Journal of Universal Mathematics
Vol.5 No.2 pp.122-128 (2022)
ISSN-2618-5660
DOI: 10.33773/jum.1134765

A NOTE ON KENMOTSU MANIFOLDS ADMITTING GENERALIZED TANAKA-WEBSTER CONNECTION

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Abstract. In the present paper, we investigate some tensor conditions of Kenmotsu manifolds with the generalized Tanaka-Webster connection. Using the Q tensor whose trace is the well-known Z-tensor, we prove the conditions $\xi - Q^*$ flat, $\phi - Q^*$ flat Kenmotsu manifold with respect to the generalized Tanaka-Webster connection.

1. INTRODUCTION

In 2013, Mantica et al. defined the $Q-$tensor notation. This tensor is the name given whose trace the Z tensor defined by these authors [1, 2]. Along with this new tensor, they defined pseudo $Q-$symmetric Riemannian manifolds, named $(PQS)_n$, which are a new type of manifolds that include both pseudo-symmetric manifolds $(PS)_n$ and pseudo-concircular symmetric manifolds $(PCS)_n$. They studied various properties such manifolds and obtained important results [1]. Recently, Yılmaz and Yıldırım investigated some curvature condition Sasakian manifolds and Kenmotsu manifolds with Q tensor, respectively [3, 4] .

A generalized $(0, 2)$ symmetric Z tensor defined as [1]

$$Z(X,Y) = S(X,Y) + \lambda g(X,Y), \tag{1.1}$$

where X, Y are vector fields, g Riemannian metric on M^{2n+1}, S is the Ricci tensor of (M^{2n+1}, g) and λ is an arbitrary scalar function.

The $(1, 3)$ Q tensor whose trace is the Z tensor expressed as

$$Q(X,Y)Z = R(X,Y)Z - \frac{\lambda}{2n}\left\{g(Y,Z)X - g(X,Z)Y\right\}, \tag{1.2}$$

where R denotes the curvature tensor of (M^{2n+1}, g) and λ is an arbitrary scalar function.

Kenmotsu manifolds [5] have proved to be an important area for contact geometry with many valuable studies carried out in the last 30 years. Another reason to

Date: **Received:** 2022-06-23; **Accepted:** 2022-07-29 .
2000 *Mathematics Subject Classification.* 53C15;53C40.
Key words and phrases. Kenmotsu manifolds, Generalized Tanaka-Webster connection, Q tensor.

study these manifolds is that these are in some sense complementary to Sasakian manifolds. Indeed, while some features of Kenmotsu manifolds and Sasakian manifolds are similar to each other, they are generally different in terms of structure [6]. Kenmotsu manifolds studied by many authors [7, 8, 9, 10, 11, 12].

On the other hand, the Tanaka-Webster connection [13, 14] is the canonical of fine connection defined on a non-degenerate pseudo-Hermitian CR-manifold. Tanno [15] defined the generalized Tanaka-Webster connection for contact metric manifolds by the canonical connection which coincides with the Tanaka-Webster connection if the associated CR-structure is integrable.

In the present paper, we study some curvature conditions of Kenmotsu manifolds with respect to generalized Tanaka-Webster connection. In section 2, we give some basic properties of Kenmotsu manifolds and give some conditions with respect to the generalized Tanaka-Webster connection. Then in Main results, using the Q tensor whose trace is the well-known Z-tensor, we prove the conditions $\xi - Q^*$ flat, $\phi - Q*$ flat Kenmotsu manifold with respect to the generalized Tanaka-Webster connection and obtain some important results.

2. Preliminaries

In this section, we recall some general definitions and basic formulas for late use. Let M be a Riemannian manifold of dimension $(2n+1)$. If there are the tensor field $\phi \in T_1^1(M^{2n+1})$, $\xi \in \chi(M^{2n+1})$ and a $1-$form η satisfying the conditions

$$\phi^2 = -I + \eta \otimes \xi,\ \eta(\xi) = 1, \tag{2.1}$$

then M^{2n+1} admits the almost contact structure $(\phi, \xi, , \eta)$, where ξ is the structure vector field of the almost contact manifold M^{2n+1}. If (ϕ, ξ, η) is an almost contact structure on the M^{2n+1} then: [16]

$$\phi\xi = 0,\ \eta \circ \phi = 0. \tag{2.2}$$

Also, given the M^{2n+1} manifold equipped with the almost contact structure (ϕ, ξ, η), then if there exist a Riemannian metric g on M^{2n+1} which satisfies the condition

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),\ g(X, \xi) = \eta(X), \tag{2.3}$$

then the manifold (M^{2n+1}, g) with the structure (ϕ, ξ, η) is called almost contact metric manifold, for any $X, Y \in \chi(M^{2n+1})$. Also, we note that g is a metric compatible with the almost contact structure on M^{2n+1}.

In addition these equations, if there exist

$$(\nabla_X \phi)(Y) = -g(X, \phi Y)\xi - \eta(Y)\phi X, \tag{2.4}$$

$$\nabla_X \xi = X - \eta(X)\xi, \tag{2.5}$$

where ∇ denotes the operator of covariant differentiation with respect to the Riemannian metric g, then (M^{2n+1}, g) is called a Kenmotsu manifold [5]. Moreover, an almost contact metric manifold is an almost Kenmotsu manifold if the following condition are satisfied

$$d\eta = 0,\ d\Phi = 2\eta \wedge \Phi$$

where Φ is fundamental form and defined as $\Phi(X,Y) = g(\phi X, Y)$. If the Nijenhuis tensor of ϕ vanishes, then almost Kenmotsu manifold is said to be Kenmotsu manifold. Some conditions provided in a Kenmotsu manifold are as follows: [5]:

$$(\nabla_X \eta) Y = g(X,Y) - \eta(X)\eta(Y), \tag{2.6}$$

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.7}$$

$$R(\xi,X)Y = \eta(Y)X - g(X,Y)\xi, \tag{2.8}$$

$$S(\phi X, \phi Y) = S(X,Y) + 2ng\eta(X)\eta(Y), \tag{2.9}$$

$$S(X,\xi) = -2n\eta(X), \tag{2.10}$$

$$\eta(R(X,Y)Z) = g(X,Z)\eta(Y) - g(Y,Z)\eta(X), \tag{2.11}$$

for any vector fields X, Y, Z where R is the Riemannian curvature tensor and S is the Ricci tensor.

If Ricci tensor S satisfies condition

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \tag{2.12}$$

then manifold is said to be $\eta-$Einstein manifold, where a, b certain scalars. If $b = 0$, then the manifold is an Einstein manifold.

Througout this paper we associate $*$ with the quantities with respect to generalized Tanaka-Webster connection. The generalized Tanaka-Webster connection ∇^* associated to the Levi- Civita connection ∇ is given by [18, 19]

$$\nabla^*_X Y = \nabla_X Y - \eta(Y)\nabla_X \xi + (\nabla_X \eta)(Y)\xi - \eta(X)\phi Y \tag{2.13}$$

for any vector fields X, Y on M^{2n+1}.

Using (2.5) and (2.6) the generalized Tanaka-Webster connection ∇^* for a Kenmotsu manifold is given by

$$\nabla^*_X Y = \nabla_X Y + g(X,Y)\xi - \eta(Y)X - \eta(X)\phi Y.$$

In a Kenmotsu manifold which admits the generalized Tanaka-Webster connection the following relations hold [17]:

$$\nabla^*_X \xi = 0, \tag{2.14}$$

$$R^*(X,Y)Z = R(X,Y)Z + g(Y,Z)X - g(X,Z)Y, \tag{2.15}$$

$$R^*(X,Y)\xi = R^*(X,\xi)Z = R^*(\xi,Y)Z = 0, \tag{2.16}$$

$$S^*(X,Y) = S(X,Y) + 2ng(X,Y), \tag{2.17}$$

$$S^*(\phi X, \phi Y) = S(\phi X, \phi Y) + 2ng(\phi X, \phi Y), \tag{2.18}$$

$$L^* X = LX + 2nX, \tag{2.19}$$

$$r^* = r + 2n(2n+1), \tag{2.20}$$

for any $X, Y, Z \in \chi(M^{2n+1})$, where R^* is the Riemannian curvature tensor, S^* is the Ricci tensor, L^* is the Ricci operator and r^* is scalar curvature with respect to the generalized Tanaka-Webster connection.

3. Main Results

Definition 3.1. The Q tensor with respect to the generalized Tanaka-Webster connection ∇^* defined by

$$Q^*(X,Y)Z = R^*(X,Y)Z - \frac{\lambda}{2n}\{g(Y,Z)X - g(X,Z)Y\} \tag{3.1}$$

for all vector fields X, Y, Z on M.

Proposition 1. Let ∇^* be Tanaka-Webster connection on M^{2n+1}. The Q^* tensor of ∇^* satisfies the following first Bianchi identity:

$$Q^*(X,Y)Z + Q^*(Y,Z)X + Q^*(Z,X)Y = 0.$$

Proof. Using equation (3.1), the result is clear. □

Proposition 2. The Q^* tensor in a Kenmotsu manifold (M^{2n+1}, g) which admits the generalized Tanaka-Webster connection satisfies the relations:

$$Q^*(X,Y)\xi = \frac{-\lambda}{2n}\{\eta(Y)X - \eta(X)Y\}, \tag{3.2}$$

$$Q^*(\xi,X)Y = \frac{-\lambda}{2n}\{g(X,Y)\xi - \eta(Y)X\}, \tag{3.3}$$

$$Q^*(X,\xi)Y = \frac{-\lambda}{2n}\{\eta(Y)X - g(X,Y)\xi\}, \tag{3.4}$$

for all vector fields X, Y, Z on M.

Proof. By using (2.3) and (2.16) in equation (3.1) the results is clear. □

Definition 3.2. A Kenmotsu manifold with respect to the generalized Tanaka-Webster connection ∇^* is said to be $\xi - Q^*$ flat if $Q^*(X,Y)\xi = 0$.

Theorem 3.3. *Let M be a Kenmotsu manifold with generalized Tanaka-Webster connection. In M, the following two conditions are equivalent:*

i) M is $\xi - Q^*$ flat,
ii) $\lambda = 0$.

Proof. i⇒ii**:**

Now, we assume that the manifold M with respect to the generalized Tanaka-Webster connection is $\xi - Q^*$ flat, that is, $Q^*(X,Y)\xi = 0$. Then using equation (3.2), it follows that

$$Q^*(X,Y)\xi = \frac{-\lambda}{2n}\{\eta(Y)X - \eta(X)Y\}. \tag{3.5}$$

Since $Q^*(X,Y)\xi = 0$, we obtain $\lambda = 0$.

ii⇒i**:**

Let $\lambda = 0$. From equation (3.2), it follows that $Q^*(X,Y)\xi = 0$, so the manifold is $\xi - Q^*$ flat manifold. □

Theorem 3.4. *For a $\xi - Q^*$ flat Kenmotsu manifold with respect to the generalized Tanaka-Webster connection, the manifold is a special type of $\eta-$Einstein manifold.*

Proof. Let M be a $\xi - Q^*$ flat Kenmotsu manifold. From equation (3.2), we have

$$\frac{-\lambda}{2n}\left\{\eta(Y)X - \eta(X)Y\right\} = 0.$$

Taking $Y = \xi$ in above equation, we obtain

$$\frac{\lambda}{2n}\left\{X - \eta(X)\xi\right\} = 0. \tag{3.6}$$

Taking inner product of the equation (3.6) with U, we obtain

$$\frac{\lambda}{2n}\left\{g(X,U) - \eta(X)\eta(U)\right\} = 0. \tag{3.7}$$

L being Ricci operator, replacing X by LX, we obtain

$$\frac{\lambda}{2n}\left\{S(X,U) - S(X,\xi)\eta(U)\right\} = 0.$$

By using (2.10), we get

$$\frac{\lambda}{2n}\left\{S(X,U) + 2n\eta(X)\eta(U)\right\} = 0$$

yields to

$$S(X,U) = -2n\eta(X)\eta(U).$$

This implies from (2.12), the manifold is a special type of $\eta-$Einstein manifold. $\square$

Definition 3.5. A Kenmotsu manifold is said to be $\phi - Q^*$ flat with respect to the generalized Tanaka-Webster connection ∇^*if

$$g(Q^*(\phi X, \phi Y)\phi Z, \phi W) = 0, \tag{3.8}$$

for any vector fields X, Y, Z on M.

Theorem 3.6. *Let the Kenmotsu manifold M with generalized Tanaka-Webster connection be $\phi - Q^*$ flat, then M is an $\eta-$Einstein manifold.*

Proof. Using (3.1) in (3.8), we have

$$g\left(R^*(\phi X, \phi Y)\phi Z - \frac{\lambda}{2n}\left\{g(\phi Y, \phi Z)\phi X - g(\phi X, \phi Z)\phi Y\right\}, \phi W\right) = 0. \tag{3.9}$$

Let $\{e_1, e_2, ..., e_{2n+1}\}$ be a local orthonormal basis of vector fields in M. Then $\{\phi e_1, \phi e_2, ..., \phi e_{2n+1}\}$ is also a local orthonormal basis. If we put $X = W = e_i$ in (3.9) and summing up with respect to i, $1 \leq i \leq 2n+1$, we obtain
(3.10)

$$\sum_{i=1}^{2n} R^*\left(\phi e_i, \phi Y, \phi Z, \phi e_i\right) = \frac{\lambda}{2n}\sum_{i=1}^{2n}\left\{g(\phi Y, \phi Z)g\left(\phi e_i, \phi e_i\right) - g(\phi e_i, \phi Z)g\left(\phi Y, \phi e_i\right)\right\}.$$

Using equation (2.17), above result is equal to

$$S^*\left(\phi Y, \phi Z\right) = \frac{\lambda}{2n}\left(2n-1\right)g\left(\phi Y, \phi Z\right).$$

Then, by view of (2.9) and (2.18), we obtain

$$S\left(Y, Z\right) = \left\{\frac{\lambda}{2n}\left(2n-1\right) - 2n\right\}g(\phi Y, \phi Z) - 2n\eta(Y)\eta(Z). \tag{3.11}$$

By using equation (2.3), we get

$$S\left(Y,Z\right)=\left\{\frac{\lambda}{2n}\left(2n-1\right)-2n\right\}g(Y,Z)-\frac{\lambda}{2n}\left(2n-1\right)\eta(Y)\eta(Z). \tag{3.12}$$

With this result, it is seen that the manifold is $\eta-$Einstein. □

Corollary 1. Let the Kenmotsu manifold M with generalized Tanaka-Webster connection be $\phi-Q^{*}$ flat, then the scalar curvature is

$$r=\lambda(2n-1)-2n(2n+1).$$

Proof. Let $\{e_1,e_2,...,e_{2n+1}\}$ be a local orthonormal basis of vector fields in M. If we put $Y=Z=e_i$ in (3.12) and summing up with respect to i, $1\leq i\leq 2n+1$, we obtain

$$\sum_{i=1}^{2n+1}S(e_i,e_i)=\sum_{i=1}^{2n+1}\left\{\frac{\lambda}{2n}\left(2n-1\right)-2n\right\}g(e_i,e_i)-\frac{\lambda}{2n}\left(2n-1\right)\eta(e_i)\eta(e_i).$$

This implies the scalar curvature is $r=\lambda(2n-1)-2n(2n+1)$. □

4. Conclusion

In this paper, are shown some curvature conditions of Kenmotsu manifolds with respect to generalized Tanaka-Webster connection. The results are given which states that $\xi-Q^{*}$ flat and $\phi-Q^{*}$ flat Kenmotsu manifold admitting the generalized Tanaka-Webster connection is a manifold of η-Einstein.

5. Acknowledgments

The authors would like to thank the reviewers and editors of Journal of Universal Mathematics.

Funding
The author(s) declared that has no received any financial support for the research, authorship or publication of this study.

The Declaration of Conflict of Interest/ Common Interest
The author(s) declared that no conflict of interest or common interest

The Declaration of Ethics Committee Approval
This study does not be necessary ethical committee permission or any special permission.

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The author(s) declared that they comply with the scientific, ethical, and citation rules of Journal of Universal Mathematics in all processes of the study and that they do not make any falsification on the data collected. Besides, the author(s) declared that Journal of Universal Mathematics and its editorial board have no responsibility for any ethical violations that may be encountered and this study has not been evaluated in any academic publication environment other than Journal of Universal Mathematics.

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