Bitlis Eren Üniversitesi Fen Bilimleri Dergisi
BİTLİS EREN UNIVERSITY JOURNAL OF SCIENCE
ISSN: 2147-3129/e-ISSN: 2147-3188
VOLUME: 11 NO: 3 PAGE: 893-901 YEAR: 2022
DOI: 10.17798/bitlisfen.1137179

On Saturated Numerical Semigroups with Multiplicity p Prime Numbers

Ahmet ÇELİK[1*]

[1]Adıyaman University, Vocational School of Technical Sciences, Depatrtment of Computer Technologies, Adıyaman, TÜRKİYE
(ORCID: 0000-0001-5980-0625)

Keywords: Saturated numerical semigroups, Isole gaps, Perfect, Pseudo-Frobenius number.

Abstract
In this paper, we will give some results for saturated numerical semigroups with multiplicity p prime number and conductor η where $p<10$.

1. Introduction

Let $\mathbb{Z}=\{...,-1,0,1,...\}$ integers set and $\Omega=\{a\in\mathbb{Z}: a\geq 0\}$ be non-negative integers set. and integers set, respectively. $\Delta\subseteq\Omega$ is called a numerical semigroup if $0\in\Delta$, $a_1+a_2\in\Delta$, for all $a_1,a_2\in\Delta$ and $\#(\Omega\backslash\Delta)$ is finite ($\#(Y)$ is cardinality the set of Y).

Let Δ be a numerical semigroup. The largest element of the set of $\mathbb{Z}\backslash\Delta$ is called the Frobenius number of Δ, and it is denoted by $\nu(\Delta)$. The smallest nonzero element of Δ is called the multiplicity of Δ and is denoted by $\mu(\Delta)$. Also, the number $\pi(\Delta)=\#(\{0,1,2,...,\nu(\Delta)\}\cap\Delta)$ is called determine number of Δ. For the numerical semigroup,

$$\Delta=<u_1,u_2,...,u_k>=\{\lambda_0=0,\lambda_1,\lambda_2,...,\lambda_{n-1},\lambda_n=\nu(\Delta)+1,\rightarrow...\},$$

where $\lambda_j<\lambda_{j+1}$ for $j=1,2,...,n=\pi(\Delta)$. Also, the arrow means $\lambda\in\Delta$, for all $\lambda\geq\nu(\Delta)+1$. In this case, the number $\eta=\nu(\Delta)+1$ is called conductor of Δ [1], [5].

Let $\Delta=<u_1,u_2,...,u_k>$ be a numerical semigroup. Then the cardinality of elements $u_1,u_2,...,u_k$, that is, k is called embedding dimension of Δ, and is denoted by $e(\Delta)$. It is known that $e(\Delta)\leq\mu(\Delta)$. So, the numerical semigroup $\Delta=<u_1,u_2,...,u_k>$ is called Maksimal Embedding Dimension (MED) if $\mu(\Delta)=e(\Delta)$. The numerical semigroup Δ is Arf if $\lambda_1+\lambda_2-\lambda_3\in\Delta$ for all $\lambda_1,\lambda_2,\lambda_3\in\Delta$ such that $\lambda_1\geq\lambda_2\geq\lambda_3$. If Δ is an Arf numerical semigroup then Δ is MED. But, its converse is not true. For example, The numerical semigroup $\Delta=<3,7,11>=\{0,3,6,7,9,10,\rightarrow...\}$ is MED but not is Arf, since $7+7-6=8\notin\Delta$ [2], [3], [4], [6].

Δ is called saturated numerical semigroup if $p+r_\Delta(x)\in\Delta$, for all $p,x\in\Delta-\{0\}$, where $r_\Delta(x)=\gcd\{\lambda\in\Delta:\lambda\leq x\}$. It known that a saturated numerical semigroup is Arf. But, an Arf numerical semigroup can not a saturated. For example, $\Delta=<5,8,11,12,14>$ is Arf but it is not saturated [7], [10].

Let Δ be a numerical semigroup and $0\neq\lambda\in\Delta$. The set $Ap(\Delta,\lambda)=\{y\in\Delta: y-\lambda\notin\Delta\}$ is Apery set of Δ according to λ. The element g is a gap of Δ if $g\in\Omega$ but $g\notin\Delta$, we denote the set of gaps of Δ, by $\rho(\Delta)$, i.e. $\rho(\Delta)=\{g\in\Omega: g\notin\Delta\}$. The element $g\in\rho(\Delta)$ is a pseudo-Frobenius number if $g+\lambda\in\Delta$, for all $\lambda\in\Delta$, $\lambda\neq 0$. And the set of all

*Corresponding author: acelik@adiyaman.edu.tr
Received: 28.06.2022, Accepted: 07.09.2022

pseudo-Frobenius number of Δ, we denote by $PF(\Delta)$. Also, the set $SG(\Delta)=\{g\in PF(\Delta): 2g\in\Delta\}$ is called the set of special gaps of Δ [6].

For a numerical semigroup Δ, $t\in\rho(\Delta)$ is called isole gap if $t-1, t+1\in\Delta$. The set of isole gaps of Δ is denoted by $I(\Delta)$, that is, $I(\Delta)=\{t\in\rho(\Delta): t-1, t+1\in\Delta\}$. Also, the numerical semigroup Δ is called perfect if $I(\Delta)=\phi$ (for details see [8], [9]).

In this study, we will give some results about the set of Pseudo-Frobenius and the set of isole gaps of Δ. Also, we will examine whether Δ will be perfect such that Δ is saturated numerical semigroup with multiplicity p prime number and conductor η where $p<10$ and $\eta\not\equiv 1(p)$.

2. Main Results

Theorem 2.1. ([6]) If $\Delta=<a_1,a_2,...,a_n>$ is a MED numerical semigroup then $Ap(\Delta,a_1)=\{0,a_2,a_3,...,a_n\}$.

Proposition 2.2. ([5]). Let $\Delta=<a_1,a_2,...,a_n>$ be a numerical semigroup. Then we have $PF(\Delta)=\{x-\lambda : x\in Ap(\Delta,\lambda),\ x>\nu(\Delta)\}$.

Theorem 2.3. ([7]) Let Δ be a numerical semigroup with $\mu(\Delta)=2$ and conductor η. Then, the Δ saturated numerical semigroup is $\Delta=<2,2\eta+1>$, for $\eta\equiv 0(\text{mod}\,2)$.

Theorem 2.4. ([7]) Let Δ be a numerical semigroup with $\mu(\Delta)=3$ and conductor η. Then, Δ is saturated if Δ is one of following numerical semigroups:

(1) $\Delta=<3,\eta+1,\eta+2>$ for $\eta\equiv 0(3)$.
(2) $\Delta=<3,\eta,\eta+2>$ for $\eta\equiv 2(3)$.

Theorem 2.5. ([7]) Let Δ be a numerical semigroup with $\mu(\Delta)=5$ and conductor η. Then, Δ is saturated if Δ is one of following numerical semigroups:

1) $\Delta=<5,\eta+1,\eta+2,\eta+3,\eta+4>$ for $\eta\equiv 0(5)$
2) $\Delta=<5,\eta,\eta+1,\eta+2,\eta+4>$ for $\eta\equiv 2(5)$
3) $\Delta=<5,\eta,\eta+1,\eta+3,\eta+4>$ for $\eta\equiv 3(5)$
4) $\Delta=<5,\eta,\eta+2,\eta+3,\eta+4>$ for $\eta\equiv 4(5)$.

Theorem 2.6. ([7]) Let Δ be a numerical semigroup with $\mu(\Delta)=7$ and conductor η. Then, Δ is saturated if Δ is one of following numerical semigroups:

1) $\Delta=\langle 7,\eta+1,\eta+2,\eta+3,\eta+4,\eta+5,\eta+6\rangle$ for $\eta\equiv 0(7)$
2) $\Delta=\langle 7,\eta,\eta+1,\eta+2,\eta+3,\eta+4,\eta+6\rangle$ for $\eta\equiv 2(7)$
3) $\Delta=\langle 7,\eta,\eta+1,\eta+2,\eta+3,\eta+5,\eta+6\rangle$ for $\eta\equiv 3(7)$
4) $\Delta=\langle 7,\eta,\eta+1,\eta+2,\eta+4,\eta+5,\eta+6\rangle$ for $\eta\equiv 4(7)$
5) $\Delta=\langle 7,\eta,\eta+1,\eta+3,\eta+4,\eta+5,\eta+6\rangle$ for $\eta\equiv 5(7)$
6) $\Delta=\langle 7,\eta,\eta+2,\eta+3,\eta+4,\eta+5,\eta+6\rangle$ for $\eta\equiv 6(7)$.

Theorem 2.7. Let $\Delta = <2, 2\eta+1>$ be a saturated numerical semigroup for $\eta \equiv 0(2)$. Then, we have, $PF(\Delta) = \{2\eta - 1\}$.

Proof. Let $\Delta = <2, 2\eta+1>$ be a saturated numerical semigroup with $\mu(\Delta) = 2$ and conductor $\eta \equiv 0(2)$. Then we wite that, $\Delta = <2, 2\eta+1> = \{0, 2, 4, 6, ..., 2\eta-2, 2\eta, \rightarrow ...\}$, and the set of gaps of Δ is $\rho(\Delta) = \{1, 3, 5, ..., 2\eta - 1\}$. Thus, we find that the set of Pseudo-Frobenius elements of Δ is

$$PF(\Delta) = \{x - 2 : x \in Ap(\Delta, 2),\ x > \nu(\Delta) = 2\eta - 1\} = \{2\eta - 1\}$$

since

$$Ap(\Delta, 2) = \{x \in \Delta : x - 2 \notin \Delta\} = \{0, 2\eta+1\}.$$

Theorem 2.8. Let Δ be a saturated numerical semigroup with prime multiplicity $\mu(\Delta) = p$ and conductor η, where $2 < p < 10$ and $\eta \equiv j(p)$.

i) If $j = 0$ then $PF(\Delta) = \{\eta-1, \eta-2, \eta-3, ..., \eta-(p-1)\}$.
ii) If $j \neq 0$ then $PF(\Delta) = \{\eta - k : k = 1, 2, ..., j-1, j+1, ..., p\}$.

Proof. Let Δ be a saturated numerical semigroup with prime multiplicity $\mu(\Delta) = p$ and conductor η, where $p < 10$ and $\eta \equiv j(p)$.

i) If $j = 0$ then we write the saturated numerical semigroup Δ as following: for $p = 3, 5, 7$ respectively:

(a) $\Delta = <3, \eta+1, \eta+2>$,
(b) $\Delta = <5, \eta+1, \eta+2, \eta+3, \eta+4>$,
(c) $\Delta = \langle 7, \eta+1, \eta+2, \eta+3, \eta+4, \eta+5, \eta+6 \rangle$.

Now, we can explain the above cases as follows:

(a) Let Δ be a saturated numerical semigroup is $\Delta = <3, \eta+1, \eta+2>$. Then we have

$$\Delta = <3, \eta+1, \eta+2> = \{0, 3, 6, ..., \eta-6, \eta-3, \eta, \rightarrow ...\}.$$

In this case, we have that,

$$\rho(\Delta) = \{1, 2, 4, 5, 7, 8, ..., \eta-7, \eta-5, \eta-4, \eta-2, \eta-1\}$$

and $Ap(\Delta, 3) = \{0, \eta+1, \eta+2\}$. So, we find that

$$PF(\Delta) = \{x - 3 : x \in Ap(\Delta, 3),\ x > \nu(\Delta) = \eta - 1\} = \{\eta - 1, \eta - 2\}.$$

(b) Let Δ be a saturated numerical semigroup is $\Delta = <5, \eta+1, \eta+2, \eta+3, \eta+4>$. Then we write

$$\Delta = <5, \eta+1, \eta+2, \eta+3, \eta+4> = \{0, 5, 10, ..., \eta-5, \eta \rightarrow ...\}$$

and

$$\rho(\Delta) = \{1, 2, 3, 4, 6, 7, 8, 9, 11, ..., \eta-4, \eta-3, \eta-2, \eta-1\}.$$

Thus, we obtain

$$PF(\Delta) = \{x - 5 : x \in Ap(\Delta, 5),\ x > \nu(\Delta) = \eta - 1\} = \{\eta-1, \eta-2, \eta-3, \eta-4\}$$

since $Ap(\Delta, 5) = \{x \in \Delta : x - 5 \notin \Delta\} = \{0, \eta+1, \eta+2, \eta+3, \eta+4\}$.

(c) Let Δ be a saturated numerical semigroup $\Delta = \langle 7, \eta+1, \eta+2, \eta+3, \eta+4, \eta+5, \eta+6 \rangle$. We have

$$\Delta = <7, \eta+1, \eta+2, \eta+3, \eta+4, \eta+5, \eta+6> = \{0, 7, 14, ..., \eta-14, \eta-7, \eta, \rightarrow ...\}$$

and

$$\rho(\Delta) = \{1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 13, 15, ..., \eta-8, \eta-6, \eta-2, \eta-1\}.$$

We find that

$$PF(\Delta)=\{x-7:\ x\in Ap(\Delta,7),\ x>\nu(\Delta)=\eta-1\}=\{\eta-1,\eta-2,\eta-3,\eta-4,\eta-5,\eta-6\}$$

since $Ap(\Delta,7)=\ \ x\in\Delta:\ x-7\notin\Delta\ \ =\{0,\eta+1,\eta+2,\eta+3,\eta+4,\eta+5,\eta+6\}$.

Considering the above explanations, we obtain

$$PF(\Delta)=\{\eta-1,\eta-2,\eta-3,...,\eta-(p-1)\}$$

for $2<p<10$ and $\eta\equiv 0(p)$.

ii) If $j\neq 0$ then we write the saturated numerical semigroup Δ as following:

(a) For $p=3$;

The saturated numerical semigroup is $\Delta=<3,\eta,\eta+2>$ for $\eta\equiv 2(3)$. Then, we write

$$\Delta=<3,\eta,\eta+2>=\{0,3,6,9,...,\eta-5,\eta-2,\eta,\rightarrow...\},$$

$$\rho(\Delta)=\{1,2,4,5,7,8,...,\eta-6,\eta-4,\eta-3,\eta-1\}$$

and

$$Ap(\Delta,3)=\{0,\eta,\eta+2\}.$$

Thus, we find that $PF(\Delta)=\{x-3:\ x\in Ap(\Delta,3),\ x>\nu(\Delta)=\eta-1\}=\{\eta-1,\eta-3\}$.

(b) For $p=5$;

(1) If the saturated numerical semigroup is $\Delta=<5,\eta,\eta+1,\eta+2,\eta+4>$ for $\eta\equiv 2(5)$, then we have $\Delta=<5,\eta,\eta+1,\eta+2,\eta+4>=\{0,5,10,...,\eta-7,\eta-2,\eta,\rightarrow...\}$, $\rho(\Delta)=\{1,2,3,...,\eta-6,\eta-5,...,\eta-3,\eta-1\}$ and $Ap(\Delta,5)=\{0,\eta,\eta+1,\eta+2,\eta+4\}$. So, we find that , $PF(\Delta)=\{x-5:\ x\in Ap(\Delta,5),\ x>\nu(\Delta)=\eta-1\}=\{\eta-1,\eta-3,\eta-4,\eta-5\}$.

(2) If the saturated numerical semigroup is $\Delta=<5,\eta,\eta+1,\eta+3,\eta+4>$ for $\eta\equiv 3(5)$, then we write

$$\Delta=<5,\eta,\eta+1,\eta+3,\eta+4>=\{0,5,10,...,\eta-8,\eta-3,\eta,\rightarrow...\}$$

$$\rho(\Delta)=\{1,2,3,...,\eta-7,\eta-6,\eta-5,\eta-4,\eta-2,\eta-1\}$$

and

$$Ap(\Delta,5)=\{0,\eta,\eta+1,\eta+3,\eta+4\}.$$

Thus, we obtain that

$$PF(\Delta)=\{x-5:\ x\in Ap(\Delta,5),\ x>\nu(\Delta)=\eta-1\}=\{\eta-1,\eta-2,\eta-4,\eta-5\}.$$

(3) If the saturated numerical semigroup is $\Delta=<5,\eta,\eta+2,\eta+3,\eta+4>$ for $\eta\equiv 4(5)$, then we find that

$$\Delta=<5,\eta,\eta+2,\eta+3,\eta+4>=\{0,5,10,...,\eta-9,\eta-4,\eta,\rightarrow...\}$$

$$\rho(\Delta)=\{1,2,3,...,\eta-8,\eta-7,\eta-6,\eta-5,\eta-3,\eta-2,\eta-1\}$$

and

$$Ap(\Delta,5)=\{0,\eta,\eta+2,\eta+3,\eta+4\}.$$

Thus, we write that

$$PF(\Delta)=\{x-5:\ x\in Ap(\Delta,5),\ x>\nu(\Delta)=\eta-1\}=\{\eta-1,\eta-2,\eta-3,\eta-5\}.$$

(c) For $p=7$;

(1) If the saturated numerical semigroup is $\Delta=\langle 7,\eta,\eta+1,\eta+2,\eta+3,\eta+4,\eta+6\rangle$ for $\eta\equiv 2(7)$, then we have

$$\Delta=<7,\eta,\eta+1,\eta+2,\eta+3,\eta+4,\eta+6>=\{0,7,14,21,...,\eta-9,\eta-2,\eta,\rightarrow...\}$$

and

$$\rho(\Delta)=\{1,2,3,...,6,8,...,13,15,...,\eta-8,\eta-7,...,\eta-3,\eta-1\}.$$

Thus, we find that

$PF(\Delta)=\{x-7: x\in Ap(\Delta,7),\ x>F(\Delta)=\eta-1\}=\{\eta-7,\eta-6,\eta-5,\eta-4,\eta-3,\eta-1\}$ since $Ap(\Delta,7)=\{0,\eta,\eta+1,\eta+2,\eta+3,\eta+4,\eta+6\}$.

Making same operations, we find followings:

(2) If the saturated numerical semigroup is $\Delta=\langle 7,\eta,\eta+1,\eta+2,\eta+3,\eta+5,\eta+6\rangle$ for $\eta\equiv 3(7)$, then we have $PF(\Delta)=\{\eta-7,\eta-6,\eta-5,\eta-4,\eta-2,\eta-1\}$.

(3) If the saturated numerical semigroup is $\Delta=\langle 7,\eta,\eta+1,\eta+2,\eta+4,\eta+5,\eta+6\rangle$ for $\eta\equiv 4(7)$, then we have $PF(\Delta)=\{\eta-7,\eta-6,\eta-5,\eta-3,\eta-2,\eta-1\}$.

(4) If the saturated numerical semigroup is $\Delta=\langle 7,\eta,\eta+1,\eta+3,\eta+4,\eta+5,\eta+6\rangle$ for $\eta\equiv 5(7)$, then we have $PF(\Delta)=\{\eta-7,\eta-6,\eta-4,\eta-3,\eta-2,\eta-1\}$.

(5) If the saturated numerical semigroup is $\Delta=\langle 7,\eta,\eta+2,\eta+3,\eta+4,\eta+5,\eta+6\rangle$ for $\eta\equiv 6(7)$, then we have $PF(\Delta)=\{\eta-7,\eta-5,\eta-4,\eta-3,\eta-2,\eta-1\}$.

Finally, we obtain following results for a saturated numerical semigroup Δ ,with $\mu(\Delta)=p$ is prime number and conductor η, where $2<p<10$ and $\eta\equiv j(p)$:

i) If $j=0$ then $PF(\Delta)=\{\eta-1,\eta-2,\eta-3,...,\eta-(p-1)\}$.
ii) If $j\neq 0$ then $PF(\Delta)=\{\eta-k: k=1,2,...,j-1,j+1,...,p\}$.

Theorem 2.9. The saturated numerical semigroup Δ given by Theorem 2.3. is not perfect,

Proof. Let Δ be a saturated numerical semigroup which given by Theorem 2.3. Then, $\Delta=<2,2\eta+1>$, with $\mu(\Delta)=2$ and conductor $\eta\equiv 0(\bmod 2)$. In this case, we write $\Delta=<2,2\eta+1>=\{0,2,4,6,..,2\eta,\rightarrow...\}$ and $\rho(\Delta)=\{1,3,5,...,2\eta-1\}$. In this case, we obtain the set of isole gaps of Δ is $I(\Delta)=\{y\in\rho(\Delta): y-1,y+1\in\Delta\}=\{\rho(\Delta)\}\neq\phi$, that is, Δ is not perfect.

Theorem 2.10. Let Δ be a saturated numerical semigroup with $\mu(\Delta)=p$ is prime number and conductor η, where $2<p<10$ and $\eta\equiv j(p)$.

(i) If $j=2$ then $I(\Delta)=\{\nu(\Delta)\}$

(ii) If $j\neq 2$ then $I(\Delta)=\phi$.

Proof. Let Δ be a saturated numerical semigroup with prime multiplicity $\mu(\Delta)=p$ and conductor η, where $2<p<10$ and $\eta\equiv j(p)$.

(i) If $j=2$ then we have the following saturated numerical semigroups:

(1) For $p=3$;

The saturated numerical semigroup is $\Delta=<3,\eta,\eta+2>$ for $\eta\equiv 2(3)$ and we write $\Delta=<3,\eta,\eta+2>=\{0,3,6,9,...,\eta-5,\eta-2,\eta,\rightarrow...\}$. Therefore, we obtain the set of isole gaps of Δ is $I(\Delta)=\{y\in\rho(\Delta): y-1,y+1\in\Delta\}=\{\eta-1\}=\{\nu(\Delta)\}$ since

$$\rho(\Delta)=\{1,2,4,5,7,8,...,\eta-6,\eta-4,\eta-3,\eta-1\}.$$

(2) For $p=5$;

The saturated numerical semigroup is $\Delta=<5,\eta,\eta+1,\eta+2,\eta+4>$ for $\eta\equiv 2(5)$, then we have

$$\Delta=<5,\eta,\eta+1,\eta+2,\eta+4>=\{0,5,10,...,\eta-7,\eta-2,\eta,\rightarrow...\}$$

and

$$\rho(\Delta)=\{1,2,3,4,6...,\eta-6,...,\eta-3,\eta-1\}.$$

Thus, we obtain that

$$I(\Delta)=\{y\in\rho(\Delta):\ y-1,y+1\in\Delta\}=\{\eta-1\}=\{\nu(\Delta)\}.$$

(3) For $p=7$;

The saturated numerical semigroup is $\Delta=\langle 7,\eta,\eta+1,\eta+2,\eta+3,\eta+4,\eta+6\rangle$ for $\eta\equiv 2(7)$, then we have

$$\Delta=<7,\eta,\eta+1,\eta+2,\eta+3,\eta+4,\eta+6>=\{0,7,14,16,...,\eta-9,\eta-2,\eta,\rightarrow...\}$$

and

$$\rho(\Delta)=\{1,2,3,4,5,6,8,...,\eta-8,\eta-7,...,\eta-3,\eta-1\}.$$

So, we find that

$$I(\Delta)=\{y\in\rho(\Delta):\ y-1,y+1\in\Delta\}=\{\eta-1\}=\{\nu(\Delta)\}.$$

(ii) If $j\neq 2$ then we have the following saturated numerical semigroups:

(1) If $j=0$ then we write the saturated numerical semigroup Δ as following: for $p=3,5,7$ respectively:

(a) $\Delta=<3,\eta+1,\eta+2>$,

(b) $\Delta=<5,\eta+1,\eta+2,\eta+3,\eta+4>$,

(c) $\Delta=\langle 7,\eta+1,\eta+2,\eta+3,\eta+4,\eta+5,\eta+6\rangle$.

Now, we can explain the above cases as follows:

(a) Let Δ be a saturated numerical semigroup is $\Delta=<3,\eta+1,\eta+2>$. Then we have we obtain the set of isole gaps of Δ is $I(\Delta)=\{y\in\rho(\Delta):\ y-1,y+1\in\Delta\}=\phi$ since

$$\Delta=<3,\eta+1,\eta+2>=\{0,3,6,...,\eta-6,\eta-3,\eta,\rightarrow...\}$$

and

$$\rho(\Delta)=\left\{1,2,4,5,7,8,...,\eta-7,\eta-5,\eta-4,\eta-2,\eta-1\right\}.$$

(b) The saturated numerical semigroup is $\Delta=<5,\eta+1,\eta+2,\eta+3,\eta+4>$**.** Then we write

$$\Delta=<5,\eta+1,\eta+2,\eta+3,\eta+4>=\{0,5,10,...,\eta-5,\eta,\rightarrow...\}$$

and

$$\rho(\Delta)=\left\{1,2,3,4,6,7,8,9,...,\eta-4,\eta-3,\eta-2,\eta-1\right\}.$$

So, we find that $I(\Delta)=\{y\in\rho(\Delta):\ y-1,y+1\in\Delta\}=\phi$.

(c) Let Δ be a saturated numerical semigroup is $\Delta=\langle 7,\eta+1,\eta+2,\eta+3,\eta+4,\eta+5,\eta+6\rangle$ **.**

In this case, we obtain that $I(\Delta)=\phi$ since $\Delta=\{0,7,14,...,\eta-14,\eta-7,\eta,\rightarrow...\}$ and $\rho(\Delta)=\left\{1,2,3,4,5,6,8,9,10,11,12,13,...,\eta-8,\eta-6,\eta-2,\eta-1\right\}$.

(2) If $j=3$ then we write the saturated numerical semigroup Δ as following: for $p=5,7$ respectively:

(a) $\Delta=<5,\eta,\eta+1,\eta+3,\eta+4>$,

(b) $\Delta=\langle 7,\eta,\eta+1,\eta+2,\eta+3,\eta+5,\eta+6\rangle$.

Now, we can explain the above cases as follows:

(a) If the saturated numerical semigroup is $\Delta=<5,\eta,\eta+1,\eta+3,\eta+4>$ for $\eta\equiv 3(5)$ then we write $\Delta=<5,\eta,\eta+1,\eta+3,\eta+4>=\{0,5,10,...,\eta-8,\eta-3,\eta,\rightarrow...\}$
$\rho(\Delta)=\{1,2,3,...,\eta-7,\eta-6,\eta-5,\eta-4,\eta-2,\eta-1\}$. Thus, we obtain

$$I(\Delta)=\{y\in\rho(\Delta):\ y-1,y+1\in\Delta\}=\phi.$$

(b) If the saturated numerical semigroup is $\Delta=\langle 7,\eta,\eta+1,\eta+2,\eta+3,\eta+5,\eta+6\rangle$ for $\eta\equiv 3(7)$, then we have $I(\Delta)=\{y\in\rho(\Delta):\ y-1,y+1\in\Delta\}=\phi$ since
$\Delta=\langle 7,\eta,\eta+1,\eta+2,\eta+3,\eta+5,\eta+6\rangle=\{0,7,14,21,...,\eta-10,\eta-3,\eta,\rightarrow...\}$ and

$$\rho(\Delta)=\{1,2,3,4,5,6,8,...,\eta-5,\eta-4,\eta-2,\eta-1\}.$$

(3) If $j=4$ then we write the saturated numerical semigroup Δ as following: for $p=5,7$ respectively:

(a) $\Delta=<5,\eta,\eta+2,\eta+3,\eta+4>$,

(b) $\Delta=\langle 7,\eta,\eta+1,\eta+2,\eta+4,\eta+5,\eta+6\rangle$.

Now, we can explain the above cases as follows:

(a) If the saturated numerical semigroup is $\Delta=<5,\eta,\eta+2,\eta+3,\eta+4>$ for $\eta\equiv 4(5)$ then we write $\Delta=<5,\eta,\eta+2,\eta+3,\eta+4>=\{0,5,10,...,\eta-9,\eta-4,\eta,\rightarrow...\}$,
$\rho(\Delta)=\{1,2,3,4,6,...,\eta-8,\eta-7,\eta-6,\eta-5,\eta-3,\eta-2,\eta-1\}$. Thus, we obtain

$$I(\Delta)=\{y\in\rho(\Delta):\ y-1,y+1\in\Delta\}=\phi.$$

(b) If the saturated numerical semigroup is $\Delta=\langle 7,\eta,\eta+1,\eta+2,\eta+4,\eta+5,\eta+6\rangle$ for $\eta\equiv 4(7)$, then we have $I(\Delta)=\{y\in\rho(\Delta):\ y-1,y+1\in\Delta\}=\phi$ since

$$\Delta=\langle 7,\eta,\eta+1,\eta+2,\eta+4,\eta+5,\eta+6\rangle=\{0,7,14,21,...,\eta-11,\eta-4,\eta,\rightarrow...\}$$

and

$$\rho(\Delta)=\{1,2,3,4,5,6,8,...,\eta-10,...,\eta-5,\eta-3,\eta-2,\eta-1\}.$$

(4) If $j=5$ then we write the saturated numerical semigroup $\Delta=\langle 7,\eta,\eta+1,\eta+3,\eta+4,\eta+5,\eta+6\rangle=\{0,7,14,21,...,\eta-12,\eta-5,\eta,\rightarrow...\}$.In this case, we obtain that $I(\Delta)=\{y\in\rho(\Delta):\ y-1,y+1\in\Delta\}=\phi$ since

$$\rho(\Delta)=\{1,2,3,4,5,6,8,...,\eta-11,...,\eta-6,\eta-4,\eta-3,\eta-2,\eta-1\}.$$

(5) If $j=6$ then we write the saturated numerical semigroup $\Delta=\langle 7,\eta,\eta+2,\eta+3,\eta+4,\eta+5,\eta+6\rangle=\{0,7,14,21,...,\eta-13,\eta-6,\eta,\rightarrow...\}$. Thus, we find that $I(\Delta)=\{y\in\rho(\Delta):\ y-1,y+1\in\Delta\}=\phi$ since

$$\rho(\Delta)=\{1,2,3,4,5,6,8,...,\eta-14,...,\eta-5,\eta-4,\eta-3,\eta-2,\eta-1\}.$$

Corollary 2.11. Let Δ be a saturated numerical semigroup with prime multiplicity $\mu(\Delta)=p$ and conductor η, where $2<p<10$ and $\eta\equiv j(p)$.

(i) If $j=2$ then Δ not perfect

(ii) If $j\neq 2$ then Δ is perfect.

Proof. It is clear.

Corollary 2.12 Let Δ be a saturated numerical semigroup with prime multiplicity $\mu(\Delta)=p$ and conductor η, where $p<10$ and $\eta\equiv j(p)$. Then, we have $PF(\Delta)=SG(\Delta)$.

Proof Let Δ be a saturated numerical semigroup with prime multiplicity $\mu(\Delta)=p$ and conductor η, where $p<10$ and $\eta \equiv j(p)$. Then, it is clear that $SG(\Delta) \subseteq PF(\Delta)$.

(1) For $p=2$;
If $\eta \equiv 0(2)$ then $\Delta = <2, 2\eta+1> = \{0,2,4,6,...,2\eta-2,2\eta,\rightarrow ...\}$ and we write $PF(\Delta) = 2\eta-1$ from Theorem 2.7.
If $x \in PF(\Delta) \Rightarrow x = 2\eta-1 \Rightarrow 2x = 4\eta-2 = (3\eta)+(\eta-2) \in \Delta \Rightarrow x \in SG(\Delta)$.

(2) For $p=3$;
(a) If $\eta \equiv 0(3)$ then $\Delta = <3, \eta+1, \eta+2> = \{0,3,6,9,..,\eta-6,\eta-3,\eta,\rightarrow ...\}$ and we write $PF(\Delta) = \{\eta-1, \eta-2\}$ from Theorem 2.8/(i). Let $x \in PF(\Delta)$.
If $x = \eta-1 \Rightarrow 2x = 2\eta-2 = (\eta-3)+(\eta+1) \in \Delta \Rightarrow x \in SG(\Delta)$

or

if $x = \eta-2 \Rightarrow 2x = 2\eta-4 = (\eta-6)+(\eta+2) \in \Delta \Rightarrow x \in SG(\Delta)$.

(b) If $\eta \equiv 2(3)$ then $\Delta = <3, \eta, \eta+2> = \{0,3,6,9,...,\eta-8,\eta-5,\eta-2,\eta,\rightarrow ...\}$ and we write $PF(\Delta) = \{\eta-1, \eta-3\}$ from Theorem 2.8/(ii). Let $x \in PF(\Delta)$
if $x = \eta-1 \Rightarrow 2x = 2\eta-2 = (\eta-2)+(\eta) \in \Delta \Rightarrow x \in SG(\Delta)$

or

if $x = \eta-3 \Rightarrow 2x = 2\eta-6 = (\eta-8)+(\eta+2) \in \Delta \Rightarrow x \in SG(\Delta)$.

If we make same operations for $p=5$ and $p=7$, we obtain $PF(\Delta) \subseteq SG(\Delta)$. Thus, the proof is completed.

Example 2.13. We put $p=3$ and $\eta=9$, in Theorem 2.4/(1) Then, we write $\Delta = <3,10,11> = \{0,3,6,9,\rightarrow ...\}$ MED and saturated numerical semigroup since $\mu(\Delta)=e(\Delta)=3$. Here, $f(\Delta)=8$, $n(\Delta)=3$, $\rho(\Delta)=\{1,2,4,5,7,8\}$ and $Ap(\Delta,3)=\{0,10,11\}$. In this case, we find that

$$PF(\Delta) = \{x-3 : x \in Ap(\Delta,3),\ x > F(\Delta)=8\} = \{10-3, 11-3\} = \{9,8\}$$

and

$$I(\Delta) = \{y \in H(\Delta) : y-1, y+1 \in \Delta\} = \phi .$$

That is, $\Delta = <3,10,11> = \{0,3,6,9,\rightarrow ...\}$ numerical semigroup is perfect. Also,

$$SG(\Delta) = \{x \in PF(\Delta) : 2x \in \Delta\} = \{8,9\} = PF(\Delta) .$$

Example 2.14. We put $p=5$ and $\eta=12$, in Theorem 2.5/(2) Then, we write $\Delta = <5,12,13,14,16> = \{0,5,10,12,\rightarrow ...\}$ saturated numerical semigroup. Here, $\mu(\Delta)=5$, $f(\Delta)=11$, $n(S)=3$, $\rho(\Delta)=\{1,2,3,4,6,7,8,9,11,\rightarrow ...\}$ and $Ap(\Delta,5)=\{0,12,13,14,16\}$.

Thus, we obtain

$$PF(\Delta) = \{x-5 : x \in Ap(\Delta,5),\ x > F(\Delta)=11\} = \{7,8,9,11\}$$

and

$$I(\Delta) = \{y \in H(\Delta) : y-1, y+1 \in \Delta\} = \{11\} .$$

Therefore, the numerical semigroup Δ is not perfect and, we find that

$$SG(\Delta) = \{ x \in PF(\Delta) : 2x \in \Delta \} = \{7,8,9,11\} = PF(\Delta)$$

Statement of Research and Publication Ethics
The study is complied with research and publication ethics

References

[1] R. Froberg, C. Gotlieb and R. Haggkvist, "On numerical semigroups," *Semigroup Forum*, vol. 35, pp.63-68, 1987.
[2] S. İlhan and H.İ. Karakaş, "Arf numerical semigroups," *Turkish J. Math.*, vol.41, pp.1448-1457, 2017.
[3] J.C.Rosales, P.A.García-Sánchez, J.I.García-García and M.B.Branco, "Numerical semigroups with maximal embedding dimension," *Int. J. Commut. Rings*, vol. 2, pp.47–53, 2003.
[4] J. C. Rosales, P.A.García-Sánchez, J.I.García-García, M.B.Branco, "Arf numerical semigroups," *J. Algebra*, vol. 276, pp. 3–12, 2004.
[5] J.C.Rosales and P.A.Garcia-Sanchez, Numerical semigroups, New York: Springer 181, 2009.
[6] A. Assi and P.A.Garcia-Sanchez, Numerical semigroups and applications, https://hal.archives-ouvertes.fr/hal-01085760, pp 47, 2014.
[7] A.Çelik, Some results on saturated numerical semigroups, Doctoral thesis, Dicle University, 2018.
[8] M. A. Moreno Frias and J.C.Rosales, "Perfect numerical semigroups," *Turkish J. Math*., vol. 43,pp. 1742 – 1754, 2019.
[9] J. S. Harold, "On isolated gaps in numerical semigroups," *Turkish J. Math.*, vol.46, pp.123-129, 2022.
[10] S. İlhan, "On a family of telescopic numerical semigroups," *Acta Univ. Apulensis Math. Inform*, vol. 67, pp.13-18, 2021