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New classes of control functions for nonlinear contractions and applications

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Abstract

We initiate the use of sub and super homogeneous control functions for nonlinear contractions in complete metric spaces and establish new fixed point theorems. Moreover, we develop other variants of control functions for the fixed point theorems of Boyd-Wong [4] and Matkowski [8]. As applications, we present new sufficient conditions ensuring the existence of solutions to some classes of integral equations of Fredholm and Volterra types.

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1. Introduction

The famous Banach's contraction principle [2] is one of the most important tools for establishing sufficient conditions of existence and uniqueness of the solution to a large class of nonlinear equations. Due to its usefulness in different branches of mathematics, this principle has been developed in various ways. For instance, Boyd-Wong [4] and Matkowski [8] generalized the Banach contraction principle by using a control function ψ as follows:

Theorem 1.1 ([4]). *Let (X, d) be a complete metric space and $T\colon X \to X$ be a mapping. Assume there exists a function $\psi\colon \overline{P} \to \mathbb{R}^+$ such that*

$$d(Tx, Ty) \leq \psi(d(x, y)) \;\; \text{for all } x, y \in X, \tag{C1}$$

$$\psi(t) < t \;\; \text{for all } t \in \overline{P}^*, \tag{1}$$

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$$\limsup_{s\to t^+} \psi(s) < t \ \ \textit{for all } t \in \overline{P}^*,$$

where $P := \{d(x,y) : x,y \in X\}$, $\overline{P}$ *is the closure of* P *and* $\overline{P}^* := \overline{P} \setminus \{0\}$. *Then,* T *has a unique fixed point* z, *and the sequence* $\{T^n x\}$ *converges to* z *for all* $x \in X$.

Theorem 1.2 ([8])**.** *Let* (X,d) *be a complete metric space and* $T\colon X \to X$ *be a mapping. Assume there exists an increasing function* $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ *such that* (C1) *holds and*

$$\lim_{n\to\infty} \psi^n(t) = 0 \ \ \textit{for all} \ \ t > 0.$$

Then, T *has a unique fixed point* z, *and the sequence* $\{T^n x\}$ *converges to* z *for all* $x \in X$.

For a detailed exploration of the interconnection between different classes of control functions, we direct the reader to the paper [6] and the book [1]. It is worth noting that when the control function ψ is applied to the left side of the contraction (see for instance [5]) it must satisfies the following condition:

$$t < \psi(t) \ \ \text{for all } t \in \overline{P}^*. \tag{2}$$

In this paper, we consider a complete metric space (X,d) and a self-mapping $T\colon X \to X$ that satisfies (C1) or one of the following inequalities:

$$d(Tx,Ty) < \psi(d(x,y)) \ \ \text{for all } (x,y) \in Y, \tag{C2}$$

$$\psi(d(Tx,Ty)) \le d(x,y) \ \ \text{for all } \ x,y \in X, \tag{C3}$$

$$\psi(d(Tx,Ty)) < d(x,y) \ \ \text{for all } (x,y) \in Y, \tag{C4}$$

where $Y := X \times X \setminus \{(x,x) : x \in X\}$ and ψ is a given control function that may satisfies neither (1) nor (2). We first establish new fixed point theorems for mappings satisfying nonlinear contractions involving sub or super homogeneous control functions. Then, we investigate the existence of fixed points for mappings fulfilling a nonlinear contraction of Boyd-Wong or Matkowski type via several variants of control functions. As applications, we discuss the existence of solutions to certain classes of integral equations of Fredholm or Volterra types.

In the present paper, we will assume that (X,d) is a complete metric space, $x_0 \in X$ and $T\colon X \to X$ is a mapping, and will use the notations $P := \{d(x,y) : x,y \in X\}$, $P^* := P \setminus \{0\}$, $\overline{P}$ is the closure of P, $\overline{P}^* := \overline{P} \setminus \{0\}$, $x_n := T^n x_0$ and $d_n := d(x_n, x_{n+1})$, where T^n is the n-th iterate of T with T^0 the identity mapping and $n \in \mathbb{N}$. The organization of the paper is as follows. In Sections 2 and 3, we establish new fixed point theorems via sub and super homogeneous control functions. In Sections 4 and 5, we present new extensions of Boyd-Wong's and Matkowski's fixed point theorems. We discuss in Section 6 the existence of solutions to new classes of integral equations of Fredholm and Volterra types.

2. Subhomogeneous control functions

In this section, we present new fixed point theorems for mappings satisfying the nonlinear contraction (C1) or (C2), where ψ belongs to the following class of functions.

Definition 2.1. *A function* $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ *is said to be subhomogeneous if*

$$\psi(\lambda t) \le \lambda \psi(t) \ \ \textit{for all } \lambda \in (0,1) \textit{ and all } t \in \mathbb{R}^+.$$

Remark 2.2. *If* ψ *is increasing and subhomogeneous, then it is continuous at* 0 *and satisfies* $\psi(0) = 0$.

Example 2.3. *Consider the function* $\psi(t) = \frac{t^2}{1+t}$ *for all* $t \in \mathbb{R}^+$, *then* ψ *is increasing, subhomogeneous, continuous and satisfies* $\psi(t) < t$ *for all* $t > 0$.

The first main result is the following.

Theorem 2.4. *Assume that there exists an increasing subhomogeneous function ψ such that* (C1) *holds. If there exists $x_0 \in X$ such that*

$$\psi(d_0) < d_0, \tag{3}$$

then the sequence $\{T^n x_0\}$ converges to a fixed point z of T. If for all $t \in P^*$, we have

$$\psi(t) < t, \tag{4}$$

then z is the unique fixed point of T, and the sequence $\{T^n x\}$ converges to z for all $x \in X$.

Proof. It is clear from (3) that x_0 is not a fixed point of T, and that $\lambda_0 := d_0^{-1}\psi(d_0) \in (0,1)$, which is combined with (C1) gives

$$d_1 \leq \psi(d_0) = \lambda_0\, d_0.$$

Using (C1), together with the monotony and the homogeneity of ψ, we deduce

$$d_2 \leq \psi(d_1) \leq \psi(\lambda_0\, d_0) \leq \lambda_0\, \psi(d_0) = \lambda_0^2\, d_0.$$

Hence, by induction, we easily get

$$d_n \leq \lambda_0^n d_0 \ \text{ for all } n \in \mathbb{N}. \tag{5}$$

We deduce that the sequence $\{d_n\} \to 0$ as $n \to \infty$. Next, for all $m, n \in \mathbb{N}$ such that $m > n$, it follows by (5) that

$$d(x_n, x_m) \leq \sum_{k=n}^{m-1} d_k \leq d_0 \sum_{k=n}^{m-1} \lambda_0^k,$$

which implies that $\{x_n\}$ is a Cauchy sequence, and therefore converges to some $z \in X$. Using (C1), we obtain $d(Tx_n, Tz) \leq \psi(d(x_n, z))$ for all $n \in \mathbb{N}$. Hence, using Remark 2.2, we conclude that z is a fixed point of T. Now, assume that (4) holds. If there exist two distinct fixed points z_1 and z_2 of T, we deduce from (C1) that

$$d(z_1, z_2) = d(Tz_1, Tz_2) \leq \psi(d(z_1, z_2)) < d(z_1, z_2),$$

which is absurd, so $z_1 = z_2$ and the fixed point of T is unique. Finally, we take $x_0 = x$ for every $x \in X$ and deduce as above that $\{T^n x\}$ converges to the unique fixed point z. □

Example 2.5. *Consider $X = \mathbb{R}$ endowed with the Euclidean metric d. Clearly* 1 *is a fixed point of the function $T\colon X \to X$ defined by $Tx = x^2$. We show that Theorem 2.4 is applicable for this example. Observe that condition* (C1) *is satisfied, where $\psi(t) = t^2$ for all $t \in \mathbb{R}^+$, which is increasing and subhomogeneous. Moreover, the inequality* (3) *is fulfilled for $x_0 = \frac{1}{2}$. Hence, from Theorem 2.4 it follows that T has a fixed point. Clearly, the condition $\psi(t) < t$ for all $t > 0$ is not satisfied and the fixed point of T is not unique.*

The second main result of this section is the following.

Theorem 2.6. *Assume that there exists an increasing subhomogeneous function ψ such that* (C2) *holds. If there exists $x_0 \in X$ such that*

$$\psi(d_0) < d_0, \tag{6}$$

then the sequence $\{T^n x_0\}$ converges to a fixed point z of T. If for all $t \in P^*$, we have

$$\psi(t) \leq t, \tag{7}$$

then z is the unique fixed point of T, and the sequence $\{T^n x\}$ converges to z for all $x \in X$.

Proof. Let $x_0 \in X$ such that (6) is satisfied, so $d_0 \neq 0$, and x_0 is not a fixed point of T. Thus, $\lambda := d_0^{-1}\psi(d_0) \in (0,1)$, which is combined with (C2) gives

$$d_1 < \psi(d_0) = \lambda\, d_0.$$

By (C2), the monotony and the homogeneity of ψ, we deduce

$$d_2 < \psi(d_1) \leq \psi(\lambda\, d_0) \leq \lambda\, \psi(d_0) = \lambda^2\, d_0.$$

Hence, by induction, we get

$$d_n < \lambda^n d_0 \;\text{ for all } n \in \mathbb{N}. \tag{8}$$

We deduce that the sequence $\{d_n\} \to 0$ as $n \to \infty$. Now, by using the triangle inequality and (8), we obtain

$$d(x_n, x_m) \leq \sum_{k=n}^{m-1} d_k < d_0 \sum_{k=n}^{m-1} \lambda^k,$$

which implies that $\{x_n\}$ is a Cauchy sequence, and therefore converges to some $z \in X$. If there exists an integer $N > 0$ such that $x_n = z$ for all $n > N$, then clearly z is a fixed point of T. Otherwise, if there exists a subsequence $\{x_{n(k)}\}$ such that $x_{n(k)} \neq z$ for all $k \in \mathbb{N}$, then it follows from (C2) that

$$d(Tx_{n(k)}, Tz) < \psi(d(x_{n(k)}, z)).$$

Hence, as $k \to \infty$, we deduce from Remark 2.2 that z is a fixed point of T. Assume now that (7) holds. If there exist two distinct fixed points z_1 and z_2 of T, then from (C2) we deduce

$$d(z_1, z_2) = d(Tz_1, Tz_2) < \psi(d(z_1, z_2)) \leq d(z_1, z_2),$$

which is absurd, so necessarily $z_1 = z_2$ and therefore the fixed point is unique. Finally, we take $x_0 = x$ for every $x \in X$ and as above we deduce that $\{T^n x\}$ converges to z. □

3. Superhomogeneous control functions

In this section, we present new fixed point theorems for mappings satisfying the nonlinear contraction (C3) or (C4), where ψ belongs to the following classes of functions.

Definition 3.1. *A function $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ is said to be superhomogeneous if*

$$\psi(\lambda t) \geq \lambda \psi(t) \;\text{ for all }\; \lambda \in (0,1) \text{ and all } t \in \mathbb{R}^+.$$

A function $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ is said to be strictly superhomogeneous if

$$\psi(\lambda t) > \lambda \psi(t) \;\text{ for all }\; \lambda \in (0,1) \text{ and all } t \in \mathbb{R}^+.$$

Definition 3.2. *A function $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ is said to be regular at zero if*

$$\{t_n\} \subset \mathbb{R}^+ \text{ is a sequence such that } \psi(t_n) \to 0, \text{ then } t_n \to 0.$$

Example 3.3. *Consider the function $\psi(t) = t(1 + e^{-t})$ for all $t \in \mathbb{R}^+$, then ψ is strictly superhomogeneous, increasing, regular at zero and satisfies $t < \psi(t)$ for every $t \neq 0$. However, if $\psi(t) = t^{-1}$ for all $t > 0$ and $\psi(0) = 0$, then ψ is strictly superhomogeneous and satisfies $t < \psi(t)$ for all $t > 0$, but not regular at zero.*

Remark 3.4. *If ψ is subhomogeneous and bijective on $\mathbb{R}^+$, then ψ^{-1} is superhomogeneous. However, not all increasing subhomogeneous functions are bijective. Due to this observation, it is of interest to study the contractions* (C3) *and* (C4).

Theorem 3.5. *Assume that there exists an increasing strictly superhomogeneous function ψ such that* (C3) *holds and ψ is regular at zero. If there exists $x_0 \in X$ such that*

$$d_0 < \psi(d_0), \tag{9}$$

then the sequence $\{T^n x_0\}$ converges to a fixed point z of T. If for all $t \in P^*$, we have

$$t < \psi(t), \tag{10}$$

then z is the unique fixed point of T, and the sequence $\{T^n x\}$ converges to z for all $x \in X$.

Proof. Let $x_0 \in X$ such that (9) is satisfied. If x_0 is a fixed point of T then it is done. Assume that $x_0 \neq Tx_0$, then from (9), we obtain $\psi(d_0) \neq 0$ and $\lambda_0 := d_0\psi(d_0)^{-1} \in (0, 1)$, and deduce from (C3) and the homogeneity of ψ that

$$\psi(d_1) \leq d_0 = \lambda_0\, \psi(d_0) < \psi(\lambda_0\, d_0).$$

This implies by monotony of ψ that

$$d_1 \leq \lambda_0\, d_0 = \lambda_0^2\, \psi(d_0).$$

By induction, we obtain $d_n \leq \lambda_0^{n+1}\psi(d_0)$ for all $n \in \mathbb{N}$. Hence, we deduce the sequence $\{d_n\}$ converges to 0. Next, by using the triangle inequality, it is not difficult to see that

$$d(x_n, x_m) \leq \psi(d_0) \sum_{k=n}^{m-1} \lambda_0^{k+1}.$$

Hence, it follows that $\{x_n\}$ is a Cauchy sequence, and therefore converges to some $z \in X$. It follows by using (C3) that $\psi(d(Tx_n, Tz)) \leq d(x_n, z)$ for all $n \in \mathbb{N}$. As $n \to \infty$, we deduce by regularity of ψ at zero that z is a fixed point of T. Finally, assume that (10) holds. If there exist two fixed points of T, then from (C3) and (10), we deduce a contradiction, and as above we obtain that the sequence $\{T^n x\}$ converges to z for all $x \in X$. □

Example 3.6. *Consider $X = \mathbb{R}$ endowed with the Euclidean metric d and $T\colon \mathbb{R} \to \mathbb{R}$ be the function defined by $Tx = x^2$. Indeed,* (C3) *holds for all $x, y \in \mathbb{R}$, where $\psi(t) = \sqrt{t}$ is increasing, strictly subhomogeneous and regular at* 0*. Moreover,*

$$d(2, T2) = 2^{\frac{1}{2}} \cdot 5^{\frac{1}{4}} < \psi(d(2, T2)) = 2 \cdot 5^{\frac{1}{2}}.$$

Hence, it follows from Theorem 3.5 that T has at least one fixed point.

Theorem 3.7. *Assume that there exists an increasing superhomogeneous function ψ such that* (C4) *holds and ψ is regular at zero. If there exists $x_0 \in X$ such that*

$$d_0 < \psi(d_0), \tag{11}$$

then the sequence $\{T^n x_0\}$ converges to a fixed point z of T. In addition if

$$t \leq \psi(t) \ \text{ for all } t \in P^*, \tag{12}$$

then z is the unique fixed point of T, and the sequence $\{T^n x\}$ converges to z for all $x \in X$.

Proof. Let $x_0 \in X$ such that (11) is satisfied. If x_0 is a fixed point of T then it is done. We assume that x_0 is not a fixed point of T. We obtain from (11) that $\lambda_0 := d_0\psi(d_0)^{-1} \in (0, 1)$ and deduce from (C4) that

$$\psi(d_1) < d_0 = \lambda_0\, \psi(d_0).$$

Thus, by monotony and the homogeneity of ψ, it follows that $\psi(d_1) < \lambda_0\, \psi(d_0) \leq \psi(\lambda_0\, d_0)$, which implies by the monotony and the homogeneity of ψ that $d_1 \leq \lambda_0\, d_0 = \lambda^2\, \psi(d_0)$. By induction, we obtain $d_n \leq \lambda_0^{n+1}\psi(d_0)$

for all $n \in \mathbb{N}$. Hence, we deduce the sequence $\{d_n\}$ converges to 0. Next, by using the triangle inequality, it is not difficult to see that

$$d(x_n, x_m) \le \psi(d_0) \sum_{k=n}^{m-1} \lambda_0^{k+1}.$$

Hence, it follows that $\{x_n\}$ is a Cauchy sequence, and so converges to some $z \in X$. If there exists an integer $N > 0$ such that $x_n = z$ for all $n > N$, then clearly z is a fixed point of T. Otherwise, if there exists a subsequence $\{x_{n(k)}\}$ such that $x_{n(k)} \ne z$ for all $k \in \mathbb{N}$, then it follows from (C4) that

$$\psi(d(Tx_{n(k)}, Tz)) < d(x_{n(k)}, z).$$

We deduce from the regularity of ψ at zero that z is a fixed point of T. The rest of the proof is similar to that of Theorem 3.5. □

4. Control functions of Boyd-Wong type

In this section, we extend the Theorem 1.1 by relaxing the conditions on the control function.

Theorem 4.1. *Assume that there exists a function* $\psi\colon \overline{P} \to \mathbb{R}^+$ *such that* (C1) *holds. If there exists* $x_0 \in X$ *such that*

$$\limsup_{s\to t^+} \psi(s) < t \quad \text{for all } t \in \overline{P} \cap (0, d_0], \tag{13}$$

$$\sup_{s\le t} \psi(s) < t \quad \text{for all } t \in (0, d_0], \tag{14}$$

then the sequence $\{T^n x_0\}$ *converges to a fixed point of* T.

Proof. Assume that x_0 is not a fixed point of T, otherwise nothing to prove. Now, from (C1) and (14), we deduce that the sequence $\{d_n\}$ is decreasing and hence converges to some $d_* \ge 0$. Assume that $d_* \ne 0$, then by (C1) and (14), we get $d_{n+1} \le \psi(d_n) < d_n \le d_0$ for all $n \in \mathbb{N}$. Hence, from (13) we obtain

$$d_* \le \limsup_{n\to\infty} \psi(d_n) < d_*,$$

which is a contradiction, so we have $d_* = 0$. Hence, for $\varepsilon \in (0, d_0]$, there is $p \in \mathbb{N}$ such that $d_p \le \varepsilon - \sup_{s\le\varepsilon} \psi(s)$. Define the set $B(x_p, \varepsilon) \coloneqq \{x \in X : d(x, x_p) \le \varepsilon\}$, which is nonempty since $x_{p+1} \in B(x_p, \varepsilon)$. Let $z \in B(x_p, \varepsilon)$, then it follows from (C1) that

$$\begin{aligned} d(Tz, x_p) &\le d(Tz, x_{p+1}) + d(x_{p+1}, x_p) \\ &\le \psi(d(z, x_p)) + d_p \\ &\le \sup_{s\le d(z,x_p)} \psi(s) + \varepsilon - \sup_{s\le\varepsilon} \psi(s) \\ &\le \sup_{s\le\varepsilon} \psi(s) + \varepsilon - \sup_{s\le\varepsilon} \psi(s) = \varepsilon. \end{aligned}$$

Thus, T maps $B(x_p, \varepsilon)$ into itself, which implies that

$$d(x_n, x_m) \le d(x_n, x_p) + d(x_p, x_m) \le 2\,\varepsilon \quad \text{for } n, m \ge p.$$

Hence, the sequence $\{x_n\}$ is Cauchy, so converges to some $z \in X$.

Now, since $\{x_n\}$ converges to z, then there is an integer $N > 0$ such that $d(x_n, z) \le d_0$ for all $n > N$. Using the triangle inequality, we have

$$\begin{aligned} d(z, Tz) &\le d(z, x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, Tz) \\ &\le d(z, x_n) + d_n + \psi(d(x_n, z)) \\ &\le d(z, x_n) + d_n + \sup_{s\le d(x_n,z)} \psi(s) \\ &< 2d(z, x_n) + d_n. \end{aligned}$$

for all $n \ge N$. As $n \to \infty$, we obtain $z = Tz$. □

Theorem 4.2. *Assume that there exists a function $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ regular at zero such that T satisfies* (C3). *If*

$$\liminf_{s\to t^+} \psi(s) > t \quad \text{for all } t \in \overline{P}^*, \tag{15}$$

$$\inf_{s\ge t} \psi(s) > t \quad \text{for all } t > 0, \tag{16}$$

then T has a unique fixed point z, and the sequence $\{T^n x\}$ converges to z for all $x \in X$.

Proof. Let $x_0 \in X$ such that $d_0 > 0$, otherwise if $d_0 = 0$, then x_0 is a fixed point of T and it is done. From (C3) and (16), we deduce that the sequence $\{d_n\}$ is decreasing and hence converges to some $d_* \ge 0$. Assume that $d_* \neq 0$, then by (C3) and (16), we get $d_n < \inf_{s\ge d_n} \psi(s) \le \psi(d_n) < d_{n-1}$ for all $n \ge 1$. Hence, from (15) we obtain

$$d_* < \liminf_{n\to\infty} \psi(d_n) \le d_*,$$

which is a contradiction, unless $d_* = 0$. Hence, for $\varepsilon > 0$ there is $p \in \mathbb{N}$ such that

$$d_p < \min\{\inf_{s\ge\varepsilon} \psi(s) - \varepsilon, \varepsilon\}.$$

Define the set $B(x_{p+1}, \varepsilon) \coloneqq \{x \in X : d(x, x_{p+1}) \le \varepsilon\}$, which contains x_p. Let $z \in B(x_{p+1}, \varepsilon)$, then it follows from (C3) that

$$\begin{aligned}\inf\nolimits_{s\ge d(Tz,x_{p+1})} \psi(s) &\le \psi(d(Tz, x_{p+1})) \le d(z, x_p)\\ &\le d(z, x_{p+1}) + d(x_{p+1}, x_p) \le \varepsilon + d_p\\ &< \varepsilon + \inf_{s\ge\varepsilon} \psi(s) - \varepsilon = \inf_{s\ge\varepsilon} \psi(s)\end{aligned}$$

Using the monotonicity of the function $t \mapsto \inf_{s\ge t} \psi(s)$, we deduce that $d(Tz, x_{p+1}) \le \varepsilon$, which implies that T maps $B(x_{p+1}, \varepsilon)$ into itself and this means that

$$d(x_n, x_m) \le d(x_n, x_{p+1}) + d(x_{p+1}, x_m) \le 2\,\varepsilon \quad \text{for } n, m \ge p.$$

Hence, the sequence $\{x_n\}$ is Cauchy, and thus converges to some $z \in X$. Now, from (C3), we deduce

$$\psi(d(x_{n+1}, Tz)) \le d(x_n, z),$$

for all $n > N$. Hence, as $n \to \infty$ we deduce from the regularity of ψ at zero that $\{x_n\}$ converges simultaneously to z and Tz, which is possible only if $z = Tz$. If there exist two fixed points z_1 and z_2 of T, then from (C3) and (16), we infer a contradiction. Then, we may take $x_0 = x$ for every $x \in X$ and deduce as above that $\{T^n x\}$ converges to the unique fixed point z. □

Remark 4.3. *Theorem 4.2 extends [5, Theorem 2].*

5. Control functions of Matkowski type

In this section, we present three fixed point theorems of Matkowski type.

Theorem 5.1. *Assume that there exists an increasing function $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that* (C1) *holds. If there exists $x_0 \in X$ such that*

$$\lim_{n\to\infty} \psi^n(d_0) = 0, \tag{17}$$

then the sequence $\{T^n x_0\}$ converges to a fixed point of T.

Proof. There is nothing to prove if x_0 is a fixed point. We assume therefore that x_0 is not a fixed point of T. Observe that from (C1), we deduce that $d_n \le \psi^n(d_0)$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} d_n = 0$. Thus,

$$\psi(s) < s \text{ for all } s \in (0, d_0], \tag{18}$$

otherwise we obtain $0 < s \le \psi^n(s) \le \psi^n(d_0)$ which contradict (17) as n tends to infinity. Hence, for every $\varepsilon \in (0, d_0]$ there is $p \in \mathbb{N}$ such that $d_p \le \varepsilon - \psi(\varepsilon)$. Define the set $B(x_p, \varepsilon) := \{x \in X : d(x, x_p) \le \varepsilon\}$, which is nonempty since it contains x_{p+1}. Let $z \in B(x_p, \varepsilon)$, then using (C1) and the monotony of ψ it follows that

$$\begin{aligned} d(Tz, x_p) &\le d(Tz, x_{p+1}) + d(x_{p+1}, x_p) \\ &\le \psi(d(z, x_p)) + d_p \\ &\le \psi(\varepsilon) + \varepsilon - \psi(\varepsilon) = \varepsilon. \end{aligned}$$

Thus, T maps $B(x_p, \varepsilon)$ into itself and this means that

$$d(x_n, x_m) \le d(x_n, x_p) + d(x_p, x_m) \le 2\,\varepsilon \text{ for } n, m \ge p.$$

Hence $\{x_n\}$ is Cauchy sequence and so converges to some $z \in X$. If there exists an integer $N > 0$ such that $x_n = z$ for $n > N$, then $z = Tz$. Otherwise, assume there exists a subsequence $\{x_{n(k)}\}$ such that $x_{n(k)} \ne z$ for all integer $k > 0$. It follows from the convergence of $\{x_n\}$ to z that there is $N \in \mathbb{N}$ such that $d(x_{n(k)}, z) \le d_0$ for all $k \ge N$. Hence, we obtain from (C1) and (18) that

$$d(x_{n(k)+1}, Tz) \le \psi(d(x_{n(k)+1}, Tz)) < d(x_{n(k)}, z) \le d_0.$$

Thus, as $k \to \infty$, the subsequence $\{x_{n(k)+1}\}$ converges to Tz, where the subsequence $\{x_{n(k)}\}$ converges to z. So, we deduce that $z = Tz$, since $\{x_n\}$ converges to z. □

Theorem 5.2. *Assume there exists an increasing function* $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ *such that* (C3) *holds. If*

$$\lim_{n\to\infty} \psi^n(t) = \infty \text{ for all } t > 0, \tag{19}$$

then T *has a unique fixed point* z*, and the sequence* $\{T^n x_0\}$ *converges to* z *for all* $x_0 \in X$*.*

Proof. First, observe that $t < \psi(t)$ for all $t > 0$. Indeed, if there exists $s > 0$ such that $\psi(s) \le s$ it follows that $\psi^n(s) \le s$ for $n \in \mathbb{N}$, which contradict (19). Using the previous observation and (C3), we deduce that

$$d_{n+1} < \psi(d_{n+1}) \le d_n, \text{ for all } n \in \mathbb{N}.$$

Thus the sequences $\{d_n\}$ is decreasing, so converges to $d_* \ge 0$, say. Assume that $d_* > 0$. Now, it is not difficult to see that $\psi^n(d_n) \le d_0$ for all $n \in \mathbb{N}$. Using the monotonicity of ψ and $d_* \le d_n$ for all $n \in \mathbb{N}$, we obtain $\psi^n(d_*) \le \psi^n(d_n) \le d_0$, which contradict (19). We conclude that $d_* = 0$. Hence, for $\varepsilon > 0$, there is $p > 0$ such that

$$d_p < \psi(\varepsilon) - \varepsilon.$$

Define the set $B(x_{p+1}, \varepsilon) := \{x \in X : d(x, x_{p+1}) \le \varepsilon\}$ that contains x_p. Let $z \in B(x_{p+1}, \varepsilon)$, so it follows from (C3) that

$$\psi(d(Tz, x_{p+1})) \le d(z, x_p) \le d(z, x_{p+1}) + d_p < \varepsilon + \psi(\varepsilon) - \varepsilon = \psi(\varepsilon).$$

Hence, by monotonicity of ψ, we deduce that $d(Tz, x_{p+1}) \le \varepsilon$. Therefore, T maps $B(x_{p+1}, \varepsilon)$ into itself and this means that

$$d(x_n, x_m) \le d(x_n, x_{p+1}) + d(x_{p+1}, x_m) \le 2\,\varepsilon \text{ for } n, m \ge p.$$

Hence $\{x_n\}$ is Cauchy sequence and so converges to some $z \in X$. If there exists an integer $N > 0$ such that $x_{n+1} = Tz$ for all $n > N$, then z and Tz become limit of the same sequence, which implies that $z = Tz$. Otherwise, there exists a subsequence $\{x_{n(k)}\}$ such that $x_{n(k)} \ne Tz$ for all k from certain order, and we have

$$d(x_{n(k)}, Tz) < \psi(d(x_{n(k)}, Tz)) \le d(x_{n(k)-1}, z).$$

Thus, as $k \to \infty$, $\{x_{n(k)}\}$ converges to Tz and $\{x_{n(k)-1}\}$ converges to z, and we deduce that $z = Tz$, since the sequence $\{x_n\}$ converges to z. The uniqueness of the fixed point follows immediately from (C3) and the first observation. □

Theorem 5.3. *Assume there exists an increasing function* $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ *such that* (C3) *holds. If there exists* $x_0 \in X$ *such that*

$$\lim_{n\to\infty} \psi^n(t) = d_0 \quad \textit{for all } t \in (0, d_0), \tag{20}$$

then the sequence $\{T^n x_0\}$ *converges to a fixed point of* T.

Proof. There is nothing to prove if x_0 is a fixed point. We assume that x_0 is not a fixed point of T. Observe that $t < \psi(t)$ for all $t \in (0, d_0)$. Indeed, if there exists $t \in (0, d_0)$ such that $\psi(t) \le t$, then by monotonicity of ψ, we deduce that $\psi^n(t) \le t < d_0$. So by taking n to infinity, we obtain from (20) a contradiction. The proof of convergence of the sequence $\{x_n\}$ to some $z \in X$ is exactly the same as that furnished in the proof of Theorem 5.2. If there exists an integer $N > 0$ such that $x_{n+1} = Tz$ for all $n > N$, then z and Tz become limit of the same sequence, which implies that $z = Tz$. Otherwise, there exists a subsequence $\{x_{n(k)}\}$ such that $x_{n(k)} \neq Tz$ for all k from certain order N, we have $d(x_n, z) < d_0$ for all $n > N$ and so we obtain

$$d(x_{n(k)}, Tz) < \psi(d(x_{n(k)}, Tz)) \le d(x_{n(k)-1}, z) < d_0.$$

Thus, as $k \to \infty$, the subsequence $\{x_{n(k)}\}$ converges to Tz and the subsequence $\{x_{n(k)-1}\}$ converges to z. So, we deduce that $z = Tz$, since $\{x_n\}$ converges to z. □

6. Application to integral equations of Fredholm and Volterra type

In this section, we discuss the existence of solutions to certain integral equations of Fredholm and Volterra type, and furnish new sufficient conditions ensuring the existence of such solutions. These equations have been studied by many authors using various fixed point theorems, see for example [9, 11, 10, 7, 3]. Firstly, we consider the integral equation of Fredholm type:

$$x(t) = f(t) + \int_a^b k(t, s, x(s))\,\mathrm{d}\,s, \quad t \in [a, b]. \tag{21}$$

Based on Theorem 2.4, we obtain the following result.

Theorem 6.1. *Consider Eq.* (21). *Suppose that the following properties hold:*

(*i*) $k\colon [a, b] \times [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$ *and* $f\colon [a, b] \to \mathbb{R}^n$ *are continuous;*

(*ii*) *there exist a continuous function* $p\colon [a, b] \times [a, b] \to \mathbb{R}^+$ *and an increasing subhomogeneous function* ψ *such that*

$$|k(t, s, u) - k(t, s, v)| \le p(t, s)\psi(|u - v|),$$

for all $(t, s, u, v) \in [a, b] \times [a, b] \times \mathbb{R}^n \times \mathbb{R}^n$;

(*iii*) $\sup_{t\in[a,b]} \int_a^b p(t, s) ds \le 1$;

(*iv*) *there exists* $x_0 \in C([a, b], \mathbb{R}^n)$ *such that* $\psi(d_0) < d_0$, *where*

$$d_0 := \sup_{t\in[a,b]} \left| x_0(t) - f(t) - \int_a^b k(t, s, x_0(s))\,\mathrm{d}\,s \right|.$$

Then the equation (21) *has a solution in* $C([a, b], \mathbb{R}^n)$.

Proof. Consider $X := C([a,b],\mathbb{R}^n)$ endowed with the supremum norm; that is,

$$\|x\| := \sup_{t\in[a,b]}\{|x(t)|\} \text{ for all } x \in X.$$

Noting that (X,d) is a complete metric space, where $d(x,y)=\|x-y\|$. Define $T : X \to X$ by

$$(Tx)(t) := f(t) + \int_a^b k(t,s,x(s))\mathrm{d}\, s.$$

It follows from (ii) and (iii) that

$$\begin{aligned} |(Tx)(t)-(Ty)(t)| &\le \int_a^b |k(t,s,x(s))-k(t,s,y(s))|\mathrm{d}s \\ &\le \int_a^b p(t,s)\psi(|x(s)-y(s)|)\mathrm{d}s \\ &\le \psi(\|x-y\|)\int_a^b p(t,s)\mathrm{d}s \\ &\le \psi(\|x-y\|), \end{aligned}$$

for all $t\in[a,b]$. Hence, $\|Tx-Ty\|\le\psi(\|x-y\|)$ for all $x,y\in X$. Moreover, (3) follows from (iv). Therefore, we conclude by Theorem 2.4 that Eq. (21) has a solution in $C([a,b],\mathbb{R}^n)$. □

Remark 6.2. *Theorems 6.1 extends [9, Theorem 4.1].*

We furnish an example to highlight the utility of Theorem 6.1.

Example 6.3. *Consider the following integral equation:*

$$x(t)+t = t^2\int_0^1 s^2\, x(s)\,\mathrm{d}s, \quad t\in[0,1]. \tag{22}$$

This equation follows from (21) *by taking* $a=0$, $b=1$, $f(t)=-t$ *and* $k(t,s,u)=t^2\,s^2\,u$ *for all* $t,s\in[0,1]$ *and all* $u\in\mathbb{R}^n$. *Clearly,* (i) *holds. Define* $p\colon [0,1]\times[0,1]\to\mathbb{R}^+$ *by* $p(t,s)=3\,t^2s^2$ *and* $\psi\colon\mathbb{R}^+\to\mathbb{R}^+$ *by* $\psi(t)=\frac{1}{3}t(1+t^2)$, *which is an increasing subhomogeneous function. Moreover,* (ii) *holds, since for all* $t,s\in[0,1]$ *and all* $u,v\in\mathbb{R}^n$ *we have*

$$|k(t,s,u)-k(t,s,v)|\le t^2s^2|u-v|\le p(t,s)\psi(|u-v|).$$

Further, (iii) *holds. It is not difficult to see that* (iv) *is satisfied for* $x_0\equiv 0$. *We conclude by Theorem 6.1 that* (22) *has a solution in* $C([0,1],\mathbb{R}^n)$.

Now, consider the integral equation of Volterra type:

$$x(t) = f(t) + \int_a^t k(t,s,x(s))\mathrm{d}s, \quad t\in[a,b]. \tag{23}$$

Based on Theorem 5.1, we obtain the following result.

Theorem 6.4. *Consider Eq.* (23). *Suppose that the following properties hold:*

(i) $k\colon [a,b]\times[a,b]\times\mathbb{R}^n\to\mathbb{R}^n$ *and* $f\colon[a,b]\to\mathbb{R}^n$ *are continuous;*

(ii) *there exist* $\tau\ge 1$ *and an increasing function* $\psi\colon\mathbb{R}^+\to\mathbb{R}^+$ *such that*

$$|\,k(t,s,u)-k(t,s,v)|\le\psi(|u-v|e^{-\tau(b-a)}),$$

for all $(t,s,u,v)\in[a,b]\times[a,b]\times\mathbb{R}^n\times\mathbb{R}^n$;

(*iii*) *there exists* $x_0 \in C([a,b],\mathbb{R}^n)$ *such that*

$$\lim_{n\to\infty} \psi^n(d_0) = 0, \tag{24}$$

where $d_0 := \sup_{t\in[a,b]} \left\{ \left| x_0(t) - f(t) - \int_a^b k(t,s,x_0(s))\mathrm{d}s \right| e^{-\tau(t-a)} \right\}$.

Then the equation (23) *has a solution in* $C([a,b],\mathbb{R}^n)$.

Proof. Consider $X := C([a,b],\mathbb{R}^n)$ endowed with the Bielecki-type norm; that is,

$$\|x\|_B := \sup_{t\in[a,b]} \{|x(t)|e^{-\tau(t-a)}\} \quad \text{for all } x \in X.$$

Note that (X,d) is a complete metric space, where $d(x,y) = \|x-y\|_B$. Define $T : X \to X$ by

$$(Tx)(t) := f(t) + \int_a^t k(t,s,x(s))\mathrm{d}s, \quad t \in [a,b].$$

From (*ii*) it follows that

$$\begin{aligned}
|(Tx)(t) - (Ty)(t)| &\le \int_a^t |k(t,s,x(s)) - k(t,s,y(s))|\mathrm{d}s \\
&\le \int_a^t \psi(|x(s)-y(s)|e^{-\tau(b-a)})\mathrm{d}s \\
&\le \int_a^t e^{\tau(s-a)}\psi(|x(s)-y(s)|e^{-\tau(s-a)})\mathrm{d}s \\
&\le \psi(\|x-y\|_B)\int_a^t e^{\tau(b-a)}\mathrm{d}s \\
&\le \tau^{-1}e^{\tau(t-a)}\psi(\|x-y\|_B) \\
&\le e^{\tau(t-a)}\psi(\|x-y\|_B),
\end{aligned}$$

for all $t \in [a,b]$. Hence, $\|Tx-Ty\|_B \le \psi(\|x-y\|_B)$ for all $x,y \in X$, that is, (C1) is satisfied. Moreover, (17) follows from (*iii*). Therefore, we conclude by Theorem 5.1 that equation (23) has a solution in $C([a,b],\mathbb{R}^n)$. □

Remark 6.5. *Theorem 6.4 extends [9, Theorem 4.2] and [11, Theorem 5.2].*

Example 6.6. *Consider the following integral equation:*

$$2\,x(t) = e^t + \int_0^t e^{-s}x(s)\,\mathrm{d}s, \quad t \in [0,1]. \tag{25}$$

This equation follows from (23) *by taking* $a = 0$, $b = 1$, $f(t) = \frac{1}{2}e^t$ *and* $k(t,s,u) = \frac{1}{2}e^{-s}\,u$ *for all* $t,s \in [0,1]$ *and all* $u \in \mathbb{R}^n$. *Define* $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ *by* $\psi(t) = \frac{1}{2}t(1+t^2)$, *which is clearly an increasing function. Moreover, for* $\tau = 1$, (*ii*) *holds, since for all* $t,s \in [0,1]$ *and all* $u,v \in \mathbb{R}^n$ *we have*

$$\begin{aligned}
|k(t,s,u) - k(t,s,v)| \le \tfrac{1}{2}e^{-s}|u-v|\,\mathrm{d}s &\le \tfrac{1}{2}e^{-s}|u-v|(1+|u-v|^2e^{-2s}) \\
&= \psi(|u-v|e^{-s}).
\end{aligned}$$

Now, we shall show that for $x_0 \equiv 0$, (24) *is satisfied. For* $x_0 \equiv 0$, *we have* $d_0 = \frac{1}{2}$ *and* $\psi(\frac{1}{2}) = \frac{5}{16} < \frac{1}{2}$, *thus by the monotonicity of* ψ *it follows that the sequence* $\{\psi^n(\frac{1}{2})\}$ *is decreasing and then converges to* c, *say. Clearly* $c < \frac{1}{2}$. *Assume that* $c > 0$. *Using the continuity of* ψ, *it follows that* $c \le \frac{1}{2}c(1+c^2)$, *which implies a contradiction, thus* $c = 0$ *and* (*iii*) *holds for* $x_0 \equiv 0$. *We conclude by Theorem 6.4 that equation* (25) *has a solution in* $C([0,1],\mathbb{R}^n)$.

References

[1] R.P. Agarwal, E. Karapınar, D. O'Regan, and A.F. Roldán-López-de Hierro, Fixed point theory in metric type spaces, Springer, 2015.

[2] S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, Fund. Math. 3 (1922) 133–181.

[3] M. Berzig, First results in suprametric spaces with applications, Mediterr. J. Math. 19 (2022) 1–18.

[4] D.W. Boyd and J.S.W. Wong, On nonlinear contractions, Proc. Amer. Math. Soc. 20 (1969) 458–464.

[5] K.-J. Chung, Remarks on nonlinear contractions, Pacific J. Math. 101 (1982) 41–48.

[6] J. Jachymski, Equivalent conditions for generalized contractions on (ordered) metric spaces, Nonlinear Anal. 74 (2011) 768–774.

[7] I. Kedim and M. Berzig, Fixed point theorems for Maia α-ψ contractive type mappings with applications, J. Math. Anal. Appl. 504 (2021) 125381.

[8] J. Matkowski, Integrable solutions of functional equations, Discuss. Math. 127, 1975.

[9] D. O'Regan and A. Petruşel, Fixed point theorems for generalized contractions in ordered metric spaces, J. Math. Anal. Appl. 341 (2008) 1241–1252.

[10] H. Rahimi, P. Vetro and G. Soleimani Rad, Coupled fixed-point results for T-contractions on cone metric spaces with applications, Math. Notes, 98 (2015) 158–167.

[11] M. Sgroi and C. Vetro, Multi-valued F-contractions and the solution of certain functional and integral equations, Filomat, 27 (2013) 1259–1268.