2}t_1t_3+\frac{1}{2}t_2^2-t_4\right|+\frac{1}{7680}|t_1|^5+\frac{1}{120}|t_1|\,|t_4|+\frac{1}{240}|t_2|\,|t_3|.\end{aligned}$$

From (1.8) and Lemma 1.4 for the values $k=\frac{11}{192}$, $r=\frac{1}{2}$, $\ell=\frac{1}{4}$, and $j=\frac{480}{1536}$, the above inequality implies that

$$|\beta_5|\leq\frac{4}{80}+\frac{32}{7680}+\frac{4}{120}+\frac{4}{240}=\frac{5}{48}.$$

Rearranging the terms of (5.12) we have

$$\begin{aligned}\beta_6=&-\frac{t_1^2}{160}\left(\frac{31}{648}t_1^4+\frac{588}{672}t_2^2+\frac{169}{252}t_1t_3-\frac{983}{2016}t_1^2t_2-t_4\right)+\frac{t_2^2}{672}\left(t_2-\frac{89}{160}t_1^2\right)\\ &+\frac{1}{112}t_1t_2t_3-\frac{1}{672}t_3^2-\frac{1}{336}t_2t_4-\frac{1}{336}t_1t_5-\frac{1}{560}t_1^2t_4,\end{aligned}$$

and from the triangle inequality it follows that

$$\begin{aligned}|\beta_6|\leq&\frac{|t_1|^2}{160}\left|\frac{31}{648}t_1^4+\frac{588}{672}t_2^2+\frac{169}{252}t_1t_3-\frac{983}{2016}t_1^2t_2-t_4\right|+\frac{|t_2|^2}{672}\left|t_2-\frac{89}{160}t_1^2\right|\\ &+\frac{1}{112}|t_1||t_2||t_3|+\frac{1}{672}|t_3|^2+\frac{1}{336}|t_2||t_4|+\frac{1}{336}|t_1||t_5|+\frac{1}{560}|t_1|^2|t_4|\end{aligned}$$

Now we will use the inequalities (1.8), (1.10) of Lemma 1.2, together with the inequality (1.11) of Lemma 1.3, and (1.12) of Lemma 1.4. Since it is easy to check that the assumption of Lemma 1.4 holds the above case, from the previous inequality we conclude that

$$|\beta_6|\leq\frac{4}{160}\cdot2+\frac{4}{672}\cdot2+\frac{1}{112}\cdot2\cdot2\cdot2+\frac{1}{672}\cdot4+\frac{1}{336}\cdot4+\frac{1}{336}\cdot4+\frac{1}{560}\cdot8=\frac{149}{840},$$

and the theorem is completely proved. □

Theorem 5.2. *If $f\in\mathcal{S}^*_{\cos}$ is given by* (1.1)*, then*

$$\beta_1=0,\quad |\beta_2|\leq\frac{1}{8},\quad |\beta_3|\leq\frac{1}{6},\quad |\beta_4|\leq\frac{85}{384},\quad |\beta_5|\leq\frac{2}{5},\quad |\beta_6|\leq\frac{25}{24}.$$

Proof. If $f\in\mathcal{S}^*_{\cos}$ has the form (1.1), by substituting (2.3)–(2.8) into (5.2)–(5.7), we get

$$\beta_1=0,$$

$$\beta_2=-\frac{1}{32}t_1^2, \tag{5.13}$$

$$\beta_3=\frac{1}{48}t_1^3-\frac{1}{24}t_1t_2, \tag{5.14}$$

$$\beta_4=-\frac{35}{3072}t_1^4-\frac{1}{64}t_2^2+\frac{3}{64}t_2t_1^2-\frac{1}{32}t_1t_3, \tag{5.15}$$

$$\beta_5=\frac{11\,t_1^5}{1920}-\frac{7\,t_2t_1^3}{192}-\frac{t_3t_2}{40}+\frac{3\,t_3t_1^2}{80}+\frac{3\,t_1{t_2}^2}{80}-\frac{t_1t_4}{40}, \tag{5.16}$$

$$\begin{aligned}\beta_6=&-\frac{1501}{552960}t_1^6-\frac{35}{768}t_2^2t_1^2+\frac{55}{2304}t_2t_1^4-\frac{35}{1152}t_3t_1^3+\frac{1}{16}t_3t_1t_2\\ &-\frac{1}{48}t_1t_5-\frac{1}{48}t_4t_2+\frac{1}{32}t_4t_1^2-\frac{1}{96}t_3^2+\frac{1}{96}t_2^3.\end{aligned} \tag{5.17}$$

According to (5.13), since from (1.8) we have $|t_1|\leq2$, we get

$$|\beta_2|\leq\frac{1}{8}.$$

The relation (5.14) implies

$$|\beta_3| = \left|\frac{t_1}{24}\right| \left|t_2 - \frac{t_1^2}{2}\right|,$$

and using (1.8) and (1.10) we obtain

$$|\beta_3| \leq \frac{2}{24} \cdot 2 = \frac{1}{6}.$$

From (5.15) by using triangle inequality it follows that

$$|\beta_4| \leq \frac{|t_1|}{16} \left|\frac{35}{192}t_1^3 - \frac{3}{4}t_1 t_2 + \frac{1}{2}t_3\right| + \frac{1}{64}|t_2|^2,$$

and using (1.8) and Lemma 1.3 for the appropriate values $\alpha = \dfrac{35}{192}$, $\beta = \dfrac{3}{4}$, and $\gamma = \dfrac{1}{2}$, the above inequality implies that

$$|\beta_4| \leq \frac{2}{16} \cdot \frac{122}{96} + \frac{4}{64} = \frac{85}{384}.$$

If we add and subtract $\dfrac{t_1 t_4}{20}$ from the righthand side of (5.16), and using then the triangle inequality we have

$$\begin{aligned}
|\beta_5| &= \left|\frac{t_1}{20}\left(\frac{11}{96}t_1^4 - \frac{140}{192}t_1^2 t_2 + \frac{3}{4}t_1 t_3 + \frac{3}{4}t_2^2 - t_4\right) + \frac{1}{40}t_1 t_4 - \frac{1}{40}t_2 t_3\right| \\
&\leq \left|\frac{t_1}{20}\right| \left|\frac{11}{96}t_1^4 - \frac{140}{192}t_1^2 t_2 + \frac{3}{4}t_1 t_3 + \frac{3}{4}t_2^2 - t_4\right| + \frac{1}{40}|t_1||t_4| + \frac{1}{40}|t_2||t_3|.
\end{aligned}$$

From (1.8) and Lemma 1.4 for the values $k = \dfrac{11}{96}$, $r = \dfrac{3}{4}$, $\ell = \dfrac{3}{8}$, and $j = \dfrac{280}{576}$, the above inequality implies that

$$|\beta_5| \leq \frac{4}{20} + \frac{4}{40} + \frac{4}{40} = \frac{2}{5}.$$

Rearranging the terms of (5.17) we have

$$\begin{aligned}
\beta_6 = &-\frac{t_1^2}{32}\left(\frac{1501}{17280}t_1^4 + \frac{19}{24}t_2^2 + \frac{35}{36}t_1 t_3 - \frac{55}{72}t_1^2 t_2 - t_4\right) + \frac{1}{16}t_1 t_2\left(t_3 - \frac{14}{48}t_1 t_2\right) \\
&+ \frac{1}{96}t_2^2\left(t_2 - \frac{2}{8}t_1^2\right) - \frac{1}{96}t_3^2 - \frac{1}{48}t_2 t_4 - \frac{1}{48}t_1 t_5,
\end{aligned}$$

and from the triangle inequality it follows that

$$\begin{aligned}
|\beta_6| \leq &\frac{|t_1|^2}{32}\left|\frac{1501}{17280}t_1^4 + \frac{19}{24}t_2^2 + \frac{35}{36}t_1 t_3 - \frac{55}{72}t_1^2 t_2 - t_4\right| + \frac{1}{16}|t_1||t_2|\left|t_3 - \frac{14}{48}t_1 t_2\right| \\
&+ \frac{1}{96}|t_2|^2\left|t_2 - \frac{2}{8}t_1^2\right| + \frac{1}{96}|t_3|^2 + \frac{1}{48}|t_2||t_4| + \frac{1}{48}|t_1||t_5|.
\end{aligned}$$

Now we will use the inequalities (1.8), (1.9) and (1.10) of Lemma 1.2 and (1.12) of Lemma 1.4. Since it is easy to check that the assumption of Lemma 1.4 holds the above case, the previous inequality leads to

$$|\beta_6| \leq \frac{4}{32} \cdot 2 + \frac{1}{16} \cdot 8 + \frac{1}{96} \cdot 8 + \frac{1}{96} \cdot 4 + \frac{1}{48} \cdot 4 + \frac{1}{48} \cdot 4 = \frac{25}{24},$$

and all the estimations are proved. □

6. Initial consecutive coefficients module difference estimates for the classes $\mathcal{S}^*_{\cos}$ and $\mathcal{S}^{\mathbf{c}}_{\cos}$

In this section we use the following lemma for finding the upper bounds of some initial coefficients module difference.

Lemma 6.1 ([39, Proposition 1]). *Let $p \in \mathcal{P}$ be given by (1.2). Let $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{B}_3$ be numbers such that $\mathcal{B}_1 \geq 0$, $\mathcal{B}_2 \in \mathbb{C}$, and $\mathcal{B}_3 \in \mathbb{R}$. Define $\psi_+(t_1,t_2)$ and $\psi_-(t_1,t_2)$ by*

$$\psi_+(t_1,t_2) = \left|\mathcal{B}_2 t_1^2 + \mathcal{B}_3 t_2\right| - \left|\mathcal{B}_1 t_1\right|,$$

and $\psi_-(t_1,t_2) = -\psi_+(t_1,t_2)$. Then

$$\psi_+(t_1,t_2) \leq \begin{cases} |4\mathcal{B}_2 + 2\mathcal{B}_3| - 2\mathcal{B}_1, & \text{when} \quad |2\mathcal{B}_2 + \mathcal{B}_3| \geq |\mathcal{B}_3| + \mathcal{B}_1, \\ 2|\mathcal{B}_3|, & \text{otherwise,} \end{cases}$$

and

$$\psi_-(t_1,t_2) \leq \begin{cases} 2\mathcal{B}_1 - \mathcal{B}_4, & \text{when} \quad \mathcal{B}_1 \geq \mathcal{B}_4 + 2|\mathcal{B}_3|, \\ 2\mathcal{B}_1\sqrt{\dfrac{2|\mathcal{B}_3|}{\mathcal{B}_4 + 2|\mathcal{B}_3|}}, & \text{when} \quad \mathcal{B}_1^2 \leq 2|\mathcal{B}_3|\left(\mathcal{B}_4 + 2|\mathcal{B}_3|\right), \\ 2|\mathcal{B}_3| + \dfrac{\mathcal{B}_1^2}{\mathcal{B}_4 + 2|\mathcal{B}_3|}, & \text{otherwise,} \end{cases}$$

where $\mathcal{B}_4 = |4\mathcal{B}_2 + 2\mathcal{B}_3|$. All the inequalities are sharp.

Theorem 6.2. *If $f \in \mathcal{S}^*_{\cos}$ has the form (1.1), then*

$$-\frac{1}{4} \leq |a_4| - |a_3| \leq \frac{1}{3}.$$

Proof. If $f \in \mathcal{S}^*_{\cos}$, from (2.4) and (2.5) we have

$$|a_4| - |a_3| = \left|\frac{1}{24}t_1^3 - \frac{1}{12}t_1 t_2\right| - \left|\frac{1}{16}t_1^2\right| = |t_1|\psi_+(t_1,t_2), \tag{6.1}$$

where

$$\psi_+(t_1,t_2) := \left|\frac{1}{24}t_1^2 - \frac{1}{12}t_2\right| - \left|\frac{1}{16}t_1\right|.$$

Using the inequality (1.8) we have $|t_1| \leq 2$, and the relation (6.1) leads to

$$|a_4| - |a_3| \leq 2\psi_+(t_1,t_2). \tag{6.2}$$

Letting $\mathcal{B}_1 = \dfrac{1}{16}$, $\mathcal{B}_2 = \dfrac{1}{24}$, and $\mathcal{B}_3 = -\dfrac{1}{12}$, then $|2\mathcal{B}_2 + \mathcal{B}_3| \ngeq |\mathcal{B}_3| + \mathcal{B}_1$. Hence, from Lemma 6.1 we have

$$\psi_+(t_1,t_2) \leq 2|\mathcal{B}_3| = \frac{1}{6},$$

consequently, the inequality (6.2) leads to

$$|a_4| - |a_3| \leq \frac{1}{3}.$$

From (6.1) we have

$$|a_3| - |a_4| = -|t_1|\,\psi_+(t_1,t_2) = |t_1|\,\psi_-(t_1,t_2). \tag{6.3}$$

Letting $\mathcal{B}_1 = \dfrac{1}{16}$, $\mathcal{B}_2 = \dfrac{1}{24}$ and $\mathcal{B}_3 = -\dfrac{1}{12}$, then $\mathcal{B}_4 = |4\mathcal{B}_2 + 2\mathcal{B}_3| = 0$, and $\mathcal{B}_1^2 \leq 2|\mathcal{B}_3|\left(\mathcal{B}_4 + 2|\mathcal{B}_3|\right)$. Hence, from Lemma 6.1 we have

$$\psi_-(t_1,t_2) \leq 2\mathcal{B}_1\sqrt{\frac{2|\mathcal{B}_3|}{\mathcal{B}_4 + 2|\mathcal{B}_3|}} = \frac{1}{8}.$$

Therefore, using the inequality (1.8) we have $|t_1| \leq 2$, and according to (6.3) the above inequality implies that

$$|a_3| - |a_4| = |t_1|\,\psi_-(t_1, t_2) \leq \frac{1}{4},$$

which completes our proof. $\square$

Theorem 6.3. *If $f \in \mathcal{S}^{\mathbf{c}}_{\cos}$ has the form* (1.1)*, then*

$$-\frac{1}{12} \leq |a_4| - |a_3| \leq \frac{1}{12}.$$

Proof. If $f \in \mathcal{S}^{\mathbf{c}}_{\cos}$, from (2.10) and (2.11) we have

$$|a_4| - |a_3| = \left|\frac{1}{96}t_1^3 - \frac{1}{48}t_1t_2\right| - \left|\frac{1}{48}t_1^2\right| = |t_1|\psi_+(t_1, t_2), \tag{6.4}$$

where

$$\psi_+(t_1, t_2) := \left|\frac{1}{96}t_1^2 - \frac{1}{48}t_2\right| - \left|\frac{1}{48}t_1\right|.$$

Using the inequality (1.8) we have $|t_1| \leq 2$, and the relation (6.4) leads to

$$|a_4| - |a_3| \leq 2\psi_+(t_1, t_2). \tag{6.5}$$

Letting $\mathcal{B}_1 = \frac{1}{48}$, $\mathcal{B}_2 = \frac{1}{96}$ and $\mathcal{B}_3 = -\frac{1}{48}$, then $|2\mathcal{B}_2 + \mathcal{B}_3| \ngeq |\mathcal{B}_3| + \mathcal{B}_1$. Therefore, according to Lemma 6.1 we get

$$\psi_+(t_1, t_2) \leq 2\,|\mathcal{B}_3| = \frac{1}{24},$$

and using the inequality (6.2) we conclude that

$$|a_4| - |a_3| \leq \frac{1}{12}.$$

From (6.4) it follows

$$|a_3| - |a_4| = -|t_1|\,\psi_+(t_1, t_2) = |t_1|\,\psi_-(t_1, t_2). \tag{6.6}$$

Letting $\mathcal{B}_1 = \frac{1}{48}$, $\mathcal{B}_2 = \frac{1}{96}$, and $\mathcal{B}_3 = -\frac{1}{48}$, then $\mathcal{B}_4 = |4\mathcal{B}_2 + 2\mathcal{B}_3| = 0$, and $\mathcal{B}_1^2 \leq 2\,|\mathcal{B}_3|\,(\mathcal{B}_4 + 2\,|\mathcal{B}_3|)$, therefore, from Lemma 6.1 we obtain that

$$\psi_-(t_1, t_2) \leq 2\mathcal{B}_1\sqrt{\frac{2\,|\mathcal{B}_3|}{\mathcal{B}_4 + 2\,|\mathcal{B}_3|}} = \frac{1}{24}.$$

Finally, from the inequality (1.8) we have $|t_1| \leq 2$, and according to (6.6) the above inequality implies that

$$|a_3| - |a_4| = |t_1|\,\psi_-(t_1, t_2) \leq \frac{1}{12},$$

and the proof is complete. $\square$

Remark 6.4. For the functions

$$\widehat{f}(z) = z\exp\left(\int_0^z \frac{\cos t - 1}{t}\,\mathrm{d}t\right) = z - \frac{1}{4}z^3 + \frac{1}{24}z^5 + \ldots,\ z \in \mathbb{U},$$

and

$$\widetilde{f}(z) = \int_0^z \left[\exp\left(\int_0^x \frac{\cos t - 1}{t}\,\mathrm{d}t\right)\right]\mathrm{d}x = z - \frac{1}{12}z^3 + \frac{1}{120}z^5 + \ldots,\ z \in \mathbb{U},$$

the left hand side of the inequalities of the Theorem 6.2 and Theorem 6.3 are attained, respectively, hence these are the best possible in both cases. To find the right hand side sharp bounds of $|a_4| - |a_3|$ for the classes $\mathcal{S}^*_{\cos}$ and $\mathcal{S}^{\mathbf{c}}_{\cos}$ remains an interesting open question.

7. Conclusion

This paper mainly focuses on finding the upper bounds of the third and fourth-order Hankel determinant for the classes $\mathcal{S}^*_{\cos}$ and $\mathcal{S}^{\mathbf{c}}_{\cos}$ of starlike and convex functions connected with cosine function. Also, we obtained the estimate for Fekete-Szegő and Zalcman functionals for these classes for the cases $n = 3$ and $n = 4$. Moreover, we gave an upper bound for the fourth Hankel determinant for the functions of $\mathcal{S}^*_{\cos}$ and $\mathcal{S}^{\mathbf{c}}_{\cos}$. In addition, by using a recent result, we determined lower and upper bounds for the difference $|a_4| - |a_3|$ of the coefficients for the functions that belong to these classes.

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