Uluslararası Anadolu Sosyal Bilimler Dergisi
… (akademik, hakemli, indexli, uluslararası dergi)…………………
Cilt: 6
Sayı: 3
Yıl: 2022

RECONSIDERING THE PARAMETERS OF CONSTANT ELASTICITY OF SUBSTITUTION PRODUCTION FUNCTION

Ata ÖZKAYA*

ABSTRACT

This study investigates the properties of constant elasticity of substitution production function. Solving the differential equation which defines elasticity of substitution, this study exactly defines the model parameters of production function. In the literature, in previous variant models, those parameters have been widely accepted as free variables. However, this paper demonstrates that the distribution of production factors and technology (efficiency) parameters are not arbitrarily free constants, but are indeed functions of elasticity of substitution and moreover they determine the limiting behavior of factor productivities. This finding contributes to the literature by reconsidering and revising the relationship between model parameters, factor productivities and rate of growth for output of the economy. The results of this study suggest that the economic implications of limiting behavior of factor productivities are deeper than previously considered. This gives more room to designing alternative growth-enhancing policies across the world.

Keywords: Production function; constant elasticity of substitution; growth rate of output per capita; economic growth; efficiency; production factors

SABİT İKAME ESNEKLİĞİ ÜRETİM FONKSİYONUNUN PARAMETRELERİNİN YENİDEN DEĞERLENDİRİLMESİ

ÖZET

Bu çalışmamız sabit ikame esnekliği üretim fonksiyonunun özelliklerini incelemektedir. Sabit ikame esnekliğini tanımlayan diferansiyel denklemin çözülmesiyle, bu çalışmada üretim fonksiyonuna ait yapısal değişkenler tam olarak elde edilmektedir. Literatürde ve daha önce modellenmiş sabit ikame esnekliği üretim fonksiyonlarında bu parametrelerin serbest olduğu varsayılmaktaydı. Hâlbuki bu çalışma üretim faktörlerinin dağılımı ve verimlilik parametrelerinin gelişigüzel sabitler olmadığını, ikame esnekliğinin birer fonksiyonu olduklarını ispat etmektedir. Ayrıca, bu iki değişken böylelikle, faktör üretkenliğinin sınır değer davranışlarını da açıklamaktadır. Bu bulgu, model parametreleri, faktör üretkenliği ve ekonomik büyüme hızı arasındaki ilişkinin tekrar ele alınması açısından literatüre katkı sunmaktadır. Bu çalışmanın sonuçları ek olarak faktör üretkenliklerinin sınır koşullardaki davranışını bilmenin daha zengin ekonomi-politik uygulamalara imkân tanıdığını göstermektedir. Böylelikle politika yapıcılar açısından, ekonomik büyüme hızını artırmayı hedefleyen alternatif politikalara alan açılmaktadır.

Anahtar Kelimeler: Üretim fonksiyonu; ikamenin sabit esnekliği; kişi başına çıktı büyüme oranı; ekonomik büyüme; yeterlik; üretim faktörleri

---

* Associate Professor, Galatasaray University, Department of Economics, Istanbul, Turkey, ORCID No:000-0001-7974-5600, ataozk@yahoo.com; aozkaya@gsu.edu.tr
Research Article Pp: 1074- 1090
Application Date: 22.06.2022 Acceptance Date: 20.08.2022 Publication Date: 31.08.2022

1. INTRODUCTION

In its chronological order, the evolution of constant elasticity of substitution (CES) production function can be summarized as follows. First, Solow (1956) proposed an example of CES function, without giving any definition of coefficient which multiply capital. This model coefficient is an arbitrary term and positively unrestricted. This function has been developed in Pitchford (1960). The author added second coefficient which multiply labor. David & van de Klundert (1965) further improved CES production function in order to include the case of Harrod neutrality of technical progress as well as cases of non-neutrality. Similar to the model in Pitchford (1960), the author introduced two coefficients multiplying capital and labor respectively, and representing the temporal levels of efficiency of capital and labor inputs. However, these studies are a modelling of production type of economy and neither of them proved how these "free coefficients" can mathematically be derived from neoclassical assumptions. On the other hand, based on the empirical findings that the value added per unit of labor used within a given industry varies across countries with the wage rate, Arrow,Chenery,Minhas, & Solow (1961) proposed a constant elasticity of substitution production function. In this function, the coefficients multiplying production factors are assumed to amount to unity, and another parameter representing efficiency multiplies both. Compounding the functions in Arrow et al. (1961) and David & van de Klundert (1965), Barro & Sala-i-Martin (1995:43) added two restricted parameters which measure the efficiency of both production factors, and a distribution parameter is restricted as in the Arrow et al. (1961). In addition, another parameter (arbitrarily free) multiplies both factors, representing neutral technical progress in the sense of Hicks. Even though this latter study is based on differential equation linking marginal product of labor and output per capita, the free coefficients (integration constants) are not mathematically derived, but stayed as model parameters. In a recent study, Klump & Preissler (2000) also showed that above-described variants of constant elasticity of substitution are mathematically derived from a second-order differential equation which defines elasticity of substitution. The CES variants are based on a functional form containing two arbitrary constants of integration, which emerge from order of differential equation. In this sense, the present study is in line with Klump & Preissler (2000). The authors proved that, apart from the model in Barro and Sala-i-Martin (1995), all other models are variants of each other. However, the authors do not identify what are these coefficients, and what do they imply in economic sense. The first aim of this study is to fill this gap.

On the other hand, Klump&Preissler (2000) report that "Knowing that different functional variants share a common ancestor is one thing. Describing the characteristics of particular ancestors that are responsible for the differences between particular families of CES functions, however, is altogether another matter. Different families of CES functions emerge if the two arbitrary constants of integration and thus also the different distribution and efficiency parameters are further specified." To propose a method explaining this query, Klump & Preissler (2000) used the approach introduced in de La Grandville (1989). This approach is the "normalizing" the parameters of CES utility and production functions in order to distinguish different families.

In this perspective, the analysis in this study differs from that of Klump & Preissler (2000). For a meaningful specification of these parameters, the authors use definition of the elasticity of substitution as a point elasticity always relying on three given arbitrary baseline values: a given capital intensity, a given marginal rate of substitution, and a given level of per capita production. Therefore, the second aim of this study is to directly resolve above-mentioned second-differential equation, to mathematically derive the integration constants, and to reveal their economic implications. These economic implications are directly related to marginal factor productivities. The findings of this study show that "normalization" approach is a special case of the result of the present. The present study contributes to the literature in this vein.

In this sense, the approach used in the present study is related to the analyses given in some other studies as Barelli & Abreu Pessôa (2003), Litina & Palivos (2008) and Ozkaya (2021) respectively. Specifically, Barelli & Abreu Pessôa (2003) show that every twice-continuously differentiable and strictly concave function can be bracketed between two CES functions at each open interval. Litina & Palivos (2008) claim that Barelli & Abreu Pessôa (2003) wrongly identified the class of functions with elasticity of substitution asymptotically equal to one as the Cobb–Douglas class. However, based on the setup of Barelli & Abreu Pessôa (2003), Ozkaya (2021) proves that the result of Litina & Palivos (2008) is wrong and that Inada conditions make production function turn into Cobb-Douglas. This study further develops the finding in Ozkaya (2021).

The study is organized as follows. Section 2 proposes a literature review on theoretical and empirical studies which focus on the economic growth rate across different countries and its relation with elasticity of substitution among production factors, and hence with the technological progress in those countries. As it can be easily seen, although our results give support to findings of certain studies, our results are in contrast to findings reported in some other studies. Section 2 enables to state the place of our study in the literature. The findings of the study are introduced in section 3. We clearly compare our initial results with those given in the literature. Finally, section 4 reconsiders a new approach known as "normalization", and investigates whether this approach is capable of aligning the variants of constant elasticity of substitution functions mentioned in the literature.

2. LITERATURE REVIEW

The elasticity of substitution between capital and labor is a parameter which defines the dynamical properties of production function. Beginning from the influential studies in last two decades (de La Grandville 1989 and 1997, Yuhn 1991, Klump & de La Grandville 2000, Saam 2008), the concept of elasticity of substitution between capital and labor has been widely analyzed from theoretical perspective. As long as the theoretical uncertainties and inconsistencies are resolved (Klump & de La Grandville 2000), the researchers reveal richer economic interpretation of the elasticity of substitution. This enables the rise of empirical methods which can be applied to modeling the differences between countries from the perspective of economic growth and development (Nakamura & Nakamura, 2008). In dynamic macroeconomics, the empirical studies mainly focused on the crucial role of the elasticity of substitution in technological developments accelerating countries' economic activity. Integration into global markets and increase in competition have been shown as an efficient way for countries to obtain sustainable growth path driven by elasticity of substitution between capital and labor (Ventura 1997, Klump 2001, Klump et al., 2012). Taken inversely, the inefficiency in substitution elasticity may result with income traps (Azariadis 1996, Guo & Lansing, 2009). If it is considered under the conditions where the demand for production factors fluctuates, especially the labor demand, then the labor economics is another discipline where the elasticity of substitution plays central role (Chirinko 2002). In public finance, conditional on the elasticity of substitution, varying share of labor force and its revenues are determinants of the optimal tax policy (Rowthorn, 1999).

Among production functions exercised in the literature, the Cobb-Douglas production function has long been attired more attention. However, as it has been pointed out by the empirical studies, the Cobb-Douglas production function is not satisfactory in terms of economic implications and does not consistent with novel observations. In their recent study Klump et al. (2012) suggest that "*…Indeed, there is now mounting empirical evidence that aggregate production is better characterized by a non-unitary (and in particular below unitary) elasticity of substitution (see e.g., Chirinko et al.1999; Klump et al.2007; Le´on-Ledesma et al.2010a). Chirinko (2008)'s recent survey suggests that most evidence favors elasticities ranges of 0.4–0.6 for the United States. Moreover, Jones (2005) argued that capital shares exhibit such protracted swings and trends in many countries as to be inconsistent with Cobb–Douglas or CES with Harrod-neutral technical progress (see also Blanchard 1997; McAdam and Willman 2013). Such variability would also suggest the presence of biases in technical change...*" . As we indicate above, the elasticity of substitution defines the dynamics of production function. Then we have to query the range of this parameter observed across actual country datasets.

Yuhn, K. (1991) reports that "*…the United States and South Korea. The test results support the de la Grandville hypothesis that the elasticity of substitution is a potent explanatory variable of economic growth. This inquiry also provides a clue to the puzzle that the U.S. elasticity of substitution between labor and capital is well below unity whereas that of South Korea is close to unity; nonetheless, the U.S. Factor shares have tended to remain fairly stable whereas the distributive shares of South Korea seem to have changed in favor of capital. Their findings indicate that a high elasticity of substitution is a bad signal for the distribution of income under the paradigm of modern technology*."

In a related study for the observed value of substitution parameter across countries Palivos & Karagiannis (2010) state that "*…Hence, an elasticity of substitution that eventually becomes greater than unity can counteract the role of diminishing returns to capital. This renders factor substitution a powerful engine of growth. Duffy and Papageorgiou (2000), estimating a CES production function, find that the elasticity of substitution is greater than one in a subsample of 21 high-capital countries, whereas it is less than one in a subsample of 23 low-capital countries*."

Among the recent studies, Mallick (2012) estimates the elasticity of substitution value for 90 countries using the CES production function. The author reports that about a fifth to a quarter of the growth rate differential between East Asia and Sub-Saharan Africa can be explained by substitution elasticity alone. Hence, a higher value of substitution is shown to broaden the efficient production possibilities.

Based on the accumulating empirical literature, we see that the level of accumulated capital per labor and economic development may be major determinants of the elasticity between capital and labor. Moreover, Klump et al. (2012) report that under the case where elasticity of substitution is greater than one their findings shed light on why some countries have enjoyed a high rate of per capita growth without exhibiting much technical progress.

More recently, Knoblach et al. (2020) show that heterogeneity in reported estimates is driven by the choice of estimation equations, the modeling of technological dynamics, and data characteristics. Based on the underlying meta-regression sample the authors find out that the long-run elasticity of U.S economy stays in the range of 0.6 to 0.7. For all estimated elasticities Knoblach et al. (2020) reject the hypothesis of a Cobb-Douglas production function.

We see that the empirical studies agree with the range of observed value of elasticity of substitution in developed countries. However, the literature has much to do in order to explain how the accumulation of capital per labor ratio coincides with these values. Our study contributes to the literature in this vein.

3. THE INTEGRATION CONSTANTS OF THE CES PRODUCTION FUNCTION

To further progress, first, let us write Cobb-Douglas technology function, $Y = F(K, L) = A.K^{\alpha}.L^{1-\alpha}$. There are two input factors for production, capital services and labor. The exponential parameter $0 < \alpha < 1$ denotes the capital share of the production, while A is exogeneous technology parameter. This type of function is a "neoclassical" type production function, where all input factors are essential and Inada conditions are satisfied (Barro & Sala-i-Martin,1995, Irmen & Maussner, 2017). As it is known, constant elasticity of substitution production function does not satisfy these requirements.

Proposition 1.

Following the results of both Barelli & Abreu Pessôa (2003) and Ozkaya (2021), any production function which can be squeezed between two CES functions, can be written in the form given by (1). Let $f(k)$ be per-capita production function, where the term $\frac{f(k_1)}{k_1.f'(k_1)^{\sigma}}$ is an integration constant and σ is the elasticity of substitution parameter, $\sigma \in [0, \infty)$.

$$f(k) = \frac{f(k_1)}{k_1.f'(k_1)^{\sigma}}.k.f'(k)^{\sigma}$$

(1)

For the proof please see Appendix I.

Proposition 2.

Following the result in Ozkaya (2021), the differential equation (1) has general solution for both $\sigma > 1$ and $\sigma < 1$, which are shown in (3) and (4), respectively.

Proof 2.

Let $u = f(k)^{1-\frac{1}{\sigma}}$ and $u' = \left(1-\frac{1}{\sigma}\right) f'.f^{-\frac{1}{\sigma}}$.

Substituting these terms into (1) and integrating, yields (2).

$$u(k) = \left(\frac{k_1.f'(k_1)^{\sigma}}{f(k_1)}\right)^{\frac{1}{\sigma}}.k^{1-\frac{1}{\sigma}} + C \qquad (2)$$

where $0 < C < \infty$.We note that if integration constant C is equal to zero, $f'(k)$ will be constant also. On the other hand, if C is infinite, then $f(k)$ will be undefined. However, by definition of the production function these cases are not possible. Therefore, C satisfies the following limiting cases, a.) and b.) :

a.) $u(0) = C$ if $\sigma > 1$ and b.) $u(\infty) = C$ if $\sigma < 1$.

The general solution for $f(k)$ can be obtained as shown below.

That is, since $u = f(k)^{1-\frac{1}{\sigma}}$, the per-capita production function is

$$f(k) = \left(\left(\frac{k_1.f'(k_1)^{\sigma}}{f(k_1)}\right)^{\frac{1}{\sigma}}.k^{1-\frac{1}{\sigma}} + C\right)^{\frac{1}{1-\frac{1}{\sigma}}}$$

Inserting the possible values of C, we have (3) and (4), respectively. For the case $\sigma > 1$, the expression (3) determines the per-capita production function depending on the initial condition, namely $f(0)$.

$$f(k) = \left(\left(\frac{k_1.f'(k_1)^{\sigma}}{f(k_1)}\right)^{\frac{1}{\sigma}}.k^{1-\frac{1}{\sigma}} + f(0)^{1-\frac{1}{\sigma}}\right)^{\frac{1}{1-\frac{1}{\sigma}}} \qquad (3)$$

For the case $\sigma < 1$, the expression (4) determines the per-capita production function depending on the terminal condition, namely $f(\infty)$.

$$f(k) = \left(\left(\frac{k_1.f'(k_1)^{\sigma}}{f(k_1)}\right)^{\frac{1}{\sigma}}.k^{1-\frac{1}{\sigma}} + f(\infty)^{1-\frac{1}{\sigma}}\right)^{\frac{1}{1-\frac{1}{\sigma}}} \qquad (4)$$

First, we rearrange (3) and (4) in terms of two production factors and second, we compute the term $\left(\frac{k_1.f'(k_1)^{\sigma}}{f(k_1)}\right)^{\frac{1}{\sigma}}$ which identifies different conditions for the case $\sigma > 1$ and $\sigma < 1$, respectively.

Computing the limiting cases, one can have:

1. For $\sigma > 1$, the initial conditions of the CES function

$\lim_{K\to 0} F(K,L) = F_L(0,L).L$ and $\lim_{L\to 0} F(K,L) = F_K(K,0).K$

The initial conditions with respect to input factors, $F_K(K,0)$ and $F_L(0,L)$ are obtained:

$f(0) = F_L(0,L)$

This expression is defined for all $0 < L < \infty$, and is constant with respect to the input factors and does not vary with the elasticity of substitution.

$$\left(\frac{k_1.f'(k_1)^{\sigma}}{f(k_1)}\right)^{\frac{1}{\sigma}} = F_K(K,0)^{\frac{\sigma-1}{\sigma}}$$
(5)

is defined for all $0 < K < \infty$, and is constant with respect to the input factors.

2. For $\sigma < 1$,

$\lim_{K\to\infty} F(K,L) = F_L(\infty,L).L$ and $\lim_{L\to\infty} F(K,L) = F_K(K,\infty).K$
(6)

The terminal conditions $F_K(K,\infty)$ and $F_L(\infty,L)$ are obtained as given below.

$f(\infty) = F_L(\infty,L)$ is defined for all $0 < L < \infty$, and is constant with respect to the input factors and does not vary with the elasticity of substitution.

$\left(\frac{k_1.f'(k_1)^{\sigma}}{f(k_1)}\right)^{\frac{1}{\sigma}} = F_K(K,\infty)^{\frac{\sigma-1}{\sigma}}$ is defined for all $0 < K < \infty$, and is constant with respect to the input factors.

The terms $F_K(K,0)^{\frac{\sigma-1}{\sigma}}$ and $F_L(0,L)^{\frac{\sigma-1}{\sigma}}$ correspond to the free parameters that have already been introduced in previous models in Solow (1956), Pitchford (1960), Arrow et al. (1961) and David & van de Klundert (1965). As it is already explained, Solow's variant was generalized by Pitchford (1960), the term multiplying capital was considered as “the constant attached to capital”, and a similar term multiplying labor is a constant “attached to labor” (Pitchford,1960,Klump&Preissler 2000). According to Pitchford, both terms depend on the substitution parameter. When it is compared to Pictchford’s coefficients, our finding is consistent with the author’s model, the terms $F_K(K,0)^{\frac{\sigma-1}{\sigma}}$ and $F_L(0,L)^{\frac{\sigma-1}{\sigma}}$ have elasticity of substitution as exponent. However, different from previous models, this study proved the existence of these coefficients. On the other hand, for the case of $\sigma < 1$ there should be another set of terms. Neither Pitchford nor other models did not explain what happens when elasticity of substitution is $\sigma < 1$. Our results indicate that the free terms should become $F_K(K,\infty)^{\frac{\sigma-1}{\sigma}}$ and $F_L(\infty,L)^{\frac{\sigma-1}{\sigma}}$ respectively. These coefficients represent different type of behavior of marginal products than those shown for the case $\sigma > 1$. Therefore, our results generalize the findings of above-mentioned previous studies. We consider that each of the initial and the terminal conditions represents the structural properties of the production process and cannot be chosen arbitrarily.

In order to compare different economies in perspective of production capabilities, there can be three cases based on the initial and terminal conditions and on the elasticity of substitution level as well.

i.) $F_{iK}(K,0) \neq F_{jK}(K,0)$ and/or $F_{iL}(0,L) \neq F_{jL}(0,L)$, and $\sigma_i = \sigma_j > 1$, where, *i*,*j*=1,..,n and *i*≠*j* denote the country.

ii.) $\sigma_i \neq \sigma_j > 1$ and $F_{iK}(K,0) = F_{jK}(K,0)$ and/or $F_{iL}(0,L) = F_{jL}(0,L)$.

iii.) $\sigma_i \neq \sigma_j > 1$ and $F_{iK}(K,0) \neq F_{jK}(K,0)$ and/or $F_{iL}(0,L) \neq F_{jL}(0,L)$.

In order to identify a common root for variants of constant elasticity of substitution function, the following analysis is presented, and the expressions (7),(8) and (9) are introduced.

Recall that the CES function proposed by Arrow et al. (1961) is:

$$Y = F(K,L) = (\beta.K^{-\rho} + \alpha.L^{-\rho})^{\frac{-1}{\rho}}$$

This production function is written out more symmetrically as:

$$Y = F(K,L) = \gamma[\delta.K^{-\rho} + (1-\delta).L^{-\rho}]^{\frac{-1}{\rho}}$$

where $\alpha + \beta = \gamma^{-\rho}$; $\beta.\gamma^{\rho} = \delta$; $\frac{\beta}{\alpha} = \frac{\delta}{1-\delta}$; $\rho = \frac{1}{\sigma} - 1$ and σ is the elasticity of substitution.

Thus, the efficiency parameter appears to be $\gamma = \left[\left(\frac{K.F_K(K,L)^{\sigma}}{F(K,L)}\right)^{\frac{1}{\sigma}} + \left(\frac{L.F_L(K,L)^{\sigma}}{F(K,L)}\right)^{\frac{1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}$ and the distribution parameter, δ should satisfy $\frac{\delta}{1-\delta} = \frac{F_K(K,L)}{F_L(K,L)}\left(\frac{K}{L}\right)^{\frac{1}{\sigma}}$.

We have the following precise result: (5) and (6) demonstrate that the efficiency parameter depends only on the value of elasticity of substitution. The efficiency parameter is obtained:

$$\gamma = \begin{cases} \left(F_K(K,0)^{\frac{\sigma-1}{\sigma}} + F_L(0,L)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}} & \text{for} \quad \sigma > 1 \\ \left(F_K(K,\infty)^{\frac{\sigma-1}{\sigma}} + F_L(\infty,L)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}} & \text{for} \quad \sigma < 1 \end{cases}$$

(7)

In addition, (3) and (5) demonstrate that for $\sigma > 1$, $\beta = \delta.\gamma^{-\rho}$ must be equal to $F_K(K,0)^{\frac{\sigma-1}{\sigma}}$, which depends on the elasticity of substitution. Similarly, from (3) and (6) it can be easily seen that $\alpha = (1-\delta).\gamma^{-\rho}$ must be equal to $F_L(0,L)^{\frac{\sigma-1}{\sigma}}$, which depends only on the elasticity of substitution as well. Above arguments lead to (8). Then, the distribution parameter, δ is:

$$\delta = \begin{cases} F_K(K,0)^{\frac{\sigma-1}{\sigma}} / \left(F_K(K,0)^{\frac{\sigma-1}{\sigma}} + F_L(0,L)^{\frac{\sigma-1}{\sigma}}\right) & \text{for} \quad \sigma > 1 \\ F_K(K,\infty)^{\frac{\sigma-1}{\sigma}} / \left(F_K(K,\infty)^{\frac{\sigma-1}{\sigma}} + F_L(\infty,L)^{\frac{\sigma-1}{\sigma}}\right) & \text{for} \quad \sigma < 1 \end{cases}$$
(8)

From (3) and (4), it is precise that whenever $\sigma > 1$, $\beta^{\frac{\sigma}{\sigma-1}} = F_K(K,0)$ is the minimum marginal product of capital and $\alpha^{\frac{\sigma}{\sigma-1}} = F_L(0,L)$ is the minimum marginal product of labor. On the other hand, for $\sigma < 1$, $\beta^{\frac{\sigma}{\sigma-1}} = F_K(K,\infty)$ denotes the maximum marginal product of capital whereas $\alpha^{\frac{\sigma}{\sigma-1}} = F_L(\infty,L)$ denotes the maximum marginal product of labor.

From (7) we easily see that the efficiency parameter imposed in Arrow et al. (1961) is equal to a particular point, namely $\gamma = F(1,1)$. Moreover, homogeneity makes it equal to $\gamma = \frac{F(n,n)}{n}$. By the definition, we have

$$\delta.\gamma^{-\rho} = F_K(K,0)^{\frac{\sigma-1}{\sigma}} = \left(\frac{F(K,L)}{K}\right)^{\frac{-1}{\sigma}} F_K(K,L)$$

Finally, the distribution parameter is obtained in (9).

$$\delta = \left(\frac{F_K(K,0)}{F(1,1)}\right)^{\frac{\sigma-1}{\sigma}} = \left(\frac{F(1,0)}{F(1,1)}\right)^{\frac{\sigma-1}{\sigma}}$$
(9)

However, it is hard to understand why the particular set of points $F(1,1)$ should be defined as the efficiency parameter and that $\frac{F(1,0)}{F(1,1)}$ as the distribution parameter. According to the result of the present study, the finding in Arrow et al.(1961) is misleading.

The behavior of the CES function as ρ takes the limiting values, 0 and ∞ (infinite) is considered as follow. The findings of this study (7), (8) and (9) demonstrate that both efficiency parameter and distribution parameter depend upon the elasticity of substitution, and that above-given argument is false and should be corrected.[*] Thus, it is not mathematically consistent to relate the CES production function to the Theorem 3 of Mean theory of Hardy et al. (1934). In proof, taking logarithms of the production functions and then setting $\sigma \to 1$ in (3) and (4), is sufficient to reach the result. Similarly, Arrow et al. (1961) claim that: "*And as $\rho \to \infty$, the elasticity of substitution tends to zero and we approach the case of fixed proportions. We may prove this by making the appropriate limiting process on (13). And once again the general theory of mean values assures us that as a mean value of order $-\infty$ we have….*" Unfortunately, this argument is false too. This result of the present study is in sharp contrast with the finding

[*] Similarly, the approach of Pitchford (1960) is also misleading

in Arrow et al. Note again that the CES production function does not satisfy the assumptions for Theorem 4 of Mean theory of Hardy et al. (1934).

In proof, we allow $\sigma \to 0$ in (6), and we obtain

$$F(K,L) = min(F_K(K,\infty).K, F_L(\infty,L).L),$$

which is the minimum of the production shares having maximum marginal product. The future research agenda should be concerned with the revision of the related results* stated in modern textbooks (i.e., Acemoglu 2009, Barro & Sala-i-Martin, 2003, de La Grandville, 2009) on the economic growth theory, which are particularly based on constant elasticity of substitution. Here we restrict ourselves to demonstrate that, for any K, L and any given value of the parameters of the production function, income (and income per capita) are decreasing functions of the elasticity of substitution. Taking logarithms of (3) and (4), and then differentiating with respect to elasticity of substitution immediately give the desired result. Hence, our actual knowledge on link between elasticity of substitution and economic growth should be revised.

4. NORMALIZATION OF CONSTANT ELASTICITY OF SUBSTITUTION FUNCTION

The “normalization” of the CES function at some arbitrarily chosen baseline values is initialized by de La Grandville (1989). In their theoretical study, Klump & de La Grandville (2000) apply the normalization method to compare the rate of growth of the economies which are distinguished by their elasticity of substitution.

In a related study, Klump&Preissler (2000) claim that there are certain inconsistencies and controversies arising from the variants of the CES production function in growth models. To align the use of the variants of the CES functions, the authors propose normalization of the CES parameters in terms of arbitrary chosen values for the capital-labor ratio, the per capita production, and the marginal rate of substitution. Even though we give support to above-mentioned studies in the sense of investigating the implication and the interpretation of the CES parameters, the analysis in this study differs from those by the methodology, by the structure of the function and by the results. The findings in previous section reveal a query: is it necessary to “normalize” the parameters of the CES function at some arbitrary chosen values?

As Arrow et al. (1961) did, Klump & de La Grandville (2000) employ another partitioning of the CES coefficients. That is: Klump & La Grandville (2000) propose the normalized CES per-capita production function** which is given below.

* In a forthcoming study, we revise the findings of the literature on the effects of the elasticity of substitution on the rate of growth, and we analyze the steady-state threshold values for saving rate, population growth rate and elasticity of substitution.

** Klump & de La Grandville (2000) define the substitution parameter as $\frac{\sigma}{\sigma-1}$.

$$y = A(\sigma)\left(a(\sigma)k^{\frac{\sigma}{\sigma-1}} + \left(1 - a(\sigma)\right)\right)^{\frac{\sigma-1}{\sigma}}$$

In above-given function $A(\sigma)$ and $a(\sigma)$ depend on arbitrarily chosen values of three variables, namely capital-labor ratio, marginal rate of substitution value and per-capita production. Adopting (3) and (4) to above-given normalized CES function, we determine the normalization parameters in terms of initial and terminal conditions:

For $\sigma > 1$, $A(\sigma)^{\frac{\sigma}{\sigma-1}}.a(\sigma) = F_K(K,0)^{\frac{\sigma}{\sigma-1}}$. On the other hand, for $\sigma < 1$, $A(\sigma)^{\frac{\sigma}{\sigma-1}}.a(\sigma) = F_K(K,\infty)^{\frac{\sigma}{\sigma-1}}$. Since, for $\sigma > 1$ the relationship between marginal product and average product of capital services is shown by

$F_K(K,0)^{\frac{\sigma}{\sigma-1}} = \left(\frac{F(K,L)}{K}\right)^{\frac{1}{\sigma-1}}.F_K(K,L)$. This expression shows that $A(\sigma)$ and $a(\sigma)$ denote the particular points:

$$A(\sigma) = F(1,1) \quad \text{and} \quad a(\sigma) = \left(\frac{F_K(K,0)}{F(1,1)}\right)^{\frac{\sigma}{\sigma-1}} = \left(\frac{F(1,0)}{F(1,1)}\right)^{\frac{\sigma}{\sigma-1}}$$
(10)

One can easily compute the normalized parameters for $\sigma < 1$.

From (8) and (10), it is clear that Klump & de La Grandville (2000) model is same as (reduced form) the variants in Solow (1956), Pitchford (1960), Arrow et al. (1961), David & van de Klundert (1965). Moreover, the findings in this study demonstrate that the inconsistencies and controversies reported by Klump & Preissler (2000) are eliminated by (3) and (4) (see the expressions A.4 and A.5 in Appendix II). Therefore, the results of this study point out that normalizing the CES parameters does not create any novel knowledge, and that the initial and terminal conditions inherently identify the aforementioned differences in economies.

5. CONCLUSION AND DISCUSSION

The findings (7), (8) and (9) of the present study demonstrate that both the efficiency parameter and the distribution parameter depend upon the elasticity of substitution. In this sense, this finding confirms the model in Pitchford (1960). On the other hand, the result of this study explains how the variants in Solow (1956), Pitchford (1960), Arrow et al. (1961), David & van de Klundert (1965), Klump & de La Grandville (2000) can be unified and reduced to one functional from.

In addition, the present study proves that the distribution and the efficiency parameters are composed of the initial and the terminal conditions, which are omitted in previous studies. Finally, the findings in this study show that defining the initial and terminal conditions is capable of aligning the variants of the CES production function.

A widely accepted property of constant elasticity of substitution function in the literature is that this function is a class of function defined as “mean value of order -ρ” (Barro&Sala-i-Martin 1995, and 2003, Acemoglu,2009, de La Grandville, 2009). However, one of the crucial results of this study is to prove that the CES function is not a class of function defined as a mean value of order -ρ, substitution parameter. Secondly, to make the efficiency parameter equal to 1, one has to choose the initial and the terminal conditions such that $F(n, n) = n$, for all $n > 0$, which is a particular case. However, it can be easily seen from (7) and (8) that both efficiency and distribution parameters depend on elasticity of substitution. This dependence violates the fundamental hypothesis of both Weighted means theorem and Ordinary means theorem in Hardy et al. (1934).

Finally, unless the minimum (maximum) marginal product of input factors are shown to have more elementary components, all possible variants of the CES function should be written out in terms of the initial and terminal conditions.

It is important to establish the linkage between theoretical results in the present study and empirical findings in the literature. Especially, the policy makers focusing on growth increasing economic-policies should identify the effects of marginal productivities and elasticity of substitution on rate of growth. As an example, the U.S. economy historically has elasticity of substitution level lower-than-unity. A recent survey in Chirinko (2008) suggests that most evidence favors elasticities ranges of 0.4–0.6 for the United States. Knoblach et al. (2020) report that the elasticities for the US economy stay between 0.45 and 0.87. For all estimated elasticities Knoblach et al. (2020) reject the hypothesis of a Cobb-Douglas production function. These findings suggest that in an economy where elasticity of substitution is lower-than-unity should address maximum marginal productivity of capital services and labor as well. The results given in (3) and (4) indicate that this policy should be different for developing economies where elasticity level is higher-than-unity (Palivos& Karagiannis, 2010).

The future researches should focus on: first, whether the elasticity of substitution promotes the rate of growth, second on the sensitivity of steady-state capital per labor ratio to the elasticity of substitution; third, on the threshold for the elasticity of substitution to entail ever-sustained growth. These analyses will shed light on sustainability conditions for growth enhancing policies and help to increase the efficiency of existing monetary and fiscal policies amid Covid-19 supply shocks.

Appendix I

The elasticity of substitution is defined (see de La Grandville (1997))

$$\sigma(k) = -\frac{f'(k)[f(k) - k.f'(k)]}{k.f''(k).f(k)}$$

(A.1)

such that $\sigma(k) \geq 0$ for $k \geq 0$. Reorganizing $\sigma(k)$, we get

$$\sigma(k) = \frac{1}{G'(k)} . \frac{G(k)}{k} \ ,$$

where $G(k) = k - \frac{f(k)}{f'(k)} < 0$ and $G'(k) < 0$.

Suppose that $\sigma(k)$ is bounded and continuous on $k \in [0, k_1]$; that is, $\sigma(0) - \varepsilon \leq \sigma(k) \leq \sigma(0) + \varepsilon$ .Let $\sigma(0) - \varepsilon \leq \sigma(k) \leq \sigma(0) + \varepsilon \leq \mu$ be satisfied and $\sigma(0) = \mu$ may or may not be equal to 1. To analyse the case $\mu = 1$ is out of scope of the present study. For the case $\mu = 1$, please refer to Ozkaya (2021). For notational ease, let σ_- and σ^+ signify $\sigma_- = \sigma(0) - \varepsilon$ and $\sigma^+ = \sigma(0) + \varepsilon$, respectively. The same applies for $\sigma(\infty)$, and we suppose that $\sigma(k)$ is bounded and continuous on $k \in [k_1, \infty)$; hence , $\sigma(\infty) - \varepsilon \leq \sigma(k) \leq \sigma(\infty) + \varepsilon$.

In particular, suppose that $\sigma(0) + \varepsilon \leq \mu$. Since $\sigma_- \leq \sigma(k) \leq \sigma^+ \leq \mu$, which after plugging in elasticity of substitution (A.1) leads to $\sigma_- \leq \frac{1}{\frac{dG(k)}{dk}} . \frac{G(k)}{k} \leq \mu$. Rearranging this as $\frac{1}{\sigma_-} . \frac{dk}{k} \geq \frac{dG(k)}{G(k)} \geq \frac{1}{\mu} \frac{dk}{k}$ and integrating over the $k \in [0, k_1]$ yields

$$k - G(k_1)\left(\frac{k}{k_1}\right)^{\frac{1}{\sigma_-}} \leq \frac{f(k)}{f'(k)} \leq k - G(k_1)\left(\frac{k}{k_1}\right)^{\frac{1}{\mu}}.$$

Reorganizing this formulation and re-integrating, we get the per-capita production function demonstrated between lower and upper bounds

$$\frac{f(k_1)}{k_1}\left(\frac{\left(k^{1-\frac{1}{\sigma_-}} - \frac{G(k_1)}{k_1^{\frac{1}{\sigma_-}}}\right)^{\frac{1}{1-\frac{1}{\sigma_-}}}}{\left(1 - \frac{G(k_1)}{k_1}\right)^{\frac{1}{1-\frac{1}{\sigma_-}}}}\right) \leq f(k) \leq \frac{f(k_1)}{k_1}\left(\frac{\left(k^{1-\frac{1}{\mu}} - \frac{G(k_1)}{k_1^{\frac{1}{\mu}}}\right)^{\frac{1}{1-\frac{1}{\mu}}}}{\left(1 - \frac{G(k_1)}{k_1}\right)^{\frac{1}{1-\frac{1}{\mu}}}}\right).$$

Similarly, suppose that $\sigma(0) - \varepsilon \geq \mu$. The aforementioned calculation steps immediately follow that

$$\frac{f(k_1)}{k_1}\left(\frac{\left(k^{1-\frac{1}{\mu}} - \frac{G(k_1)}{k_1^{\frac{1}{\mu}}}\right)^{\frac{1}{1-\frac{1}{\mu}}}}{\left(1 - \frac{G(k_1)}{k_1}\right)^{\frac{1}{1-\frac{1}{\mu}}}}\right) \leq f(k) \leq \frac{f(k_1)}{k_1}\left(\frac{\left(k^{1-\frac{1}{\sigma^+}} - \frac{G(k_1)}{k_1^{\frac{1}{\sigma^+}}}\right)^{\frac{1}{1-\frac{1}{\sigma^+}}}}{\left(1 - \frac{G(k_1)}{k_1}\right)^{\frac{1}{1-\frac{1}{\sigma^+}}}}\right)$$

Squeezing $\sigma(0) + \varepsilon \leq \mu$ and $\sigma(0) - \varepsilon \geq \mu$ and then, looking for the common condition satisfying $\sigma(k) = \mu$ for $k \in [0, k_1]$, yields the following equations (A.2) and (A.3). For conventional use, let us replace μ with σ.

$$\frac{k.f'(k)}{f(k)} = \frac{1}{1 - \frac{G(k_1)}{k_1^{\frac{1}{\sigma}}} k^{\frac{1}{\sigma} - 1}}$$

(A.2)

$$f(k)=\frac{f(k_1)}{k_1}.k.\left(\frac{\left(1-\frac{G(k_1)}{k_1^{\frac{1}{\sigma}}}k^{\frac{1}{\sigma}-1}\right)^{\frac{1}{1-\frac{1}{\sigma}}}}{\left(1-\frac{G(k_1)}{k_1}\right)^{\frac{1}{1-\frac{1}{\sigma}}}}\right)$$

(A.3)

Inserting (A.2) into (A.3) and rearranging gives (1). That is,

$$f(k)=\frac{f(k_1)}{k_1}.k.\left(\frac{\left(\frac{f(k)}{k.f'(k)}\right)^{\frac{1}{1-\frac{1}{\sigma}}}}{\left(\frac{f(k_1)}{k_1.f'(k_1)}\right)^{\frac{1}{1-\frac{1}{\sigma}}}}\right) \quad \text{and} \quad \frac{f(k)}{k.f'(k)}=\frac{f(k_1)}{k_1}.\frac{1}{f'(k)}.\left(\frac{\left(\frac{f(k)}{k.f'(k)}\right)^{\frac{1}{1-\frac{1}{\sigma}}}}{\left(\frac{f(k_1)}{k_1.f'(k_1)}\right)^{\frac{1}{1-\frac{1}{\sigma}}}}\right)$$

Thus, reinserting $\frac{f(k)}{k.f'(k)}$ into $f(k)$, we have

$$f(k)=\frac{f(k_1)}{k_1.f'(k_1)^{\sigma}}.k.f'(k)^{\sigma}$$

Appendix II

Klump and Preissler (2000:45) propose the normalized CES function:

$$F_t(K,L)=Y_0\left[\pi_0\left(\frac{K_t}{K_0}\right)^{\frac{\sigma-1}{\sigma}}+(1-\pi_0)\left(\frac{L_t}{L_0}\right)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}},$$

where π_0, Y_0, K_0, L_0 are arbitrary initial values. π_0 denotes the capital share in total income at the point of normalization Y_0 :

$$\pi_0=\frac{r_0K_0}{Y_0}$$

whether $\pi_0=\frac{r_0K_0}{r_0K_0+w_0L_0}$ then,

$$F_t(K,L)=F(K_0,L_0)\left[\frac{r_0K_0}{r_0K_0+w_0L_0}\left(\frac{K_t}{K_0}\right)^{\frac{\sigma-1}{\sigma}}+\frac{w_0L_0}{r_0K_0+w_0L_0}\left(\frac{L_t}{L_0}\right)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}$$

$$F_t(K,L)=\left[F(K_0,L_0)^{\frac{\sigma-1}{\sigma}}\frac{r_0K_0}{F(K_0,L_0)}\left(\frac{K_t}{K_0}\right)^{\frac{\sigma-1}{\sigma}}+F(K_0,L_0)^{\frac{\sigma-1}{\sigma}}\frac{w_0L_0}{F(K_0,L_0)}\left(\frac{L_t}{L_0}\right)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}$$

where $F(K_0,L_0)=r_0K_0+w_0L_0$

Under perfect competition, π_0 this distribution parameter is equal to the capital income share but, under imperfect competition with non-zero aggregate mark-up, it equals the share of capital income over total factor income.

These are $\pi_0=\frac{r_0K_0}{r_0K_0+w_0L_0}$ and $\pi_0=\frac{r_0K_0}{Y_0}$.

$$F_t(K,L) = \left[\left(\frac{K_0}{F(K_0,L_0)}\right)^{\frac{1}{\sigma}} r_0(K_t)^{\frac{\sigma-1}{\sigma}} + \left(\frac{L_0}{F(K_0,L_0)}\right)^{\frac{1}{\sigma}} w_0(L_t)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}$$

Then $\left(\frac{K_0}{F(K_0,L_0)}\right)^{\frac{1}{\sigma}} r_0 = F_K(K,0)^{\frac{\sigma-1}{\sigma}}$
(A.4)

where r_0 should be equal to $F_K(K,L)$ at a given normalization point (chosen by Klump & Preissler 2000). Since for any $0 < K < \infty$, $F_K(K,0) = F_K(K^*,0)$, this leads back our definition.

$\left(\frac{L_0}{F(K_0,L_0)}\right)^{\frac{1}{\sigma}} w_0 = F_L(0,L)^{\frac{\sigma-1}{\sigma}}$
(A.5)

likewise w_0.

Equation (A.4) and (A.5) depict the normalized parameters, which correspond to initial conditions of the CES function with the elasticity of substitution greater than unity.

REFERENCES

Acemoglu, D. (2009). *Introduction To Modern Economic Growth,* (Princeton University Press).

Arrow, K.J., Chenery, H.B., Minhas, B.S., & Solow, R.M. (1961). Capital-Labor Substitution and Economic Efficiency. *The Review of Economics and Statistics*, 43, 3, 225-250.

Azariadis, C. (1996). The economics of poverty traps - Part One: Complete markets. *Journal of Economic Growth* 1: 449–486.

Barelli, P.,& Abreu Pessôa, S.(2003).Inada conditions imply that production function must be asymptotically Cobb–Douglas. *Economics Letters*, 81, 3,361-363, https://doi.org/10.1016/S0165-1765(03)00218-0.

Barro, R. J. & Sala-i-Martin, X. (1995). Economic Growth, McGraw-Hill, New York.

Barro, R. J. & Sala-i-Martin, X. (2003). Economic Growth, Second edition, McGraw-Hill, New York.

Blanchard, O. J. (1997). The medium run. *Brookings Papers on Economic Activity*, 2: 89–158.

Chirinko, R. S., Fazzari, S. M., & Meyer, A. P. (1999). How responsive is business capital formation to its user cost? *Journal of Public Economics*, 74(1): 53–80.

Chirinko, R. S.(2002). Corporate taxation, capital formation, and the substitution elasticity between labor and capital. *National Tax Journal*, 60: 339–355.

Chirinko, R. S. (2008). σ: The long and short of it. *Journal of Macroeconomics*, 30(2): 671–686.

David, P. A. & van de Klundert, T. (1965), Biased efficiency growth and capital labor substitution in the U.S., 1899 – 1960. *American Economic Review*, 55, 357-394.

Duffy, J., & Papageorgiou, C. (2000). A cross-country empirical investigation of the aggregate production function specification. *Journal of Economic Growth* 5, 87–120.

De La Grandville, O.(1989). In Quest of the Slutsky Diamond. *The American Economic Review*, 79, 3, 468-481.

De La Grandville, O. (1997). Curvature and the elasticity of substitution: Straightening it out. *Journal of Economics,* 66 (1): 23-34.

De La Grandville, O. (2009). Economic growth: a unified approach, 1th Edition. (Cambridge University Press).

Guo, J. T., & Lansing, K. J. (2009). Capital-labour substitution and equilibrium indeterminacy. *Journal of Economic Dynamics and Control,* 33(12): 1991–2000.

Hardy, G.H., Littlewood, J.E.,& Polya, G.(1934). *Inequalities* (Cambridge University Press, England).

Irmen, A., & Maußner, A. (2017). A note on the characterization of the neoclassical production function. *Macroeconomic Dynamics, 21*(7), 1827-1835. doi:10.1017/S136510051500098X

Jones, C. I. (2005). The shape of production functions and the direction of technical change. *Quarterly Journal of Economics,* 120(2): 517–549.

Klump, R., & de La Grandville, O. (2000). Economic Growth and Elasticity of Substitution: Two Theorems and Some Suggestions. *The American Economic Review*, 90, 282-291.

Klump, R. (2001). Trade, money and employment in intertemporal optimizing models of growth. *Journal of International Trade and Economic Development*,10(4): 411–428.

Klump, R., McAdam, P., & Willman, A. (2007). Factor substitution and factor-augmenting technical progress in the US. *Review of Economics and Statistics,* 89(1): 183–92.

Klump, R., McAdam, P., & Willman, A. (2012). The normalized CES production function: Theory and empirics. *Journal of Economic Surveys*, 26: 769-799. https://doi.org/10.1111/j.1467-6419.2012.00730.x

Knoblach, M., Roessler, M. & Zwerschke, P. (2020). The elasticity of substitution between capital and labour in the US economy: A meta‐regression analysis. *Oxford Bulletin of Economics and Statistics, 82*(1), 62–82. https://doi.org/10.1111/obes.12312

Le ́on-Ledesma, M. A., McAdam, P. & Willman, A. (2010) Identifying the elasticity of substitution with biased technical change. *American Economic Review*, 100(4): 1330–1357.

Mallick, D. (2012). The role of the elasticity of substitution in economic growth: A cross-country investigation. *Labour Economics, 19*, 682-694.

McAdam, P., & Willman, A. (2013). Medium run redux. *Macroeconomic Dynamics, 17*(4), 695-727. doi:10.1017/S1365100511000459

Ozkaya, A. (2021). Inada conditions asymptotically transform production function into the Cobb–Douglas, *Economics Letters*, 109786.

Palivos, T., & Karagiannis, G. (2010). The elasticity of substitution as an engine of growth. *Macroeconomic Dynamics, 14*, 617 - 628.

Pitchford, J. D. (1960). Growth and the Elasticity of Substitution, Economic Record 36,491-503.

Rowthorn, R. (1999). Unemployment, wage bargaining and capital-labour substitution. Cambridge Journal of Economics 23(4): 413–425.

Saam, M. (2008). Openness to trade as a determinant of the macroeconomic elasticity of substitution. *Journal of Macroeconomics*, 30: 691–702

Solow, Robert M. A. (1956). Contribution to the Theory of Economic Growth. *The Quarterly Journal of Economics*, 70,1: 65-94.

Ventura, J. (1997). Growth and interdependence. *Quarterly Journal of Economics*, 62: 57–84.

Yuhn, K. (1991). Economic Growth, Technical Change Biases, and the Elasticity of Substitution: A Test of the De La Grandville Hypothesis. *The Review of Economics and Statistics, 73*, 340-346.

Çatışma Beyanı: Makalenin yazarları bu çalışma ile ilgili taraf olabilecek herhangi bir kişi ya da finansal ilişkileri bulunmadığını dolayısıyla herhangi bir çıkar çatışmasının olmadığını beyan eder.

Destek ve teşekkür: Çalışmada herhangi bir kurum ya da kuruluştan destek alınmamıştır.

Etik Kurul Kararı: Bu araştırma, Etik Kurul Kararı gerektiren makaleler arasında yer almamaktadır.
Katkı Oranı: Makale tek yazarlıdır.