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INCLUSION THEOREMS IN THE FUNCTION SPACES WITH WIGNER TRANSFORM

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Abstract. In this paper, we consider inclusion relations of $CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R})$ spaces of functions whose Wigner transforms are in weighted Lebesgue spaces. We then discuss compact embeddings theorems between these function spaces.

1. Introduction

In this paper, $C_c(\mathbb{R})$ denotes the space of complex-valued continous functions on $\mathbb{R}$ with compact support and also $L^p(\mathbb{R}), (1 \leq p < \infty)$ denotes the usual Lebesgue space. Let $f : \mathbb{R} \longrightarrow \mathbb{C}$ be any function. The translation operator T_x is given by $T_x f(t) = f(t-x)$ for all $x \in \mathbb{R}$. Also we take the weight functions that is positive real valued, measurable and locally bounded functions ω on $\mathbb{R}$ which satisfy $\omega(x) \geq 1$, $\omega(x+y) \leq \omega(x)\omega(y)$ for all $x, y \in \mathbb{R}$. A weight $\omega(x,s) = (1+|x|+|s|)^a$ which is defined on $\mathbb{R}^2$ is called weight of polynomial type for $a \geq 0$. We set $L^p_\omega(\mathbb{R}) = \{f : f\omega \in L^p(\mathbb{R})\}$ for $1 \leq p < \infty$ [12]. Assume that ω_1 and ω_2 are two weight functions. We show that $\omega_1 \prec \omega_2$ if there exists $C > 0$ such that $\omega_1(x) \leq C\omega_2(x)$ for all $x \in \mathbb{R}$. Two weight function ω_1 and ω_2 are called equivalent. We write $\omega_1 \approx \omega_2$ if and only if $\omega_1 \prec \omega_2$ and $\omega_2 \prec \omega_1$ [5].

Suppose that $\tau \in [0,1]$ and take $f, g \in L^2(\mathbb{R})$, the $\tau-$Wigner transform is defined by

$$W_\tau(f,g)(x,\omega) = \int_{\mathbb{R}} f(x+\tau t)\overline{g(x-(1-\tau)t)}e^{-2\pi it\omega}dt,\ x,\omega \in \mathbb{R}.$$

If we take $\tau = \frac{1}{2}$ in the equation of the $\tau-$Wigner transform, then we obtain the cross-Wigner distribution which is

$$W(f,g)(x,\omega) = \int_{\mathbb{R}} f\left(x+\frac{t}{2}\right)\overline{g\left(x-\frac{t}{2}\right)}e^{-2\pi it\omega}dt,$$

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[1], [2], [14]. If one also takes $f = g$, then $W(f,f) = Wf$ is called the Wigner distribution of $f \in L^2(\mathbb{R})$. We know well that this transform is a quadratic time-frequency representation. The Wigner transform measures how much of the signal energy during the any time period which is concentrated in a frequency band. Thus it gives us information about the energy density in the time-frequency plane and also shows the joint probability density function of the position and momentum variables [6].

In the literature, many function spaces have been defined using various time-frequency operators [3], [8], [9], [10], [10], [11], [13]. One of them is the space $CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R})$ defined by Kulak and Ömerbeyoğlu [9] using the $\tau-$Wigner transform. Now let's give the definition of this space and talk about some of its properties.

Let ω_i $(i = 1,2,3,4)$ be weight functions on $\mathbb{R}$ and let $1 \leq p,q,r,s < \infty$, $\tau \in (0,1)$. The space $CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R})$ consists of all $(f,g) \in \left(L^p_{\omega_1} \times L^q_{\omega_2}\right)(\mathbb{R})$ such that their binary $\tau-$Wigner transforms $(W_\tau(f,.), W_\tau(.,g))$ are in $\left(L^r_{\omega_3} \times L^s_{\omega_4}\right)(\mathbb{R}^2)$. This space is equipped with the following norm:

$$\|(f,g)\|_{CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}} = \|(f,g)\|_{L^p_{\omega_1}\times L^q_{\omega_2}} + \|(W_\tau(f,.), W_\tau(.,g))\|_{L^r_{\omega_3}\times L^s_{\omega_4}}$$

In this here, there exist sum and maximum norms on the spaces $\left(L^p_{\omega_1} \times L^q_{\omega_2}\right)(\mathbb{R})$ and $\left(L^r_{\omega_3} \times L^s_{\omega_4}\right)(\mathbb{R}^2)$ [9].

Let f and g be any functions on $\mathbb{R}$. The binary translation mapping is given by

$$T_x(f,g)(t) = (T_x f(t), T_x g(t)) = (f(t-x), g(t-x)), x,t \in \mathbb{R}$$

[9]. The space $CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R})$ is invariant under binary translations and the norm inequality

$$\|T_z(f,g)\|_{CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}} \leq (u(z) + v(z,0)\,v(z\tau,0))\,\|(f,g)\|_{CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}}$$

is satisfied [9]. Also the binary translation mapping $(f,g) \longrightarrow T_z(f,g)$ is continuous from $CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R})$ into $CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R})$ for every fixed $z \in \mathbb{R}$ and the mapping $z \to T_z(f,g)$ is continuous from $\mathbb{R}$ into $CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R})$ [9].

Let f, g, h, k be Borel measurable functions on $\mathbb{R}$. The binary convolution is defined by

$$(f,g) * (h,k) = (f*h, g*k),$$

where "$*$" denotes usual convolution. The conditions $\int_{\mathbb{R}} |f(y)\,h(x-y)|\,dy < \infty$ and $\int_{\mathbb{R}} |g(y)\,k(x-y)|\,dy < \infty$ must be required for the binary convolution to be defined [9]. Let's the weights ω_3 and ω_4 be constants. It is known that the space $CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R})$ is an essential Banach module over $\left(L^1_{\omega_1} \times L^1_{\omega_2}\right)(\mathbb{R})$ under binary convolutions. Furthermore this space has an approximate identity $((e_\alpha, e_\beta))_{\alpha,\beta\in I}$ of the space $\left(L^1_{\omega_1} \times L^1_{\omega_2}\right)(\mathbb{R})$ such that

$$\lim_{\alpha,\beta\in I} (e_\alpha, e_\beta) * (f,g) = (f,g)$$

for all $(f,g) \in CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R})$ [9].

As can be understood, some important properties and approximate units of the space $CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R})$ have been investigated in [9]. In this study, inclusion relations and compact embedding theorems of this space will be discussed.

2. Inclusion Properties of The Space $CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R})$

Theorem 2.1. *Let $u = \max\{\omega_1, \omega_2\}$ and $v = \max\{\omega_3, \omega_4\}$ where ω_i $(i = 1, 2, 3)$ are weight functions. Then one has the following inequalities*

$$C(f)\,\omega_1(z) \le \|T_z(f,g)\|_{CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}} \le (u(z) + v(z,0)\,v(z\tau,0))\,\|(f,g)\|_{CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}}$$

and

$$C(g)\,\omega_2(z) \le \|T_z(f,g)\|_{CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}} \le (u(z) + v(z,0)\,v(z\tau,0))\,\|(f,g)\|_{CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}}$$

for all $(0,0) \ne (f,g) \in CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R})$.

Proof. Since the assumption $(0,0) \ne (f,g) \in CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R})$, we can write $0 \ne f \in L^p_{\omega_1}(\mathbb{R})$. Then there exists $C(f) > 0$ such that

$$C(f)\,\omega_1(z) \le \|T_z f\|_{p,\omega_1} \le \omega_1(z)\,\|f\|_{p,\omega_1},$$

[5]. Similarly we have $0 \ne g \in L^q_{\omega_2}(\mathbb{R})$ and then

$$C(g)\,\omega_2(z) \le \|T_z g\|_{q,\omega_2} \le \omega_2(z)\,\|g\|_{q,\omega_2}.$$

Thus we get

$$C(f)\,\omega_1(z) \le \|T_z f\|_{p,\omega_1} \le \max\left\{\|T_z f\|_{p,\omega_1}, \|T_z g\|_{q,\omega_2}\right\}$$

$$= \|T_z(f,g)\|_{L^p_{\omega_1}\times L^q_{\omega_2}} \le \|T_z(f,g)\|_{L^p_{\omega_1}\times L^q_{\omega_2}} + \|(W_\tau(T_z f, .), W_\tau(., T_z g))\|_{L^r_{\omega_3}\times L^s_{\omega_4}}$$

$$= \|T_z(f,g)\|_{CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}}. \tag{2.1}$$

Also if we use the inequality (see Theorem 2.8 in [9])

$$\|T_z(f,g)\|_{CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}} \le (u(z) + v(z,0)\,v(z\tau,0))\,\|(f,g)\|_{CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}}$$

and (2.1), we achieve

$$C(f)\,\omega_1(z) \le \|T_z(f,g)\|_{CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}} \le (u(z) + v(z,0)\,v(z\tau,0))\,\|(f,g)\|_{CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}}.$$

Similar way, we obtain that

$$C(g)\,\omega_2(z) \le \|T_z(f,g)\|_{CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}} \le (u(z) + v(z,0)\,v(z\tau,0))\,\|(f,g)\|_{CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}}.$$

□

Theorem 2.2. *If $CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R}) \subset CW^{p,q,r,s,\tau}_{v_1,v_2,v_3,v_4}(\mathbb{R})$, then $CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R})$ is a Banach space under the norm $\|f\|_{CW} = \|(f,g)\|_{CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}} + \|(f,g)\|_{CW^{p,q,r,s,\tau}_{v_1,v_2,v_3,v_4}}$.*

Proof. Let $((f_n, g_n))_{n\in\mathbb{N}}$ be a Cauchy sequence in $\left(CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R}), \|.\|_{CW}\right)$. So $((f_n, g_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in the spaces $\left(CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R}), \|.\|_{CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}}\right)$ and $\left(CW^{p,q,r,s,\tau}_{v_1,v_2,v_3,v_4}(\mathbb{R}), \|.\|_{CW^{p,q,r,s,\tau}_{v_1,v_2,v_3,v_4}}\right)$. It's known that these spaces are Banach spaces. Then there exist $(f,g) \in CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R})$ and $(h,k) \in CW^{p,q,r,s,\tau}_{v_1,v_2,v_3,v_4}(\mathbb{R})$ such that

$$\|(f_n, g_n) - (f,g)\|_{CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}} \longrightarrow 0, \|(f_n, g_n) - (h,k)\|_{CW^{p,q,r,s,\tau}_{v_1,v_2,v_3,v_4}} \longrightarrow 0.$$

From the inequalities $\|.\|_{L^p\times L^q} \le \|.\|_{CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}}$ and $\|.\|_{L^p\times L^q} \le \|.\|_{CW^{p,q,r,s,\tau}_{v_1,v_2,v_3,v_4}}$, we have that

$$\|(f_n, g_n) - (f,g)\|_{L^p\times L^q} \longrightarrow 0, \|(f_n, g_n) - (h,k)\|_{L^p\times L^q} \longrightarrow 0.$$

So since

$$\|(f,g)-(h,k)\|_{L^p\times L^q} \leq \|(f_n,g_n)-(f,g)\|_{L^p\times L^q} + \|(f_n,g_n)-(h,k)\|_{L^p\times L^q},$$

we find that $\|(f,g)-(h,k)\|_{L^p\times L^q}=0$ and $(f,g)=(h,k)$ a.e. Therefore we obtain that

$$\|(f_n,g_n)-(f,g)\|_{CW} \longrightarrow 0$$

and $(f,g)\in \left(CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R}), \|.\|_{CW}\right)$. □

Theorem 2.3. *Assume that ω_i and v_i $(i=1,2,3,4)$ are weight functions. If $v_i \prec \omega_i$ $(i=1,2,3,4)$, then the following inclusion is satisfied*

$$CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R}) \subset CW^{p,q,r,s,\tau}_{v_1,v_2,v_3,v_4}(\mathbb{R}).$$

Proof. Using the assumptions $v_i \prec \omega_i$ $(i=1,2,3,4)$, there exist $C_i>0$ $(i=1,2,3,4)$ such that $v_i(z) \leq C_i\omega_i(z)$ $(i=1,2)$ and $v_i(z,u) \leq C_i\omega_i(z,u)$ $(i=3,4)$ for all $z,u\in\mathbb{R}$. Take any $(f,g)\in CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R})$. Then we have $(f,g)\in \left(L^p_{\omega_1}\times L^q_{\omega_2}\right)(\mathbb{R})$ and $(W_\tau(f,.),W_\tau(.,g))\in \left(L^r_{\omega_3}\times L^s_{\omega_4}\right)(\mathbb{R}^2)$. So we can write

$$\|(f,g)\|_{L^p_{v_1}\times L^q_{v_2}} \leq C_5 \|(f,g)\|_{L^p_{\omega_1}\times L^q_{\omega_2}} < \infty$$

and

$$\|(W_\tau(f,.),W_\tau(.,g))\|_{L^r_{v_3}\times L^s_{v_4}} \leq C_6 \|(W_\tau(f,.),W_\tau(.,g))\|_{L^r_{\omega_3}\times L^s_{\omega_4}} < \infty,$$

where $C_5=\max\{C_1,C_2\}$, $C_6=\max\{C_3,C_4\}$. Thus we get

$$\|(f,g)\|_{CW^{p,q,r,s,\tau}_{v_1,v_2,v_3,v_4}} \leq C_0 \|(f,g)\|_{CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}} < \infty,$$

where $C_0=\max\{C_5,C_6\}$. That means $(f,g)\in CW^{p,q,r,s,\tau}_{v_1,v_2,v_3,v_4}(\mathbb{R})$. Therefore we achieve the inclusion $CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R}) \subset CW^{p,q,r,s,\tau}_{v_1,v_2,v_3,v_4}(\mathbb{R})$. □

Theorem 2.4. *Suppose that ω_i, v_i $(i=1,2,3,4)$ are weight functions. If we have the inclusion $CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R}) \subset CW^{p,q,r,s,\tau}_{v_1,v_2,v_3,v_4}(\mathbb{R})$, then there exists a $C>0$ such that*

$$\|(f,g)\|_{CW^{p,q,r,s,\tau}_{v_1,v_2,v_3,v_4}} \leq C \|(f,g)\|_{CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}}$$

for all $(f,g)\in CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R})$.

Proof. Now take the space $CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R})$ with the norm $\|.\|_{CW} = \|.\|_{CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}} + \|.\|_{CW^{p,q,r,s,\tau}_{v_1,v_2,v_3,v_4}}$. It is known that the space $\left(CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R}), \|.\|_{CW}\right)$ is a Banach space by Theorem 2.2. Thus from the closed graph theorem, there exists a $C>0$ such that

$$\|(f,g)\|_{CW^{p,q,r,s,\tau}_{v_1,v_2,v_3,v_4}} \leq C \|(f,g)\|_{CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}}$$

for all $(f,g)\in CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R})$. □

Theorem 2.5. *Let ω_i, v_i $(i=1,2,3,4)$ be weight functions and let $\mu_i = max\{\omega_i, v_i\}$ $(i=1,2,3,4)$. Then the following equality holds*

$$CW^{p,q,r,s,\tau}_{\mu_1,\mu_2,\mu_3,\mu_4}(\mathbb{R}) = CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R}) \cap CW^{p,q,r,s,\tau}_{v_1,v_2,v_3,v_4}(\mathbb{R}).$$

Proof. Take any $(f,g)\in CW^{p,q,r,s,\tau}_{\mu_1,\mu_2,\mu_3,\mu_4}(\mathbb{R})$. Then we can write

$$\begin{aligned}\|(f,g)\|_{CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}} &= \|(f,g)\|_{L^p_{\omega_1}\times L^q_{\omega_2}} + \|(W_\tau(f,.),W_\tau(.,g))\|_{L^r_{\omega_3}\times L^s_{\omega_4}} \\ &= \max\left\{\|f\|_{p,\omega_1}, \|g\|_{q,\omega_2}\right\} + \max\left\{\|W_\tau(f,.)\|_{r,\omega_3}, \|W_\tau(.,g)\|_{s,\omega_4}\right\} \\ &\leq \max\left\{\|f\|_{p,\mu_1}, \|g\|_{q,\mu_2}\right\} + \max\left\{\|W_\tau(f,.)\|_{r,\mu_3}, \|W_\tau(.,g)\|_{s,\mu_4}\right\}\end{aligned}$$

$$= \|(f,g)\|_{CW^{p,q,r,s,\tau}_{\mu_1,\mu_2,\mu_3,\mu_4}} < \infty.$$

So, we have $(f,g) \in CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R})$. Using similar method, we find $(f,g) \in CW^{p,q,r,s,\tau}_{v_1,v_2,v_3,v_4}(\mathbb{R})$. Thus, we get

$$CW^{p,q,r,s,\tau}_{\mu_1,\mu_2,\mu_3,\mu_4}(\mathbb{R}) \subset CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R}) \cap CW^{p,q,r,s,\tau}_{v_1,v_2,v_3,v_4}(\mathbb{R}).$$

Conversely take any $(f,g) \in CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R}) \cap CW^{p,q,r,s,\tau}_{v_1,v_2,v_3,v_4}(\mathbb{R})$. If we use the assumptions $\mu_i = max\{\omega_i, v_i\}$ $(i = 1,2,3,4)$, then we obtain easily $\|(f,g)\|_{CW^{p,q,r,s,\tau}_{\mu_1,\mu_2,\mu_3,\mu_4}} < \infty$. So, we have $(f,g) \in CW^{p,q,r,s,\tau}_{\mu_1,\mu_2,\mu_3,\mu_4}(\mathbb{R})$. Then, we get

$$CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R}) \cap CW^{p,q,r,s,\tau}_{v_1,v_2,v_3,v_4}(\mathbb{R}) \subset CW^{p,q,r,s,\tau}_{\mu_1,\mu_2,\mu_3,\mu_4}(\mathbb{R})$$

Hence, from the upstairs inclusions, we obtain that

$$CW^{p,q,r,s,\tau}_{\mu_1,\mu_2,\mu_3,\mu_4}(\mathbb{R}) = CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R}) \cap CW^{p,q,r,s,\tau}_{v_1,v_2,v_3,v_4}(\mathbb{R}).$$

□

Theorem 2.6. *Let ω_i, v_i $(i = 1,2,3,4)$ be weight functions and let $u_1 = max\{\omega_1, \omega_2\}$, $\mu_1 = max\{\omega_3, \omega_4\}$, $u_2 = max\{v_1, v_2\}$, $\mu_2 = max\{v_3, v_4\}$. If we have the inclusion $CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R}) \subset CW^{p,q,r,s,\tau}_{v_1,v_2,v_3,v_4}(\mathbb{R})$, then the following relations are satisfied*

$$v_1 \prec \left(u_1 + \tau^{\frac{-1}{2}} \mu_1(.,0)\, D^1_{\tau^{-1}} \mu_1(.,0)\right)$$

and

$$v_2 \prec \left(u_2 + \tau^{\frac{-1}{2}} \mu_2(.,0)\, D^1_{\tau^{-1}} \mu_2(.,0)\right),$$

where $D^1_{\tau^{-1}}\mu_1(z,0) = \tau^{\frac{1}{2}} \mu_1(z\tau, 0)$.

Proof. Let $(f,g) \in CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R})$. From the assumption, we write $(f,g) \in CW^{p,q,r,s,\tau}_{v_1,v_2,v_3,v_4}(\mathbb{R})$. By Theorem 2.8 in [9], there exist $C_1, C_2 > 0$ such that

$$C_1\omega_1(z) \le \|T_z(f,g)\|_{CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}} \le (u_1(z) + \mu_1(z,0)\,\mu_1(z\tau,0))\, \|(f,g)\|_{CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}}, \tag{2.2}$$

$$C_2 v_1(z) \le \|T_z(f,g)\|_{CW^{p,q,r,s,\tau}_{v_1,v_2,v_3,v_4}} \le (u_2(z) + \mu_2(z,0)\,\mu_2(z\tau,0))\, \|(f,g)\|_{CW^{p,q,r,s,\tau}_{v_1,v_2,v_3,v_4}}. \tag{2.3}$$

Moreover from Theorem 2.2, we know that the space $CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R})$ is a Banach space under the norm $\|.\|_{CW}$. Then by Theorem 2.4, there exists $C_3 > 0$ such that

$$\|(f,g)\|_{CW^{p,q,r,s,\tau}_{v_1,v_2,v_3,v_4}} \le C_3 \|(f,g)\|_{CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}}. \tag{2.4}$$

On the other hand since the space $CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R})$ is invariant under the binary translations, we have $T_z(f,g) \in CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R}) \subset CW^{p,q,r,s,\tau}_{v_1,v_2,v_3,v_4}(\mathbb{R})$. So by (2.4), we get [9]

$$\|T_z(f,g)\|_{CW^{p,q,r,s,\tau}_{v_1,v_2,v_3,v_4}} \le C_3 \|T_z(f,g)\|_{CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}}. \tag{2.5}$$

Combining the inequalities (2.2), (2.3) and (2.5), we find

$$C_2 v_1(z) \le \|T_z(f,g)\|_{CW^{p,q,r,s,\tau}_{v_1,v_2,v_3,v_4}} \le C_3 \|T_z(f,g)\|_{CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}}$$
$$\le C_3 (u_1(z) + \mu_1(z,0)\,\mu_1(z\tau,0))\, \|(f,g)\|_{CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}}.$$

From last inequality, we achieve

$$v_1(z) \le k(u_1(z) + \mu_1(z,0)\,\mu_1(z\tau,0))$$
$$= k\left(\left(u_1(z) + \tau^{\frac{-1}{2}} \mu_1(z,0)\, D^1_{\tau^{-1}} \mu_1(z,0)\right)\right),$$

where $k=\frac{C_3\|(f,g)\|_{CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}}}{C_2}$, $D^1_{\tau^{-1}}\mu_1(z,0)=\tau^{\frac{1}{2}}\mu_1(z\tau,0)$. That means

$$v_1 \prec \left(u_1+\tau^{\frac{-1}{2}}\mu_1(.,0)\,D^1_{\tau^{-1}}\mu_1(.,0)\right).$$

Similarly using the inequalities (2.2), (2.3) and (2.5), we obtain that

$$v_2 \prec \left(u_2+\tau^{\frac{-1}{2}}\mu_2(.,0)\,D^1_{\tau^{-1}}\mu_2(.,0)\right).$$

□

3. Compact Embeddings of the Space $CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R})$

Lemma 3.1. *Suppose that $((f_n,g_n))_{n\in\mathbb{N}}$ is a sequence in $CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R})$. If $((f_n,g_n))_{n\in\mathbb{N}}$ converges to zero in $CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R})$, then*

$$\int_{\mathbb{R}} f_n(x)\,h(x)\,dx \longrightarrow 0$$

and

$$\int_{\mathbb{R}} g_n(x)\,h(x)\,dx \longrightarrow 0$$

for $n\longrightarrow\infty$ and for every $h\in C_c(\mathbb{R})$.

Proof. Take any $h\in C_c(\mathbb{R})$. Let $\frac{1}{p}+\frac{1}{p_1}=1$. Since $h\in C_c(\mathbb{R})\subset L^{p_1}(\mathbb{R})$, $f_n\in L^p(\mathbb{R})$ for all $n\in\mathbb{N}$ and by Hölder inequality, we have

$$\begin{aligned}\left|\int_{\mathbb{R}} f_n(x)\,h(x)\,dx\right| &\le \|f_n\|_p\,\|h\|_{p_1}\\ &\le \max\left\{\|f_n\|_{p,\omega_1},\|g_n\|_{q,\omega_2}\right\}\|h\|_{p_1}\\ &\le \|(f_n,g_n)\|_{CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}}\,\|h\|_{p_1}. \qquad (3.1)\end{aligned}$$

Using the assumption and the inequality (3.1), we obtain that

$$\int_{\mathbb{R}} f_n(x)\,h(x)\,dx \longrightarrow 0,$$

for $n\longrightarrow\infty$ and for all $h\in C_c(\mathbb{R})$.

Now let $\frac{1}{q}+\frac{1}{q_1}=1$. Since $h\in C_c(\mathbb{R})\subset L^{q_1}(\mathbb{R})$, $g_n\in L^q(\mathbb{R})$ for all $n\in\mathbb{N}$ and by Hölder inequality, similarly we can write

$$\left|\int_{\mathbb{R}} g_n(x)\,h(x)\,dx\right| \le \|(f_n,g_n)\|_{CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}}\,\|h\|_{q_1}. \qquad (3.2)$$

Then from assumption and the inequality (3.2), we achieve that

$$\int_{\mathbb{R}} g_n(x)\,h(x)\,dx \longrightarrow 0,$$

for $n\longrightarrow\infty$ and for all $h\in C_c(\mathbb{R})$. □

Theorem 3.2. *Let ω_1, ω_2 be weight functions of polynomial type on $\mathbb{R}$ and let ω_3, ω_4 be weight functions of polynomial type on $\mathbb{R}^2$. Assume that u and v are any weight functions on $\mathbb{R}$. If $u \prec \omega_1$, $v \prec \omega_2$ and for $x \longrightarrow \infty$*

$$\frac{u(x)}{u_1(x) + v_1(x,0)\, v_1(x\tau,0)} \nrightarrow 0$$

or

$$\frac{v(x)}{u_1(x) + v_1(x,0)\, v_1(x\tau,0)} \nrightarrow 0,$$

where $u_1 = \max\{\omega_1, \omega_2\}$ and $v_1 = \max\{\omega_3, \omega_4\}$, then the embedding of the space $CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R})$ into $(L^p_u \times L^q_v)(\mathbb{R})$ is never compact.

Proof. If we use the assumptions $u \prec \omega_1$, $v \prec \omega_2$, then we can say that there exist $C_1 > 0$ and $C_2 > 0$ such that $u(x) \le C_1\omega_1(x)$ and $v(x) \le C_2\omega_2(x)$ for all $x \in \mathbb{R}$. So, we have the inclusion $CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R}) \subset (L^p_u \times L^q_v)(\mathbb{R})$. Assume that $(a_n)_{n\in\mathbb{N}}$ is a sequence with $a_n \longrightarrow \infty$ as $n \longrightarrow \infty$ in $\mathbb{R}$. Since the assumptions $\frac{u(x)}{u_1(x)+v_1(x,0)v_1(x\tau,0)}$ and $\frac{v(x)}{u_1(x)+v_1(x,0)v_1(x\tau,0)}$ do not tend to zero as $x \longrightarrow \infty$, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$\frac{u(x)}{u_1(x) + v_1(x,0)\, v_1(x\tau,0)} \ge \delta_1 > 0 \tag{3.3}$$

and

$$\frac{v(x)}{u_1(x) + v_1(x,0)\, v_1(x\tau,0)} \ge \delta_2 > 0 \tag{3.4}$$

for $x \longrightarrow \infty$. Fixed $(f,g) \in CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R})$. Define a sequence $((f_n,g_n))_{n\in\mathbb{N}}$ such that

$$(f_n, g_n) = (u_1(a_n) + v_1(a_n,0)\, v_1(a_n\tau,0))^{-1}\, T_{a_n}(f,g).$$

Now let's show that this sequence is bounded in $CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R})$. Then we can write

$$\|(f_n,g_n)\|_{CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}} = \left\|(u_1(a_n) + v_1(a_n,0)\, v_1(a_n\tau,0))^{-1}\, T_{a_n}(f,g)\right\|_{CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}}$$

$$= (u_1(a_n) + v_1(a_n,0)\, v_1(a_n\tau,0))^{-1}\, \|T_{a_n}(f,g)\|_{CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}}. \tag{3.5}$$

So by (3.5) and the selected sequence, we find

$$\|(f_n,g_n)\|_{CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}} =$$

$$\le (u_1(a_n) + v_1(a_n,0)\, v_1(a_n\tau,0))^{-1}\, (u_1(a_n) + v_1(a_n,0)\, v_1(a_n\tau,0))\, \|(f,g)\|_{CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}}$$

$$= \|(f,g)\|_{CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}}.$$

Moreover, we will show that there wouldn't exist norm convergence subsequence of $((f_n,g_n))_{n\in\mathbb{N}}$ in $(L^p_u \times L^q_v)(\mathbb{R})$. On the other hand, we have

$$\left|\int_{\mathbb{R}} f_n(x)\, h(x)\, dx\right| \le \frac{1}{(u_1(a_n) + v_1(a_n,0)\, v_1(a_n\tau,0))} \int_{\mathbb{R}} |T_{a_n} f(x)|\, |h(x)|\, dx$$

$$= \frac{1}{(u_1(a_n) + v_1(a_n,0)\, v_1(a_n\tau,0))}\, \|T_{a_n} f\|_p\, \|h\|_{p_1}$$

$$= \frac{1}{(u_1(a_n) + v_1(a_n,0)\, v_1(a_n\tau,0))}\, \|f\|_p\, \|h\|_{p_1}, \tag{3.6}$$

where $\frac{1}{p}+\frac{1}{p_1}=1$ for all $h\in C_c(\mathbb{R})$. Since the weights u_1 and v_1 are polynomial type, we obtain that the inequality (3.6) tends zero for $n\longrightarrow\infty$. That means

$$\int_{\mathbb{R}} f_n(x)\,h(x)\,dx\longrightarrow 0.$$

Using similar method, we find

$$\int_{\mathbb{R}} g_n(x)\,h(x)\,dx\longrightarrow 0.$$

By Lemma 3.1, the only possible limit of $((f_n,g_n))_{n\in\mathbb{N}}$ in $(L_u^p\times L_v^q)(\mathbb{R})$ is zero. It's known by [5] that $\|T_{a_n}f\|_{p,u}\approx u(a_n)$ and $\|T_{a_n}g\|_{q,v}\approx v(a_n)$. Then there exist $C_1,C_2,C_3,C_4>0$ such that

$$C_1u(a_n)\le\|T_{a_n}f\|_{p,u}\le C_2u(a_n)\tag{3.7}$$

and

$$C_3v(a_n)\le\|T_{a_n}g\|_{q,v}\le C_4v(a_n).\tag{3.8}$$

Then by (3.7), we get

$$\begin{aligned}\|(f_n,g_n)\|_{L_u^p\times L_v^q}&=\left\|(u_1(a_n)+v_1(a_n,0)\,v_1(a_n\tau,0))^{-1}T_{a_n}(f,g)\right\|_{L_u^p\times L_v^q}\\&=(u_1(a_n)+v_1(a_n,0)\,v_1(a_n\tau,0))^{-1}\|T_{a_n}(f,g)\|_{L_u^p\times L_v^q}\\&=(u_1(a_n)+v_1(a_n,0)\,v_1(a_n\tau,0))^{-1}\max\left\{\|T_{a_n}f\|_{p,u},\|T_{a_n}g\|_{q,v}\right\}\\&\ge C_1(u_1(a_n)+v_1(a_n,0)\,v_1(a_n\tau,0))^{-1}u(a_n).\end{aligned}\tag{3.9}$$

Similarly using by (3.9), we find

$$\|(f_n,g_n)\|_{L_u^p\times L_v^q}\ge C_3(u_1(a_n)+v_1(a_n,0)\,v_1(a_n\tau,0))^{-1}v(a_n).\tag{3.10}$$

From the inequalities (3.3) and (3.4), we write

$$\frac{u(a_n)}{u_1(a_n)+v_1(a_n,0)\,v_1(a_n\tau,0)}\ge\delta_1>0\tag{3.11}$$

and

$$\frac{v(a_n)}{u_1(a_n)+v_1(a_n,0)\,v_1(a_n\tau,0)}\ge\delta_2>0\tag{3.12}$$

for all a_n. So using the inequalities (3.9), (3.10), (3.11) and (3.12), we obtain that

$$\|(f_n,g_n)\|_{L_u^p\times L_v^q}\ge C_1(u_1(a_n)+v_1(a_n,0)\,v_1(a_n\tau,0))^{-1}u(a_n)\ge\delta_1C_1>0$$

and

$$\|(f_n,g_n)\|_{L_u^p\times L_v^q}\ge C_3(u_1(a_n)+v_1(a_n,0)\,v_1(a_n\tau,0))^{-1}v(a_n)\ge\delta_2C_2>0.$$

Hence, we conclude that there would not be possible to find norm convergent subsequence of $((f_n,g_n))_{n\in\mathbb{N}}$ in $(L_u^p\times L_v^q)(\mathbb{R})$. The desired is achieved. □

Theorem 3.3. *Let ω_1, ω_2 be weight functions of polynomial type on $\mathbb{R}$ and let ω_3, ω_4 be weight functions of polynomial type on $\mathbb{R}^2$. Assume that r_1 and r_2 are any weight functions on $\mathbb{R}$ and r_3, r_4 are any weight functions on $\mathbb{R}^2$. If $r_i \prec \omega_i$ $(i = 1, 2, 3, 4)$ and for $x \longrightarrow \infty$*

$$\frac{r_1(x)}{u_1(x) + v_1(x, 0)\, v_1(x\tau, 0)} \nrightarrow 0$$

or

$$\frac{r_2(x)}{u_1(x) + v_1(x, 0)\, v_1(x\tau, 0)} \nrightarrow 0,$$

where $u_1 = \max\{\omega_1,\ \omega_2\}$ and $v_1 = \max\{\omega_3,\ \omega_4\}$, then the embedding of the space $CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R})$ into $CW^{p,q,r,s,\tau}_{r_1,r_2,r_3,r_4}(\mathbb{R})$ is never compact.

Proof. Using the assumptions $r_i \prec \omega_i$ $(i = 1, 2, 3, 4)$ and by Theorem 2.3, we obtain that $CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R}) \subset CW^{p,q,r,s,\tau}_{r_1,r_2,r_3,r_4}(\mathbb{R})$. Also, the unit map is a continuous from $CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R})$ into $CW^{p,q,r,s,\tau}_{r_1,r_2,r_3,r_4}(\mathbb{R})$. Assume that the unit map is compact. Let $((f_n, g_n))_{n\in\mathbb{N}}$ in $CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R})$ be arbitrary bounded sequence. If there exists convergent subsequence of $((f_n, g_n))_{n\in\mathbb{N}}$ in $CW^{p,q,r,s,\tau}_{r_1,r_2,r_3,r_4}(\mathbb{R})$, this sequence also converges in $\left(L^p_{r_1} \times L^q_{r_2}\right)(\mathbb{R})$. But this is not possible by Theorem 3.2. This completes the proof. □

4. Conclusion

In some works in the literature, inclusion and compact embedding theorems of various function spaces have been studied [3], [7], [8], [10], [11]. In this paper, firstly, the inclusion theorems of the space $CW^{p,q,r,s,\tau}_{\omega_1,\omega_2,\omega_3,\omega_4}(\mathbb{R})$ have been proved. Afterwards, it has been investigated under which conditions compact embeddings would not occur and so compact embedding theorems have been obtained.

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