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Research Article

An algebraic construction technique for codes over Hurwitz integers

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Abstract

Let α be a prime Hurwitz integer. $\mathcal{H}_\alpha$, which is the set of residual class with respect to related modulo function in the rings of Hurwitz integers, is a subset of $\mathcal{H}$, which is the set of all Hurwitz integers. In this study, we present an algebraic construction technique, which is a modulo function formed depending on two modulo operations, for codes over Hurwitz integers. We consider left congruent modulo α, and the domain of related modulo function is $\mathbb{Z}_{N(\alpha)}$, which is residual class ring of ordinary integers with $N(\alpha)$ elements. Therefore, we obtain the residue class rings of Hurwitz integers with $N(\alpha)$ size. In addition, we present some results for mathematical notations used in two modulo functions, and for the algebraic construction technique formed depending upon two modulo functions. Moreover, we presented graphs obtained by graph layout methods, such as spring, high-dimensional, and spiral embedding, for the set of the residual class obtained with respect to the related modulo function in the rings of Hurwitz integers.

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1. Introduction

The set of all Gaussian integers that denoted by $\mathbb{Z}[i]$ is shown by $\mathbb{Z}[i] = \{\alpha = \alpha_1 + \alpha_2 i : \alpha_1, \alpha_2 \in \mathbb{Z}$, where $i^2 = -1\}$. In this study, we use $\mathcal{G}$ notation instead of $\mathbb{Z}[i]$. For the Gaussian integer $\alpha_1 + \alpha_2 i$, α_1 is the real part, and α_2 is the imaginary part. The conjugate of a Gaussian integer α is $\overline{\alpha} = \alpha_1 - \alpha_2 i$. The norm of a Gaussian integer α is $N(\alpha) = \alpha\overline{\alpha} = \alpha_1^2 + \alpha_2^2$. A Gaussian integer is called a prime Gaussian integer if its norm is a prime integer. In [3], Huber considered Gaussian integers such that $p \equiv 1 \mod N(\alpha)$, where p is a prime integer, and presented a technique known as the modulo function to construct block codes over Gaussian integers. In this way, Huber constructed one Mannheim error-correcting (OMEC) codes over Gaussian integers fields. In a similar technique, in [4], Huber showed how block codes over Eisenstein-Jacobi integers could be used for coding over two-dimensional vector space. In [1], Freudenberger et al. presented

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new coding techniques for codes over Gaussian integers by using the modulo function in [3].

Quaternions, a four-dimensional algebraic structure, are a number system that expands over complex numbers. The set of all quaternions that denoted by $\mathbb{H}(\mathbb{R})$ is shown by $\mathbb{H}(\mathbb{R}) = \{\alpha = \alpha_1 + \alpha_2 i + \alpha_3 j + \alpha_4 k : \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}\}$, where $i^2 = j^2 = k^2 = -1$, and $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$. For the quaternion $\alpha = \alpha_1 + \alpha_2 i + \alpha_3 j + \alpha_4 k$, α_1 is called the real part, $\alpha_2 i + \alpha_3 j + \alpha_4 k$ is called the vector part (sometimes imaginary part), and $\alpha_1, \alpha_2, \alpha_3$, and α_4 are called the components of a quaternion α. α is called a quaternion integer if $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{Z}$. The conjugate of a quaternion integer α is $\overline{\alpha} = \alpha_1 - \alpha_2 i - \alpha_3 j - \alpha_4 k$. The norm of a quaternion integer α is $N(\alpha) = \alpha\overline{\alpha} = \alpha_1^2 + \alpha_2^2 + \alpha_2^3 + \alpha_2^4$. The inverse of a quaternion integer α is $\alpha^{-1} = \frac{\overline{\alpha}}{N(\alpha)}$, where $N(\alpha) \neq 0$. The commutative property of multiplication does not hold over quaternion integers, in general. If the imaginary parts of quaternion integers are parallel to each other, then their product is commutative. In [9], Özen and Güzeltepe studied codes over some finite fields by using quaternion integers, which have the commutative property. In [10], Özen and Güzeltepe studied cyclic codes over some finite quaternion integer rings. They considered the quaternion integers which have the commutative property. In [12], Shah and Rasool established that; over quaternion integers, for a given n length cyclic code there exists a cyclic code of length $2n - 1$. A quaternion α is called a Lipschitz integer if all of its components are in integers. A Lipschitz integer α is called a primitive Lipschitz integer just if the greatest common divisor of its components is 1. In [2], Freudenberger et al. presented new block codes over Lipschitz integers. They consider primitive Lipschitz integers. A quaternion α is called a Hurwitz integer if all of its components are either in $\mathbb{Z}$ or in $\mathbb{Z} + \frac{1}{2}$. A Hurwitz integer α is called a prime Hurwitz integer if its norm is a prime integer. In this study, we consider prime Hurwitz integers. In [5], Güzeltepe obtained new classes of linear codes over Hurwitz integers. Güzeltepe considered Hurwitz integers, which have the commutative property. In [11], Rohweder et al. presented a new algebraic construction technique for codes over Hurwitz integers that is inherently accompanied by a respective modulo operation. They considered as the domain of related modulo function in [11] is $\mathbb{Z}_{N(\alpha)} \times \mathbb{Z}_{N(\alpha)}$. Therefore, they constructed new sets of residual class, which has $N^2(\alpha)$ elements, with respect to the related modulo technique in the rings of Hurwitz integers using the primitive Lipschitz integers. In this study, we present a new algebraic construction technique that obtained with two modulo function. The domain of it is $\mathbb{Z}_{N(\alpha)}$. Therefore, we constructed new sets of residual class, which has $N(\alpha)$ elements, with respect to the related algebraic construction technique in the rings of Hurwitz integers by using the prime Hurwitz integers. Other studies on codes over Hurwitz integers were presented in papers to [6], [7], and [8].

The presentation of our results is organized as follows: In the next section, we give the necessary fundamental definitions used throughout this paper. In section 3, we present an algebraic construction technique, which is a modulo function formed depending upon two modulo functions such that the domain of it is $\mathbb{Z}_{N(\alpha)}$. Therefore, we construct new set of residual class with $N(\alpha)$ elements, with respect to the related modulo function in the rings of Hurwitz integers. In section 4, we give some of the natural results for mathematical notations in two modulo functions and the modulo function. In section 5, we give examples for the algebraic construction technique given in this study. In section 6, we investigate the code rates of some codes over Hurwitz integers. In section 7, we give the points and graph on $2D$ (two-dimensional) and $3D$ (three-dimensional) the elements of the residual class obtained with respect to the related modulo function in prime Hurwitz integers using graph layout methods such as spring, high-dimensional, and spiral embedding. We conclude this paper with section 8.

2. Preliminaries

We begin with some basic definitions.

Definition 2.1. Let α be a quaternion. $\alpha = \alpha_1 + \alpha_2 i + \alpha_3 j + \alpha_4 k$ is called a Hurwitz integer just if either $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{Z}$ or $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{Z} + \frac{1}{2}$. The set of all Hurwitz integers that denoted by $\mathcal{H}$ is shown by

$$\begin{aligned} \mathcal{H} &= \left\{\alpha_1 + \alpha_2 i + \alpha_3 j + \alpha_4 k : \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{Z} \text{ or } \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{Z} + \tfrac{1}{2}\right\} \\ &= \mathcal{H}(\mathbb{Z}) \bigcup \mathcal{H}(\mathbb{Z} + \tfrac{1}{2}). \end{aligned} \tag{2.1}$$

Example 2.2. i. $\pm 1 \pm \frac{1}{2}i \pm \frac{1}{2}j \pm \frac{1}{2}k$ is not a Hurwitz integer.
ii. $\pm\frac{1}{2} \pm \frac{1}{2}j$ is not a Hurwitz integer.
iii. $\pm\frac{1}{2} \pm \frac{1}{2}i \pm \frac{1}{2}j \pm \frac{1}{2}k$ is a Hurwitz integer.

The ring of Hurwitz integers forms a subring of the ring of all quaternions because of closed under multiplication and addition. The conjugate of a Hurwitz integer α is $\overline{\alpha} = \alpha_1 - \alpha_2 i - \alpha_3 j - \alpha_4 k$. The norm of a Hurwitz integer α is $N(\alpha) = \alpha \cdot \overline{\alpha} = \alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2$. The inverse of a Hurwitz integer α is $\alpha^{-1} = \frac{\overline{\alpha}}{N(\alpha)}$, where $N(\alpha) \neq 0$.

Definition 2.3. Let $\alpha = \alpha_1 + \alpha_2 i + \alpha_3 j + \alpha_4 k$ be a Hurwitz integers such that $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{Z}$. α is called a primitive Hurwitz integer just if the greatest common divisor of $\alpha_1, \alpha_2, \alpha_3$, and α_4 is 1, i.e. $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = 1$.

Definition 2.4. Let $\alpha = \alpha_1 + \alpha_2 i + \alpha_3 j + \alpha_4 k$ be a Hurwitz integer such that $\alpha_1 = \beta_1 + \frac{1}{2}, \alpha_2 = \beta_2 + \frac{1}{2}, \alpha_3 = \beta_2 + \frac{1}{2}$ and $\alpha_4 = \beta_4 + \frac{1}{2}$, where $\beta_1, \beta_2, \beta_3, \beta_4 \in \mathbb{Z}$. α is called a primitive Hurwitz integer just if the greatest common divisor of $2\beta_1 + 1, 2\beta_2 + 1, 2\beta_3 + 1$, and $2\beta_4 + 1$ is 1, i.e., $(2\beta_1 + 1, 2\beta_2 + 1, 2\beta_3 + 1, 2\beta_3 + 1) = 1$.

Definition 2.5. Let α be a Hurwitz integer. α is called a prime Hurwitz integer just if its norm is a prime integer.

From the Definition 2.3, the Definition 2.4 and the Definition 2.5, we could say that every prime Hurwitz integer is a primitive Hurwitz integer, but the converse may not be valid. Note that we consider prime Hurwitz integers in this study.

Definition 2.6. Let α and π be Hurwitz integers. If there exists $\lambda \in \mathcal{H}$ such that $q_1 - q_2 = \lambda\pi$, then $q_1, q_2 \in \mathcal{H}$ are said to be right congruent modulo α. This relation is denoted by $q_1 \equiv_r q_2$. Here, $\equiv_r$ is represented the right congruent. This relation $q_1 \equiv_r q_2$ is an equivalence relation. The elements in the right ideal

$$\langle\alpha\rangle = \{\lambda\alpha : \lambda \in \mathcal{H}\} \tag{2.2}$$

define a normal subgroup of the additive group of the ring $\mathcal{H}$. The set of cosets to $\langle\alpha\rangle$ in $\mathcal{H}$ defines Abelian group denoted by $\mathcal{H}_\alpha = \mathcal{H}/\langle\alpha\rangle$. Analogous results are valid for left congruent modulo α [5].

Note that we use left congruent modulo α in this study. In the following section, we will present an algebraic construction technique, obtained depending on two modulo functions, used to construct block codes over Hurwitz integers. We close this section by giving the definitions of two mathematical notations used in the modulo functions before going to the next section.

Definition 2.7. A notation for rounding to the nearest integer is denoted by $\lfloor\cdot\rceil$. It is rounding a rational number to the integer closest to it. Each component is separately rounding to the integer closest to it for a quaternion.

Definition 2.8. A notation for rounding to the nearest half-integer is denoted by $\lfloor\lfloor\cdot\rceil\rceil$. It is rounding a rational number to the half-integer closest to it. Each component is separately rounding to the half-integer closest to it for a quaternion [7].

We use to the Wolfram Mathematica 10.2 program for the computations in this study. The rounding is done in the direction of odd or even numbers due to the working principle of the Wolfram Mathematica 10.2. For example, $\lfloor \frac{3}{2} \rceil = 2$, $\lfloor \frac{5}{2} \rceil = 2$, $\lfloor -\frac{1}{2} \rceil = 0$, $\lfloor \frac{1}{2} \rceil = 0$, $\lfloor -\frac{3}{2} \rceil = -2$, $\lfloor -\frac{5}{2} \rceil = -2$, and so on. In this study, for a quaternion α, if any of its components is negative or zero, the rounding is done in the direction up, and otherwise, in the direction down.

Example 2.9. i. $\lfloor \frac{3}{2} \rceil = 1$, $\lfloor -\frac{1}{2} \rceil = 0$, $\lfloor -\frac{3}{2} \rceil = -1$, and so on.
ii. $\lfloor\lfloor 0 \rceil\rceil = \frac{1}{2}$, where $0 \in \mathbb{Z}$, $\lfloor\lfloor -1 \rceil\rceil = -\frac{1}{2}$, $\lfloor\lfloor 1 \rceil\rceil = \frac{1}{2}$, and so on.
iii. $\lfloor\lfloor 0 \rceil\rceil = \frac{1}{2} + \frac{1}{2}i + \frac{1}{2}j + \frac{1}{2}k$, where $0 \in \mathcal{H}$.

3. Algebraic construction technique

The first study on codes over high-dimensional algebraic structures was got by Huber in 1994. In [3], Huber shown how block codes over Gaussian integers by the modulo function, an algebraic construction technique, can be used for code over two-dimensional signal space. The modulo function $\mu : \mathcal{G} \to \mathcal{G}_\alpha$ is defined according to

$$\mu(\xi) = \xi \mod \alpha = \eta = \xi - \left[\frac{\xi\overline{\alpha}}{\alpha\overline{\alpha}}\right]\alpha, \tag{3.1}$$

where $\xi \in \mathcal{G}$ and α is a prime Gaussian integer [3]. Here $\mathcal{G}_\alpha$ is the set of residual class with respect to modulo α. In [2], this technique is applied to obtain codes over Lipschitz integers by Freudenberger et al. In [5], this technique is applied to obtain codes over Hurwitz integers by Güzeltepe. In [11], to obtain codes over Hurwitz integers by using Lipschitz integers, Rohweder et al. presented a new algebraic construction technique that is inherently accompanied by a respective modulo operation. They considered as the domain of related modulo function in [11] is $\mathbb{Z}_{N(\alpha)} \times \mathbb{Z}_{N(\alpha)}$. Therefore, they constructed new sets of residual class, which have $N^2(\alpha)$ elements, with respect to the related modulo technique in the rings of Hurwitz integers by using the primitive Lipschitz integers. In this study, we present an algebraic construction technique based on two modulo functions, and consider as the domain of related modulo function is $\mathbb{Z}_{N(\alpha)}$. Therefore, we construct new sets of residual class, which have $N(\alpha)$ elements, with respect to the related modulo function in the rings of Hurwitz integers by using the prime Hurwitz integers. The idea of the algebraic construction technique is to construct the residue class rings of Hurwitz integers which is the norm of each element is less than or equal to $\frac{N(\alpha)}{2}$. The technique presented in this study is to guarantee this idea.

Definition 3.1. Let α be a prime Hurwitz integer, and let $z \in \mathbb{Z}_{N(\alpha)}$. The modulo function $\mu : \mathbb{Z}_{N(\alpha)} \to \mathcal{H}_\alpha$ is defined by

$$\mu_\alpha(z) = \min\{\mu_\alpha^{(1)}(z), \mu_\alpha^{(2)}(z)\} \tag{3.2}$$

such that

$$\min\{\mu_\alpha^{(1)}(z), \mu_\alpha^{(2)}(z)\} = \begin{cases} \mu_\alpha^{(1)}(z), & \text{if } N(\mu_\alpha^{(1)}(z)) \leq N(\mu_\alpha^{(2)}(z)) \\ \mu_\alpha^{(2)}(z), & \text{otherwise} \end{cases}, \tag{3.3}$$

where

$$\mu_\alpha^{(1)}(z) = z \mod \alpha = z - \alpha\lfloor \frac{\overline{\alpha}z}{N(\alpha)} \rceil \tag{3.4}$$

and

$$\mu_\alpha^{(2)}(z) = z \mod \alpha = z - \alpha\lfloor\lfloor \frac{\overline{\alpha}z}{N(\alpha)} \rceil\rceil. \tag{3.5}$$

$\mathbb{Z}_{N(\alpha)}$ is the well-known residual class ring of ordinary integers with $N(\alpha)$ elements, and $\mathcal{H}_\alpha$ is the residual class ring of prime Hurwitz integer α. The set of quotient ring of the Hurwitz integers that denoted by $\mathcal{H}_\alpha$ is shown by

$$\mathcal{H}_\alpha = \{\mu_\alpha(z) = \min\{\mu_\alpha^{(1)}(z), \mu_\alpha^{(2)}(z)\} | z \in \mathbb{Z}_{N(\pi)}\}, \tag{3.6}$$

where α is a prime Hurwitz integer.

$\mu_\alpha^{(1)}(z)$ is a version of the modulo function in eq. 3.1 arranged to the left congruent modulo α. $\mu_\alpha^{(2)}(z)$ is a version of the modulo function in eq. 3.1 arranged with respect to the left congruent modulo α and the notation $\lfloor\lfloor\cdot\rceil\rceil$ (see Definition 2.8). $\mu_\alpha(z)$ is the modulo function given the minimum remainder with respect to $\mu_\alpha^{(1)}(z)$ and $\mu_\alpha^{(2)}(z)$. In this way, we obtain the remainder whose norm is less than or equal to $\frac{N(\alpha)}{2}$. The technique in Definition 3.1 shows that it is a suitable modulo function for reaching the minimum energy because of this property.

Proposition 3.2. *Let α be a prime Hurwitz integer, and let $z = 0$. Then*

i. $\mu_\alpha^{(1)}(0) = 0$,
ii. $N(\mu_\alpha^{(2)}(0)) = N(\alpha)$,
iii. $\mu_\alpha^{(2)}(0) \equiv 0 \mod \alpha$.

Proof. Let $\alpha = \alpha_1 + \alpha_2 i + \alpha_3 j + \alpha_4 k$ be a prime Hurwitz integer, and $z = 0$. Then

i. The proof can be easily seen from eq. 3.4.
ii. From eq. 3.5,

$$\begin{aligned} \mu_\alpha^{(2)}(0) &= 0 - \alpha\lfloor\lfloor \tfrac{\overline{\alpha}0}{N(\alpha)} \rceil\rceil \\ &= -(\alpha_1 + \alpha_2 i + \alpha_3 j + \alpha_4 k)\lfloor\lfloor \tfrac{(\alpha_1 - \alpha_2 i - \alpha_3 j - \alpha_4 k)0}{N(\alpha)} \rceil\rceil \\ &= -(\alpha_1 + \alpha_2 i + \alpha_3 j + \alpha_4 k)\lfloor\lfloor 0 \rceil\rceil \\ &= -(\alpha_1 + \alpha_2 i + \alpha_3 j + \alpha_4 k)(\tfrac{1}{2} + \tfrac{1}{2}i + \tfrac{1}{2}j + \tfrac{1}{2}k) \\ &= -\tfrac{\alpha_1}{2} - \tfrac{\alpha_1}{2}i - \tfrac{\alpha_1}{2}j - \tfrac{\alpha_1}{2}k - \tfrac{\alpha_2}{2}i + \tfrac{\alpha_2}{2} - \tfrac{\alpha_2}{2}k + \tfrac{\alpha_2}{2}j \\ &\quad -\tfrac{\alpha_3}{2}j + \tfrac{\alpha_3}{2}k + \tfrac{\alpha_3}{2} - \tfrac{\alpha_3}{2}i - \tfrac{\alpha_4}{2}k - \tfrac{\alpha_4}{2}j + \tfrac{\alpha_4}{2}i + \tfrac{\alpha_4}{2} \\ &= \tfrac{-\alpha_1+\alpha_2+\alpha_3+\alpha_4}{2} + \tfrac{-\alpha_1-\alpha_2-\alpha_3+\alpha_4}{2}i + \tfrac{-\alpha_1+\alpha_2-\alpha_3-\alpha_4}{2}j + \tfrac{-\alpha_1-\alpha_2+\alpha_3-\alpha_4}{2}k. \end{aligned} \tag{3.7}$$

Herefrom,

$$\begin{aligned} N(\mu_\alpha^{(2)}(0)) &= N(\tfrac{-\alpha_1+\alpha_2+\alpha_3+\alpha_4}{2} + \tfrac{-\alpha_1-\alpha_2-\alpha_3+\alpha_4}{2}i + \tfrac{-\alpha_1+\alpha_2-\alpha_3-\alpha_4}{2}j + \tfrac{-\alpha_1-\alpha_2+\alpha_3-\alpha_4}{2}k) \\ &= \tfrac{4(\alpha_1^2+\alpha_2^2+\alpha_3^2+\alpha_4^2)}{4} \\ &= \alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2 \\ &= N(\alpha). \end{aligned} \tag{3.8}$$

iii. Let β be a Hurwitz integer such that $N(\beta) \leq N(\alpha)$. If $N(\beta) = N(\alpha)$, then $\alpha \mid \beta$. So, $\beta = \alpha\lambda$, where $\lambda \in \mathcal{H}$. Note that, in this study, we consider the left congruent modulo α. From $(ii.)$, $\alpha \mid \mu_\alpha^{(2)}(0)$ because of $N(\mu_\alpha^{(2)}(0)) = N(\alpha)$. Consequently,

$$\mu_\alpha^{(2)}(0) \equiv 0 \mod \alpha. \tag{3.9}$$

This completes the proof.

□

The set $\mathcal{H}_\alpha$ contains $N(\alpha)$ elements. The modulo function μ in the Definition 3.1 defines a bijective mapping from $\mathbb{Z}_{N(\alpha)}$ into $\mathcal{H}_\alpha$. In other words, there exists a ring isomorphism from $\mathbb{Z}_{N(\alpha)}$ into $\mathcal{H}_\alpha$. By the following theorems, we show that the modulo function μ in the Definition 3.1 is a ring homomorphism and a ring isomorphism, respectively.

Theorem 3.3. *Let α be a prime Hurwitz integer. The modulo function $\mu : \mathbb{Z}_{N(\alpha)} \to \mathcal{H}_\alpha$ is a ring homomorphism.*

Proof. Let α be a prime Hurwitz integer, and let $z_1, z_2 \in \mathbb{Z}_{N(\alpha)}$. From eq. 3.2, $\mu_\alpha(z_1) = \min\{\mu_\alpha^{(1)}(z_1), \mu_\alpha^{(2)}(z_1)\}$ and $\mu_\alpha(z_2) = \min\{\mu_\alpha^{(1)}(z_2), \mu_\alpha^{(2)}(z_2)\}$. There are four probable case with respect to eq. 3.2;

i. $\mu_\alpha(z_1) = \mu_\alpha^{(1)}(z_1)$ and $\mu_\alpha(z_2) = \mu_\alpha^{(1)}(z_2)$ or,
ii. $\mu_\alpha(z_1) = \mu_\alpha^{(1)}(z_1)$ and $\mu_\alpha(z_2) = \mu_\alpha^{(2)}(z_2)$ or,
iii. $\mu_\alpha(z_1) = \mu_\alpha^{(2)}(z_1)$ and $\mu_\alpha(z_2) = \mu_\alpha^{(1)}(z_2)$ or,
iv. $\mu_\alpha(z_1) = \mu_\alpha^{(2)}(z_1)$ and $\mu_\alpha(z_2) = \mu_\alpha^{(2)}(z_2)$.

Let us show that $\mu_\alpha(z_1 + z_2) = \mu_\alpha(z_1) + \mu_\alpha(z_2)$.

(*i.*) Let $\mu_\alpha(z_1) = \mu_\alpha^{(1)}(z_1)$ and $\mu_\alpha(z_2) = \mu_\alpha^{(1)}(z_2)$. From eq. 3.4,

$$\begin{aligned} \mu_\alpha^{(1)}(z_1) + \mu_\alpha^{(1)}(z_2) &= z_1 - \alpha\lfloor \tfrac{\overline{\alpha} z_1}{N(\alpha)} \rceil + z_2 - \alpha\lfloor \tfrac{\overline{\alpha} z_2}{N(\alpha)} \rceil \\ &= z_1 + z_2 - \alpha(\lfloor \tfrac{\overline{\alpha} z_1}{N(\alpha)} \rceil + \lfloor \tfrac{\overline{\alpha} z_2}{N(\alpha)} \rceil). \end{aligned} \tag{3.10}$$

There exist $\lambda_1, \lambda_2 \in \mathcal{H}$ such that $\lfloor \frac{\overline{\alpha} z_1}{N(\alpha)} \rceil = \lambda_1$ and $\lfloor \frac{\overline{\alpha} z_2}{N(\alpha)} \rceil = \lambda_2$. Hereby,

$$\mu_\alpha^{(1)}(z_1) + \mu_\alpha^{(1)}(z_2) = z_1 + z_2 - \alpha(\lambda_1 + \lambda_2). \tag{3.11}$$

Let $\lambda_1 + \lambda_2 = \lambda$, where $\lambda \in \mathcal{H}$. Therefore,

$$\mu_\alpha^{(1)}(z_1) + \mu_\alpha^{(1)}(z_2) = z_1 + z_2 - \alpha\lambda. \tag{3.12}$$

Since $\mu_\alpha(z_1 + z_2) \equiv (z_1 + z_2) \mod \alpha$, then there exists $\exists\beta \in \mathcal{H}$ such that $\mu_\alpha(z_1 + z_2) = z_1 + z_2 - \alpha\beta$. So, we have

$$\mu_\alpha(z_1 + z_2) = \mu_\alpha^{(1)}(z_1) + \mu_\alpha^{(1)}(z_2). \tag{3.13}$$

(*ii.*) Let $\mu_\alpha(z_1) = \mu_\alpha^{(1)}(z_1)$ and $\mu_\alpha(z_2) = \mu_\alpha^{(2)}(z_2)$. From eq. 3.4 and eq. 3.5,

$$\begin{aligned} \mu_\alpha^{(1)}(z_1) + \mu_\alpha^{(2)}(z_2) &= z_1 - \alpha\lfloor \tfrac{\overline{\alpha} z_1}{N(\alpha)} \rceil + z_2 - \alpha\lfloor\lfloor \tfrac{\overline{\alpha} z_2}{N(\alpha)} \rceil\rceil \\ &= z_1 + z_2 - \alpha(\lfloor \tfrac{\overline{\alpha} z_1}{N(\alpha)} \rceil + \lfloor\lfloor \tfrac{\overline{\alpha} z_2}{N(\alpha)} \rceil\rceil). \end{aligned} \tag{3.14}$$

There exist $\lambda_1, \lambda_2 \in \mathcal{H}$ such that $\lfloor \frac{\overline{\alpha} z_1}{N(\alpha)} \rceil = \lambda_1$ and $\lfloor\lfloor \frac{\overline{\alpha} z_2}{N(\alpha)} \rceil\rceil = \lambda_2$. Hereby,

$$\mu_\alpha^{(1)}(z_1) + \mu_\alpha^{(2)}(z_2) = z_1 + z_2 - \alpha(\lambda_1 + \lambda_2). \tag{3.15}$$

Let $\lambda_1 + \lambda_2 = \lambda$, where $\lambda \in \mathcal{H}$. Therefore,

$$\mu_\alpha^{(1)}(z_1) + \mu_\alpha^{(2)}(z_2) = z_1 + z_2 - \alpha\lambda. \tag{3.16}$$

Since $\mu_\alpha(z_1 + z_2) \equiv (z_1 + z_2) \mod \alpha$, then there exists $\exists\beta \in \mathcal{H}$ such that $\mu_\alpha(z_1 + z_2) = z_1 + z_2 - \alpha\beta$. So, we have

$$\mu_\alpha(z_1 + z_2) = \mu_\alpha^{(1)}(z_1) + \mu_\alpha^{(2)}(z_2). \tag{3.17}$$

Similarly, we can also show the cases in (*iii.*) and (*iv.*). Consequently,

$$\mu_\alpha(z_1 + z_2) = \mu_\alpha(z_1) + \mu_\alpha(z_2). \tag{3.18}$$

On the other hand, let us show that $\mu_\alpha(z_1 z_2) = \mu_\alpha(z_1)\mu_\alpha(z_2)$.

(*i.*) Let $\mu_\alpha(z_1) = \mu_\alpha^{(1)}(z_1)$ and $\mu_\alpha(z_2) = \mu_\alpha^{(1)}(z_2)$. From eq. 3.4,

$$\begin{aligned} \mu_\alpha^{(1)}(z_1)\mu_\alpha^{(1)}(z_2) &= (z_1 - \alpha\lfloor \tfrac{\overline{\alpha} z_1}{N(\alpha)} \rceil)(z_2 - \alpha\lfloor \tfrac{\overline{\alpha} z_2}{N(\alpha)} \rceil) \\ &= z_1 z_2 - z_1\alpha\lfloor \tfrac{\overline{\alpha} z_2}{N(\alpha)} \rceil - z_2\alpha\lfloor \tfrac{\overline{\alpha} z_1}{N(\alpha)} \rceil + \alpha\lfloor \tfrac{\overline{\alpha} z_1}{N(\alpha)} \rceil\alpha\lfloor \tfrac{\overline{\alpha} z_2}{N(\alpha)} \rceil. \end{aligned} \tag{3.19}$$

There exist $\lambda_1, \lambda_2 \in \mathcal{H}$ such that $\lfloor \frac{\overline{\alpha} z_1}{N(\alpha)} \rceil = \lambda_1$ and $\lfloor \frac{\overline{\alpha} z_2}{N(\alpha)} \rceil = \lambda_2$. Hereby,

$$\begin{aligned} \mu_\alpha^{(1)}(z_1)\mu_\alpha^{(1)}(z_2) &= z_1z_2 - z_1\alpha\lambda_2 - z_2\alpha\lambda_1 + \alpha\lambda_1\alpha\lambda_2 \\ &= z_1z_2 - \alpha(z_1\lambda_2 + z_2\lambda_1 + \lambda_1\alpha\lambda_2). \end{aligned} \tag{3.20}$$

Let $z_1\lambda_2 + z_2\lambda_1 + \lambda_1\alpha\lambda_2 = \lambda$, where $\lambda \in \mathcal{H}$. Therefore,

$$\mu_\alpha^{(1)}(z_1)\mu_\alpha^{(1)}(z_2) = z_1z_2 - \alpha\lambda. \tag{3.21}$$

Since $\mu_\alpha(z_1z_2) \equiv (z_1z_2) \mod \alpha$, then there exists $\exists\beta \in \mathcal{H}$ such that $\mu_\alpha(z_1z_2) = z_1z_2 - \alpha\beta$. So, we have

$$\mu_\alpha(z_1z_2) = \mu_\alpha^{(1)}(z_1)\mu_\alpha^{(1)}(z_2). \tag{3.22}$$

($ii.$) Let $\mu_\alpha(z_1) = \mu_\alpha^{(1)}(z_1)$ and $\mu_\alpha(z_2) = \mu_\alpha^{(2)}(z_2)$. From eq. 3.4, and eq. 3.5,

$$\begin{aligned} \mu_\alpha^{(1)}(z_1)\mu_\alpha^{(2)}(z_2) &= (z_1 - \alpha\lfloor \tfrac{\overline{\alpha} z_1}{N(\alpha)} \rceil)(z_2 - \alpha\lfloor\lfloor \tfrac{\overline{\alpha} z_2}{N(\alpha)} \rceil\rceil) \\ &= z_1z_2 - z_1\alpha\lfloor\lfloor \tfrac{\overline{\alpha} z_2}{N(\alpha)} \rceil\rceil - z_2\alpha\lfloor \tfrac{\overline{\alpha} z_1}{N(\alpha)} \rceil + \alpha\lfloor \tfrac{\overline{\alpha} z_1}{N(\alpha)} \rceil\alpha\lfloor\lfloor \tfrac{\overline{\alpha} z_2}{N(\alpha)} \rceil\rceil. \end{aligned} \tag{3.23}$$

There exist $\lambda_1, \lambda_2 \in \mathcal{H}$ such that $\lfloor \frac{\overline{\alpha} z_1}{N(\alpha)} \rceil = \lambda_1$ and $\lfloor\lfloor \frac{\overline{\alpha} z_2}{N(\alpha)} \rceil\rceil = \lambda_2$. Hereby,

$$\begin{aligned} \mu_\alpha^{(1)}(z_1)\mu_\alpha^{(2)}(z_2) &= z_1z_2 - z_1\alpha\lambda_2 - z_2\alpha\lambda_1 + \alpha\lambda_1\alpha\lambda_2 \\ &= z_1z_2 - \alpha(z_1\lambda_2 + z_2\lambda_1 + \lambda_1\alpha\lambda_2). \end{aligned} \tag{3.24}$$

Let $z_1\lambda_2 + z_2\lambda_1 + \lambda_1\alpha\lambda_2 = \lambda$, where $\lambda \in \mathcal{H}$. Therefore,

$$\mu_\alpha^{(1)}(z_1)\mu_\alpha^{(2)}(z_2) = z_1z_2 - \alpha\lambda. \tag{3.25}$$

Since $\mu_\alpha(z_1z_2) \equiv (z_1z_2) \mod \alpha$, then there exists $\exists\beta \in \mathcal{H}$ such that $\mu_\alpha(z_1z_2) = z_1z_2 - \alpha\beta$. So, we have

$$\mu_\alpha(z_1z_2) = \mu_\alpha^{(1)}(z_1)\mu_\alpha^{(2)}(z_2). \tag{3.26}$$

Similarly, we can also show the cases in ($iii.$) and ($iv.$). Consequently,

$$\mu_\alpha(z_1z_2) = \mu_\alpha(z_1)\mu_\alpha(z_2). \tag{3.27}$$

Last of all, the modulo function $\mu : \mathbb{Z}_{N(\alpha)} \to \mathcal{H}_\alpha$ is a ring homomorphism. This completes this proof. □

Theorem 3.4. *Let α be a prime Hurwitz integer. The modulo function $\mu : \mathbb{Z}_{N(\alpha)} \to \mathcal{H}_\alpha$ is a ring isomorphism ring. Namely, $\mathbb{Z}_{N(\alpha)} \cong \mathcal{H}_\alpha$.*

Proof. Let α be a prime Hurwitz integer, and let $z \in \mathbb{Z}_{N(\alpha)}$. According to Theorem 3.3, the modulo function μ that defined in the Definition 3.1 is a ring homomorphism. We should show that it is a bijective ring homomorphism, i.e., a ring isomorphism. the modulo function μ that defined in the Definition 3.1 is a surjective ring homomorphism because of $\mathrm{Im}\mu = \{\mu_\alpha(z) = \min\{\mu_\alpha^{(1)}(z), \mu_\alpha^{(2)}(z)\} : z \in \mathbb{Z}_{N(\alpha)}\} = \mathcal{H}_\alpha$. If $z = 0$, then

$$\mu_\alpha(0) = \min\{\mu_\alpha^{(1)}(0), \mu_\alpha^{(2)}(0)\}. \tag{3.28}$$

From Proposition 3.2,

$$\mu_\alpha(0) = 0. \tag{3.29}$$

If $z \neq 0$, then $\mu_\alpha(z)$ is greater than or equal to 1. Hereby, the modulo function μ that defined in the Definition 3.1 is an injective ring homomorphism because of $Ker\mu = \{z \in \mathbb{Z}_{N(\alpha)} : \mu_\alpha(z) = 0\} = \{z \in \mathbb{Z}_{N(\alpha)} : z = 0\} = \{0\}$. So, the modulo function μ that defined in the Definition 3.1 is a ring isomorphism since it is both a surjective ring homomorphism and an injective ring homomorphism, i.e. $\mathbb{Z}_{N(\alpha)} \cong \mathcal{H}_\alpha$. This completes the proof. □

Definition 3.5. Let $\mathbb{F}_q$ be a finite field with q elements. C is called a code if C is a nonempty subset of $\mathbb{F}_q^n$. An element of C is called a codeword in C. A linear code C of length n over $\mathbb{F}_q$ is defined to be a subspace of $\mathbb{F}_q^n$.

Definition 3.6. A code C of length n is a subset of the direct product $\mathcal{H}^n$ of n copies of $\mathcal{H}$. In each of the cases we consider, $\mathcal{H}$ is an Abelian group, and thus the same is true for $\mathcal{H}^n$. A code C is a group code if it is a subgroup of $\mathcal{H}^n$, or equivalently as $\mathcal{H}^n$ is a finite group,

$$c, c^{'} \in C \Rightarrow c - c^{'} \in C.$$

In the case when $\mathcal{H}$ is a finite field, and thus $\mathcal{H}^n$ is a vector space of dimension n over $\mathcal{H}$, then a linear code is a subspace C of $\mathcal{H}^n$. Here we say that a code C in $\mathcal{H}^n$ is an $(n, k) - code$ if the size of C is equal to $|\mathcal{H}|^k$ (The case $k = 0$ is of less interest, and thus left aside) [8].

Definition 3.7. Let α be a prime Hurwitz integer, and $z \in \mathbb{Z}_{N(\alpha)}$. The average energy of $\mathcal{H}_\alpha$ that denoted by $\mathcal{E}_\alpha$ is computed by

$$\mathcal{E}_\alpha = \frac{1}{N(\alpha)} \sum_{z=0}^{N(\alpha)-1} N(\mu_\alpha(z)). \tag{3.30}$$

4. Some results

This section presents some results related to the modulo function μ defined in definition 3.1, and the mathematical notations given in eq. 3.4 and eq. 3.5. $\mathcal{H}_\alpha^{(1)}$ is represented the set of residual class with respect to eq. 3.4, $\mathcal{H}_\alpha^{(2)}$ is represented the set of residual class with respect to eq. 3.5, and $\mathcal{H}_\alpha$ is represented the set of residual class with respect to eq. 3.2. $2 + i$, $3 + 2j$, and $4i + k$ are prime Hurwitz integers that have two-components, and so on. $3 + i + j$, $3 + 2j + 2k$, and $4i + 3j + 2k$ are prime Hurwitz integers that have three-components, and so on.

Corollary 4.1. *Let α be a prime Hurwitz integer that has two components, where $N(\alpha) \geq 13$. Then, we have*

$$\mathcal{H}_\alpha = \mathcal{H}_\alpha^{(1)}. \tag{4.1}$$

Proof. Let α be a prime Hurwitz integer that have two-components, where $N(\alpha) \geq 13$. Note that $\mathcal{H}_\alpha^{(1)} = \{\mu_\alpha^{(1)}(z) | z \in \mathbb{Z}_\alpha\}$, $\mathcal{H}_\alpha^{(2)} = \{\mu_\alpha^{(2)}(z) | z \in \mathbb{Z}_\alpha\}$, and $\mathcal{H}_\alpha = \{\mu_\alpha(z) = \min\{\mu_\alpha^{(1)}(z), \mu_\alpha^{(2)}(z)\} | z \in \mathbb{Z}_{N(\pi)}\}$. We must show that $\mathcal{H}_\alpha = \{\mu_\alpha(z) = \mu_\alpha^{(1)}(z) | z \in \mathbb{Z}_\alpha\}$. With respect to eq. 3.4,

$$\mu_\alpha^{(1)}(z) = z - \alpha \lfloor \frac{\overline{\alpha} z}{N(\alpha)} \rceil, \tag{4.2}$$

where $z \in \mathbb{Z}_{N(\alpha)}$. There exist $\exists \lambda_1 \in \mathcal{H}$ such that $\lfloor \frac{\overline{\alpha} z}{N(\alpha)} \rceil = \lambda_1$. Therefore,

$$\mu_\alpha^{(1)}(z) = z - \alpha \lambda_1. \tag{4.3}$$

Since α is a prime Hurwitz integer that has two-components such that $N(\alpha) \geq 13$, then $\mu_\alpha^{(1)}(z)$ is to a Hurwitz integer that has two-components. So,

$$N(\mu_\alpha^{(1)}(z)) \leq \frac{N(\alpha)}{2}. \tag{4.4}$$

For $\forall z \in \mathbb{Z}_{N(\alpha)}$, it is easily seen from eq. 3.4. Similarly, with respect to eq. 3.5,

$$\mu_\alpha^{(2)}(z) = z - \alpha \lfloor \lfloor \frac{\overline{\alpha} z}{N(\alpha)} \rceil \rceil, \tag{4.5}$$

where $z \in \mathbb{Z}_{N(\alpha)}$. There exist $\exists \lambda_2 \in \mathcal{H}$ such that $\lfloor \lfloor \frac{\overline{\alpha} z}{N(\alpha)} \rceil \rceil = \lambda_2$. Therefore,

$$\mu_\alpha^{(2)}(z) = z - \alpha \lambda_2. \tag{4.6}$$

Although α is a prime Hurwitz integer that has two-components such that $N(\alpha) \geq 13$, $\mu_\alpha^{(2)}(z)$ is to a Hurwitz integer, which its components are in $\mathbb{Z} + \frac{1}{2}$ (see the Definition 2.8).

A component of $\mu_\alpha^{(2)}(z)$ is equal to $\frac{a-b}{2}$. A component of $\mu_\alpha^{(2)}(z)$ is equal to $\frac{a+b}{2}$, too. For $\forall z \in \mathbb{Z}_{N(\alpha)}$, it is easily seen from eq. 3.5. Since the sum of these components squares is $\frac{a^2+b^2}{2}$, and $N(\mu_\alpha^{(2)}(0)) = N(\alpha)$ (see Proposition 3.2) for $z = 0$, then

$$\frac{N(\alpha)}{2} \leq N(\mu_\alpha^{(2)}(z)) \leq N(\alpha). \tag{4.7}$$

From eq. 4.4 and eq. 4.7,

$$N(\mu_\alpha^{(1)}(z)) < N(\mu_\alpha^{(2)}(z)). \tag{4.8}$$

With respect to Definition 3.1,

$$\mathcal{H}_\alpha = \{\mu_\alpha^{(1)}(z) | z \in \mathbb{Z}_{N(\alpha)}\}. \tag{4.9}$$

Last of all, $\mathcal{H}_\alpha = \mathcal{H}_\alpha^{(1)}$. This completes the proof. □

Proposition 4.2. *Let $\beta = \beta_1 + \beta_2 i + \beta_3 j + \beta_4 k$ be a Hurwitz integer. If $\beta_1, \beta_2, \beta_3, \beta_4 \in \mathbb{Z}$, then,*

$$\lfloor \beta \rceil = \beta, \tag{4.10}$$

and

$$\lfloor \lfloor \beta \rceil \rceil = \beta \pm \frac{1}{2} \pm \frac{1}{2} i \pm \frac{1}{2} j \pm \frac{1}{2} k. \tag{4.11}$$

If $\beta_1, \beta_2, \beta_3, \beta_4 \in \mathbb{Z} + \frac{1}{2}$, then,

$$\lfloor \beta \rceil = \beta \pm \frac{1}{2} \pm \frac{1}{2} i \pm \frac{1}{2} j \pm \frac{1}{2} k, \tag{4.12}$$

and

$$\lfloor \lfloor \beta \rceil \rceil = \beta. \tag{4.13}$$

Proof. Let $\beta = \beta_1 + \beta_2 i + \beta_3 j + \beta_4 k$ be a Hurwitz integer. If $\beta_1, \beta_2, \beta_3, \beta_4 \in \mathbb{Z}$, then

$$\begin{aligned} \lfloor \beta \rceil &= \lfloor \beta_1 + \beta_2 i + \beta_3 j + \beta_4 k \rceil \\ &= \lfloor \beta_1 \rceil + \lfloor \beta_2 \rceil i + \lfloor \beta_3 \rceil j + \lfloor \beta_4 \rceil k \\ &= \beta_1 + \beta_2 i + \beta_3 j + \beta_4 k \\ &= \beta. \end{aligned} \tag{4.14}$$

On the other hand,

$$\begin{aligned} \lfloor \lfloor \beta \rceil \rceil &= \lfloor \lfloor \beta_1 + \beta_2 i + \beta_3 j + \beta_4 k \rceil \rceil \\ &= \lfloor \lfloor \beta_1 \rceil \rceil + \lfloor \lfloor \beta_2 \rceil \rceil i + \lfloor \lfloor \beta_3 \rceil \rceil j + \lfloor \lfloor \beta_4 \rceil \rceil k. \end{aligned} \tag{4.15}$$

Since the property of rounding notation, then,

$$\begin{aligned} \lfloor \lfloor \beta \rceil \rceil &= \beta_1 \pm \tfrac{1}{2} + (\beta_2 \pm \tfrac{1}{2}) i + (\beta_3 \pm \tfrac{1}{2}) j + (\beta_4 \pm \tfrac{1}{2}) k \\ &= \beta_1 + \beta_2 i + \beta_3 j + \beta_4 k \pm \tfrac{1}{2} \pm \tfrac{1}{2} i \pm \tfrac{1}{2} j \pm \tfrac{1}{2} k \\ &= \beta \pm \tfrac{1}{2} \pm \tfrac{1}{2} i \pm \tfrac{1}{2} j \pm \tfrac{1}{2} k. \end{aligned} \tag{4.16}$$

If $\beta_1, \beta_2, \beta_3, \beta_4 \in \mathbb{Z} + \frac{1}{2}$, then,

$$\begin{aligned} \lfloor \beta \rceil &= \lfloor \beta_1 + \beta_2 i + \beta_3 j + \beta_4 k \rceil \\ &= \lfloor \beta_1 \rceil + \lfloor \beta_2 \rceil i + \lfloor \beta_3 \rceil j + \lfloor \beta_4 \rceil k. \end{aligned} \tag{4.17}$$

Since the property of rounding notation, then,

$$\begin{aligned} \lfloor \beta \rceil &= \beta_1 \pm \tfrac{1}{2} + (\beta_2 \pm \tfrac{1}{2}) i + (\beta_3 \pm \tfrac{1}{2}) j + (\beta_4 \pm \tfrac{1}{2}) k \\ &= \beta_1 + \beta_2 i + \beta_3 j + \beta_4 k \pm \tfrac{1}{2} \pm \tfrac{1}{2} i \pm \tfrac{1}{2} j \pm \tfrac{1}{2} k \\ &= \beta \pm \tfrac{1}{2} \pm \tfrac{1}{2} i \pm \tfrac{1}{2} j \pm \tfrac{1}{2} k. \end{aligned} \tag{4.18}$$

On the other hand,

$$\begin{aligned}\lfloor\lfloor\beta\rceil\rceil &= \lfloor\lfloor\beta_1+\beta_2 i+\beta_3 j+\beta_4 k\rceil\rceil \\ &= \lfloor\lfloor\beta_1\rceil\rceil+\lfloor\lfloor\beta_2\rceil\rceil i+\lfloor\lfloor\beta_3\rceil\rceil j+\lfloor\lfloor\beta_4\rceil\rceil k \\ &= \beta_1+\beta_2 i+\beta_3 j+\beta_4 k \\ &= \beta.\end{aligned} \tag{4.19}$$

This completes the proof. □

Corollary 4.3. *Let $\beta=\beta_1+\beta_2 i+\beta_3 j+\beta_4 k$ and $\pi=\pi_1+\pi_2 i+\pi_3 j+\pi_4 k$ be Hurwitz integers. If $\beta_1,\beta_2,\beta_3,\beta_4\in\mathbb{Z}$, and $\pi_1,\pi_2,\pi_3,\pi_4\in\mathbb{Z}$, then,*

$$\lfloor\pi\pm\beta\rceil=\lfloor\pi\rceil\pm\lfloor\beta\rceil=\pi\pm\beta, \tag{4.20}$$

and

$$\lfloor\lfloor\pi\pm\beta\rceil\rceil=\left(\pi\pm\frac{1}{2}\pm\frac{1}{2}i\pm\frac{1}{2}j\pm\frac{1}{2}k\right)\pm\left(\beta\pm\frac{1}{2}\pm\frac{1}{2}i\pm\frac{1}{2}j\pm\frac{1}{2}k\right). \tag{4.21}$$

If $\beta_1,\beta_2,\beta_3,\beta_4\in\mathbb{Z}$, and $\pi_1,\pi_2,\pi_3,\pi_4\in\mathbb{Z}+\frac{1}{2}$ (or $\beta_1,\beta_2,\beta_3,\beta_4\in\mathbb{Z}+\frac{1}{2}$, and $\pi_1,\pi_2,\pi_3,\pi_4\in\mathbb{Z}$), then,

$$\lfloor\pi\pm\beta\rceil=\lfloor\pi\rceil\pm\lfloor\beta\rceil=\pi\pm\beta\pm\frac{1}{2}\pm\frac{1}{2}i\pm\frac{1}{2}j\pm\frac{1}{2}k, \tag{4.22}$$

and

$$\lfloor\lfloor\pi\pm\beta\rceil\rceil=\lfloor\lfloor\pi\rceil\rceil\pm\lfloor\lfloor\beta\rceil\rceil=\pi\pm\beta\pm\frac{1}{2}\pm\frac{1}{2}i\pm\frac{1}{2}j\pm\frac{1}{2}k. \tag{4.23}$$

If $\beta_1,\beta_2,\beta_3,\beta_4\in\mathbb{Z}+\frac{1}{2}$, and $\pi_1,\pi_2,\pi_3,\pi_4\in\mathbb{Z}+\frac{1}{2}$, then,

$$\lfloor\pi\pm\beta\rceil=\lfloor\pi\rceil\pm\lfloor\beta\rceil=\left(\pi\pm\frac{1}{2}\pm\frac{1}{2}i\pm\frac{1}{2}j\pm\frac{1}{2}k\right)\pm\left(\beta\pm\frac{1}{2}\pm\frac{1}{2}i\pm\frac{1}{2}j\pm\frac{1}{2}k\right), \tag{4.24}$$

and

$$\lfloor\lfloor\pi\pm\beta\rceil\rceil=\lfloor\lfloor\pi\rceil\rceil\pm\lfloor\lfloor\beta\rceil\rceil=\pi\pm\beta. \tag{4.25}$$

Proof. The proof can be easily seen from the Proposition 4.2. □

Proposition 4.4. *Let $\beta=\beta_1+\beta_2 i+\beta_3 j+\beta_4 k$ and $\pi=\pi_1+\pi_2 i+\pi_3 j+\pi_4 k$ be Hurwitz integers. We consider π_1 and β_1. If $\pi_1,\beta_1\in\mathbb{Z}$, then,*

$$\lfloor\pi_1\beta_1\rceil=\pi_1\beta_1, \tag{4.26}$$

and

$$\lfloor\lfloor\pi_1\beta_1\rceil\rceil=\pi_1\beta_1\pm\frac{1}{2}. \tag{4.27}$$

Let $\pi_1\in\mathbb{Z}$ and $\beta_1\in\mathbb{Z}+\frac{1}{2}$. If π_1 is an even integer, then,

$$\lfloor\pi_1\beta_1\rceil=\pi_1\beta_1, \tag{4.28}$$

and

$$\lfloor\lfloor\pi_1\beta_1\rceil\rceil=\pi_1\beta_1\pm\frac{1}{2}. \tag{4.29}$$

If π_1 is an odd integer, then,

$$\lfloor\pi_1\beta_1\rceil=\pi_1\beta_1\pm\frac{1}{2}, \tag{4.30}$$

and

$$\lfloor\lfloor\pi_1\beta_1\rceil\rceil=\pi_1\beta_1. \tag{4.31}$$

Let $\pi_1,\beta_1\in\mathbb{Z}+\frac{1}{2}$ such that $\pi=\frac{\lambda_1}{2}$ and $\beta=\frac{\lambda_2}{2}$, where λ_1 and λ_2 are odd integers. If $\lambda_1\lambda_2\equiv 1 \mod 4$, then,

$$\lfloor\pi_1\beta_1\rceil=\pm\pi_1\beta_1\mp\frac{1}{4}, \tag{4.32}$$

and

$$\lfloor\lfloor\pi_1\beta_1\rceil\rceil=\pm\pi_1\beta_1\pm\frac{1}{4}. \tag{4.33}$$

If $\lambda_1\lambda_2 \equiv 3 \mod 4$, *then,*

$$\lfloor \pi_1\beta_1 \rceil = \pm\pi_1\beta_1 \pm \frac{1}{4}, \tag{4.34}$$

and

$$\lfloor\lfloor \pi_1\beta_1 \rceil\rceil = \pm\pi_1\beta_1 \mp \frac{1}{4}. \tag{4.35}$$

Proof. Let $\beta = \beta_1 + \beta_2 i + \beta_3 j + \beta_4 k$ and $\pi = \pi_1 + \pi_2 i + \pi_3 j + \pi_4 k$ be Hurwitz integers. We consider π_1 and β_1. Let $\pi_1, \beta_1 \in \mathbb{Z}$. $\pi_1\beta_1 \in \mathbb{Z}$ because of $\pi_1, \beta_1 \in \mathbb{Z}$. From the property of round notation,

$$\lfloor \pi_1\beta_1 \rceil = \pi_1\beta_1, \tag{4.36}$$

and

$$\lfloor\lfloor \pi_1\beta_1 \rceil\rceil = \pi_1\beta_1 \pm \frac{1}{2}. \tag{4.37}$$

Let $\pi_1 \in \mathbb{Z}$ and $\beta_1 \in \mathbb{Z} + \frac{1}{2}$. If π_1 is an even integer, then $\pi_1\beta_1 \in \mathbb{Z}$. From the property of round notation,

$$\lfloor \pi_1\beta_1 \rceil = \pi_1\beta_1, \tag{4.38}$$

and

$$\lfloor\lfloor \pi_1\beta_1 \rceil\rceil = \pi_1\beta_1 \pm \frac{1}{2}. \tag{4.39}$$

If π_1 is an odd integer, then $\pi_1\beta_1 \in \mathbb{Z} + \frac{1}{2}$. From the property of round notation,

$$\lfloor \pi_1\beta_1 \rceil = \pi_1\beta_1 \pm \frac{1}{2}, \tag{4.40}$$

and

$$\lfloor\lfloor \pi_1\beta_1 \rceil\rceil = \pi_1\beta_1. \tag{4.41}$$

Similarly, for $\pi_1 \in \mathbb{Z} + \frac{1}{2}$ and $\beta_1 \in \mathbb{Z}$, it can be shown. Let $\pi_1, \beta_1 \in \mathbb{Z} + \frac{1}{2}$, and let $\pi_1 = \frac{\lambda_1}{2}$ and $\beta_1 = \frac{\lambda_2}{2}$, where λ_1 and λ_2 are odd integers. If $\lambda_1\lambda_2 \equiv 1 \mod 4$, then, $\lambda_1\lambda_2 = 4k+1$, where $k \in \mathbb{Z}$. Hereby,

$$\begin{aligned} \lfloor \pi_1\beta_1 \rceil &= \lfloor \pm\tfrac{\lambda_1\lambda_2}{4} \rceil \\ &= \lfloor \pm\tfrac{4k+1}{4} \rceil. \end{aligned} \tag{4.42}$$

$\lfloor \frac{4k+1}{4} \rceil = \pm k$ because of $k < \lfloor \frac{4k+1}{4} \rceil < \frac{2k+1}{2}$. Therefore,

$$\lfloor \pi_1\beta_1 \rceil = \pm k \tag{4.43}$$

Since $k = \frac{\lambda_1\lambda_2}{4} - \frac{1}{4} = \pi_1\beta_1 - \frac{1}{4}$, then

$$\lfloor \pi_1\beta_1 \rceil = \pm\pi_1\beta_1 \mp \tfrac{1}{4}. \tag{4.44}$$

On the other hand,

$$\begin{aligned} \lfloor\lfloor \pi_1\beta_1 \rceil\rceil &= \lfloor\lfloor \pm\tfrac{\lambda_1\lambda_2}{4} \rceil\rceil \\ &= \lfloor\lfloor \pm\tfrac{4k+1}{4} \rceil\rceil. \end{aligned} \tag{4.45}$$

$\pm\lfloor\lfloor \frac{4k+1}{4} \rceil\rceil = \pm\frac{2k+1}{2}$ because of $k < \lfloor\lfloor \frac{4k+1}{4} \rceil\rceil < \frac{2k+1}{2}$. Therefore,

$$\begin{aligned} \lfloor\lfloor \pi_1\beta_1 \rceil\rceil &= \pm\tfrac{2k+1}{2} \\ &= \pm k \pm \tfrac{1}{2}. \end{aligned} \tag{4.46}$$

Since $k = \frac{\lambda_1\lambda_2}{4} - \frac{1}{4} = \pi_1\beta_1 - \frac{1}{4}$, then

$$\begin{aligned} \lfloor\lfloor \pi_1\beta_1 \rceil\rceil &= \pm\pi_1\beta_1 \mp \tfrac{1}{4} \pm \tfrac{1}{2} \\ &= \pm\pi_1\beta_1 \pm \tfrac{1}{4}. \end{aligned} \tag{4.47}$$

If $\lambda_1\lambda_2 \equiv 3 \mod 4$, then $\lambda_1\lambda_2 = 4k+3$, where $k \in \mathbb{Z}$. Hereby,

$$\begin{aligned} \lfloor \pi_1\beta_1 \rceil &= \lfloor \pm\tfrac{\lambda_1\lambda_2}{4} \rceil \\ &= \lfloor \pm\tfrac{4k+3}{4} \rceil. \end{aligned} \tag{4.48}$$

$\lfloor \frac{4k+3}{4} \rceil = \pm k \pm 1$ because of $\frac{2k+1}{2} < \lfloor \frac{4k+3}{4} \rceil < k+1$. Therefore,

$$\lfloor \pi_1\beta_1 \rceil = \pm k \pm 1. \tag{4.49}$$

Since $k = \frac{\lambda_1\lambda 2}{4} - \frac{3}{4} = \pi_1\beta_1 - \frac{3}{4}$, then

$$\begin{aligned}\lfloor \pi_1\beta_1 \rceil &= \pm\pi_1\beta_1 \mp \tfrac{3}{4} \pm 1 \\ &= \pm\pi_1\beta_1 \pm \tfrac{1}{4}.\end{aligned} \tag{4.50}$$

On the other hand,

$$\begin{aligned}\lfloor\lfloor \pi_1\beta_1 \rceil\rceil &= \lfloor\lfloor \pm\tfrac{\lambda_1\lambda_2}{4} \rceil\rceil \\ &= \lfloor\lfloor \pm\tfrac{4k+3}{4} \rceil\rceil.\end{aligned} \tag{4.51}$$

$\pm\lfloor\lfloor \frac{4k+3}{4} \rceil\rceil = \pm\frac{2k+1}{2}$ because of $\frac{2k+1}{2} < \lfloor\lfloor \frac{4k+3}{4} \rceil\rceil < k+1$. Therefore,

$$\begin{aligned}\lfloor\lfloor \pi_1\beta_1 \rceil\rceil &= \pm\tfrac{2k+1}{2} \\ &= \pm k \pm \tfrac{1}{2}.\end{aligned} \tag{4.52}$$

Since $k = \frac{\lambda_1\lambda_2}{4} - \frac{3}{4} = \pi_1\beta_1 - \frac{3}{4}$, then

$$\begin{aligned}\lfloor\lfloor \pi_1\beta_1 \rceil\rceil &= \pm\pi_1\beta_1 \mp \tfrac{3}{4} \pm \tfrac{1}{2} \\ &= \pm\pi_1\beta_1 \mp \tfrac{1}{4}.\end{aligned} \tag{4.53}$$

This completes the proof. □

Proposition 4.5. *Let $\alpha = \alpha_1 + \alpha_2 i + \alpha_3 j + \alpha_4 k$ be a prime Hurwitz integer. If α_1, α_2, α_3, $\alpha_4 \in \mathbb{Z}$, then*

$$\mu_\alpha^{(1)}(N(\alpha)) = 0, \tag{4.54}$$

and

$$\mu_\alpha^{(2)}(N(\alpha)) \equiv 0 \mod \alpha. \tag{4.55}$$

If $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{Z} + \frac{1}{2}$, then

$$\mu_\alpha^{(1)}(N(\alpha)) \equiv 0 \mod \alpha, \tag{4.56}$$

and

$$\mu_\alpha^{(2)}(N(\alpha)) = 0. \tag{4.57}$$

Proof. Let $\alpha = \alpha_1 + \alpha_2 i + \alpha_3 j + \alpha_4 k$ be a prime Hurwitz integer. If $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{Z}$, then from eq. 3.4,

$$\begin{aligned}\mu_\alpha^{(1)}(N(\alpha)) &= N(\alpha) - \alpha\lfloor \tfrac{\overline{\alpha}N(\alpha)}{N(\alpha)} \rceil \\ &= N(\alpha) - \alpha\lfloor \overline{\alpha} \rceil.\end{aligned} \tag{4.58}$$

From eq. 4.10,

$$\begin{aligned}\mu_\alpha^{(1)}(N(\alpha)) &= N(\alpha) - \alpha\overline{\alpha} \\ &= N(\alpha) - N(\alpha) \\ &= 0.\end{aligned} \tag{4.59}$$

And, from eq. 3.5,

$$\begin{aligned}\mu_\alpha^{(2)}(N(\alpha)) &= N(\alpha) - \alpha\lfloor\lfloor \tfrac{\overline{\alpha}N(\alpha)}{N(\alpha)} \rceil\rceil \\ &= N(\alpha) - \alpha\lfloor\lfloor \overline{\alpha} \rceil\rceil.\end{aligned} \tag{4.60}$$

From eq. 4.11,

$$\begin{aligned}\mu_\alpha^{(2)}(N(\alpha)) &= N(\alpha) - \alpha\left(\overline{\alpha} + \tfrac{1}{2} + \tfrac{1}{2}i + \tfrac{1}{2}j + \tfrac{1}{2}k\right) \\ &= N(\alpha) - N(\alpha) - \alpha\left(\tfrac{1}{2} + \tfrac{1}{2}i + \tfrac{1}{2}j + \tfrac{1}{2}k\right) \\ &= -\alpha\left(\tfrac{1}{2} + \tfrac{1}{2}i + \tfrac{1}{2}j + \tfrac{1}{2}k\right).\end{aligned} \tag{4.61}$$

Therefore, $\mu_\alpha^{(2)}(N(\alpha)) \equiv 0 \mod \alpha$. On the other hand, if $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{Z}+\frac{1}{2}$, then from eq. 3.4,

$$\begin{aligned} \mu_\alpha^{(1)}(N(\alpha)) &= N(\alpha) - \alpha \lfloor \tfrac{\overline{\alpha}N(\alpha)}{N(\alpha)} \rceil \\ &= N(\alpha) - \alpha \lfloor \overline{\alpha} \rceil. \end{aligned} \tag{4.62}$$

From eq. 4.12,

$$\begin{aligned} \mu_\alpha^{(1)}(N(\alpha)) &= N(\alpha) - \alpha \left(\overline{\alpha} + \tfrac{1}{2} + \tfrac{1}{2}i + \tfrac{1}{2}j + \tfrac{1}{2}k\right) \\ &= N(\alpha) - N(\alpha) - \alpha \left(\tfrac{1}{2} + \tfrac{1}{2}i + \tfrac{1}{2}j + \tfrac{1}{2}k\right) \\ &= -\alpha \left(\tfrac{1}{2} + \tfrac{1}{2}i + \tfrac{1}{2}j + \tfrac{1}{2}k\right). \end{aligned} \tag{4.63}$$

Therefore, $\mu_\alpha^{(1)}(N(\alpha)) \equiv 0 \mod \alpha$. And, from eq. 3.5,

$$\begin{aligned} \mu_\alpha^{(2)}(N(\alpha)) &= N(\alpha) - \alpha \lfloor\lfloor \tfrac{\overline{\alpha}N(\alpha)}{N(\alpha)} \rceil\rceil \\ &= N(\alpha) - \alpha \lfloor\lfloor \overline{\alpha} \rceil\rceil. \end{aligned} \tag{4.64}$$

From eq. 4.13,

$$\begin{aligned} \mu_\alpha^{(2)}(N(\alpha)) &= N(\alpha) - \alpha\overline{\alpha} \\ &= N(\alpha) - N(\alpha) \\ &= 0. \end{aligned} \tag{4.65}$$

This completes the proof. □

Proposition 4.6. *Let $\alpha = \alpha_1 + \alpha_2 i + \alpha_3 j + \alpha_4 k$ be a prime Hurwitz integer, and let $z \in \mathbb{Z}_{N(\alpha)}$. Then,*

$$\mu_\alpha(z) + \mu_\alpha(N(\alpha) - z) \equiv 0 \mod \alpha. \tag{4.66}$$

Proof. Let α be a prime Hurwitz integer, and let $z \in \mathbb{Z}_{N(\alpha)}$. Firstly, we suppose that $\mu_\alpha(z) = \mu_\alpha^{(1)}(z)$ and $\mu_\alpha(N(\alpha) - z) = \mu_\alpha^{(1)}(N(\alpha) - z)$. Then,

$$\mu_\alpha(z) + \mu_\alpha(N(\alpha) - z) = \mu_\alpha^{(1)}(z) + \mu_\alpha^{(1)}(N(\alpha) - z). \tag{4.67}$$

From eq. 3.4,

$$\begin{aligned} \mu_\alpha(z) + \mu_\alpha(N(\alpha) - z) &= z - \alpha \lfloor \tfrac{\overline{\alpha}z}{N(\alpha)} \rceil + N(\alpha) - z - \alpha \lfloor \tfrac{\overline{\alpha}(N(\alpha)-z)}{N(\alpha)} \rceil \\ &= N(\alpha) - \alpha \lfloor \tfrac{\overline{\alpha}z}{N(\alpha)} \rceil - \alpha \lfloor \tfrac{\overline{\alpha}N(\alpha)}{N(\alpha)} \rceil - \alpha \lfloor \tfrac{\overline{\alpha}(-z)}{N(\alpha)} \rceil \\ &= N(\alpha) - \alpha \lfloor \tfrac{\overline{\alpha}z}{N(\alpha)} \rceil - \alpha \lfloor \overline{\alpha} \rceil + \alpha \lfloor \tfrac{\overline{\alpha}z}{N(\alpha)} \rceil \\ &= N(\alpha) - \alpha \lfloor \overline{\alpha} \rceil. \end{aligned} \tag{4.68}$$

If $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{Z}$, then, from eq. 4.58 and eq. 4.59,

$$\mu_\alpha(z) + \mu_\alpha(N(\alpha) - z) = 0. \tag{4.69}$$

Therefore, $\mu_\alpha(z) + \mu_\alpha(N(\alpha) - z) \equiv 0 \mod \alpha$. If $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{Z}+\frac{1}{2}$, then, from eq. 4.62 and eq. 4.63,

$$\mu_\alpha(z) + \mu_\alpha(N(\alpha) - z) \equiv 0 \mod \alpha. \tag{4.70}$$

Secondly, we suppose that $\mu_\alpha(z) = \mu_\alpha^{(1)}(z)$ and $\mu_\alpha(N(\alpha) - z) = \mu_\alpha^{(2)}(N(\alpha) - z)$. Then,

$$\mu_\alpha(z) + \mu_\alpha(N(\alpha) - z) = \mu_\alpha^{(1)}(z) + \mu_\alpha^{(2)}(N(\alpha) - z). \tag{4.71}$$

From eq. 3.4 and eq. 3.5,

$$\begin{aligned} \mu_\alpha(z) + \mu_\alpha(N(\alpha) - z) &= z - \alpha \lfloor \tfrac{\overline{\alpha}z}{N(\alpha)} \rceil + N(\alpha) - z - \alpha \lfloor\lfloor \tfrac{\overline{\alpha}(N(\alpha)-z)}{N(\alpha)} \rceil\rceil \\ &= N(\alpha) - \alpha \lfloor \tfrac{\overline{\alpha}z}{N(\alpha)} \rceil - \alpha \lfloor\lfloor \tfrac{\overline{\alpha}N(\alpha)}{N(\alpha)} \rceil\rceil - \alpha \lfloor\lfloor \tfrac{\overline{\alpha}(-z)}{N(\alpha)} \rceil\rceil \\ &= N(\alpha) - \alpha \lfloor \tfrac{\overline{\alpha}z}{N(\alpha)} \rceil - \alpha \lfloor\lfloor \overline{\alpha} \rceil\rceil + \alpha \lfloor\lfloor \tfrac{\overline{\alpha}z}{N(\alpha)} \rceil\rceil. \end{aligned} \tag{4.72}$$

If $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{Z}$, then, from eq. 4.60 and eq. 4.61,

$$\mu_\alpha(z) + \mu_\alpha(N(\alpha) - z) = -\alpha \lfloor \tfrac{\overline{\alpha}z}{N(\alpha)} \rceil + \alpha \lfloor\lfloor \tfrac{\overline{\alpha}z}{N(\alpha)} \rceil\rceil. \tag{4.73}$$

There exist $\lambda_1, \lambda_2 \in \mathcal{H}$ such that $\lfloor \frac{\overline{\alpha}z}{N(\alpha)} \rceil = \lambda_1$ and $\alpha \lfloor\lfloor \frac{\overline{\alpha}z}{N(\alpha)} \rceil\rceil = \lambda_2$. Therefore,

$$\begin{aligned} \mu_\alpha(z) + \mu_\alpha(N(\alpha) - z) &= -\alpha\lambda_1 + \alpha\lambda_2 \\ &= \alpha(-\lambda_1 + \lambda_2). \end{aligned} \tag{4.74}$$

Therefore, $\mu_\alpha(z) + \mu_\alpha(N(\alpha) - z) \equiv 0 \mod \alpha$. If $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{Z} + \frac{1}{2}$, then, from eq. 4.64 and eq. 4.65,

$$\mu_\alpha(z) + \mu_\alpha(N(\alpha) - z) = -\alpha\lfloor \frac{\overline{\alpha}z}{N(\alpha)} \rceil + \alpha\lfloor\lfloor \frac{\overline{\alpha}z}{N(\alpha)} \rceil\rceil. \tag{4.75}$$

There exist $\lambda_1, \lambda_2 \in \mathcal{H}$ such that $\lfloor \frac{\overline{\alpha}z}{N(\alpha)} \rceil = \lambda_1$ and $\alpha \lfloor\lfloor \frac{\overline{\alpha}z}{N(\alpha)} \rceil\rceil = \lambda_2$. Therefore,

$$\begin{aligned} \mu_\alpha(z) + \mu_\alpha(N(\alpha) - z) &= -\alpha\lambda_1 + \alpha\lambda_2 \\ &= \alpha(-\lambda_1 + \lambda_2). \end{aligned} \tag{4.76}$$

Therefore, $\mu_\alpha(z) + \mu_\alpha(N(\alpha) - z) \equiv 0 \mod \alpha$. Thirdly, we suppose that $\mu_\alpha(z) = \mu_\alpha^{(2)}(z)$ and $\mu_\alpha(N(\alpha) - z) = \mu_\alpha^{(1)}(N(\alpha) - z)$. Then,

$$\mu_\alpha(z) + \mu_\alpha(N(\alpha) - z) = \mu_\alpha^{(2)}(z) + \mu_\alpha^{(1)}(N(\alpha) - z). \tag{4.77}$$

From eq. 3.5 and eq. 3.4,

$$\begin{aligned} \mu_\alpha(z) + \mu_\alpha(N(\alpha) - z) &= z - \alpha\lfloor\lfloor \frac{\overline{\alpha}z}{N(\alpha)} \rceil\rceil + N(\alpha) - z - \alpha\lfloor \frac{\overline{\alpha}(N(\alpha)-z)}{N(\alpha)} \rceil \\ &= N(\alpha) - \alpha\lfloor\lfloor \frac{\overline{\alpha}z}{N(\alpha)} \rceil\rceil - \alpha\lfloor \frac{\overline{\alpha}N(\alpha)}{N(\alpha)} \rceil - \alpha\lfloor \frac{\overline{\alpha}(-z)}{N(\alpha)} \rceil \\ &= N(\alpha) - \alpha\lfloor\lfloor \frac{\overline{\alpha}z}{N(\alpha)} \rceil\rceil - \alpha\lfloor \overline{\alpha} \rceil + \alpha\lfloor \frac{\overline{\alpha}z}{N(\alpha)} \rceil. \end{aligned} \tag{4.78}$$

If $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{Z}$, then, from eq. 4.58 and eq. 4.59,

$$\mu_\alpha(z) + \mu_\alpha(N(\alpha) - z) = -\alpha\lfloor\lfloor \frac{\overline{\alpha}z}{N(\alpha)} \rceil\rceil + \alpha\lfloor \frac{\overline{\alpha}z}{N(\alpha)} \rceil. \tag{4.79}$$

There exist $\lambda_1, \lambda_2 \in \mathcal{H}$ such that $\lfloor\lfloor \frac{\overline{\alpha}z}{N(\alpha)} \rceil\rceil = \lambda_1$ and $\alpha \lfloor \frac{\overline{\alpha}z}{N(\alpha)} \rceil = \lambda_2$. Therefore,

$$\begin{aligned} \mu_\alpha(z) + \mu_\alpha(N(\alpha) - z) &= -\alpha\lambda_1 + \alpha\lambda_2 \\ &= \alpha(-\lambda_1 + \lambda_2). \end{aligned} \tag{4.80}$$

Therefore, $\mu_\alpha(z) + \mu_\alpha(N(\alpha) - z) \equiv 0 \mod \alpha$. If $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{Z} + \frac{1}{2}$, then, from eq. 4.62 and eq. 4.63,

$$\mu_\alpha(z) + \mu_\alpha(N(\alpha) - z) = -\alpha\lfloor\lfloor \frac{\overline{\alpha}z}{N(\alpha)} \rceil\rceil + \alpha\lfloor \frac{\overline{\alpha}z}{N(\alpha)} \rceil. \tag{4.81}$$

There exist $\lambda_1, \lambda_2 \in \mathcal{H}$ such that $\lfloor\lfloor \frac{\overline{\alpha}z}{N(\alpha)} \rceil\rceil = \lambda_1$ and $\alpha \lfloor \frac{\overline{\alpha}z}{N(\alpha)} \rceil = \lambda_2$. Therefore,

$$\begin{aligned} \mu_\alpha(z) + \mu_\alpha(N(\alpha) - z) &= -\alpha\lambda_1 + \alpha\lambda_2 \\ &= \alpha(-\lambda_1 + \lambda_2). \end{aligned} \tag{4.82}$$

Therefore, $\mu_\alpha(z) + \mu_\alpha(N(\alpha) - z) \equiv 0 \mod \alpha$. Lastly, we suppose that $\mu_\alpha(z) = \mu_\alpha^{(2)}(z)$ and $\mu_\alpha(N(\alpha) - z) = \mu_\alpha^{(2)}(N(\alpha) - z)$. Then,

$$\mu_\alpha(z) + \mu_\alpha(N(\alpha) - z) = \mu_\alpha^{(2)}(z) + \mu_\alpha^{(12)}(N(\alpha) - z). \tag{4.83}$$

From eq. 3.4,

$$\begin{aligned} \mu_\alpha(z) + \mu_\alpha(N(\alpha) - z) &= z - \alpha\lfloor\lfloor \frac{\overline{\alpha}z}{N(\alpha)} \rceil\rceil + N(\alpha) - z - \alpha\lfloor\lfloor \frac{\overline{\alpha}(N(\alpha)-z)}{N(\alpha)} \rceil\rceil \\ &= N(\alpha) - \alpha\lfloor\lfloor \frac{\overline{\alpha}z}{N(\alpha)} \rceil\rceil - \alpha\lfloor\lfloor \frac{\overline{\alpha}N(\alpha)}{N(\alpha)} \rceil\rceil - \alpha\lfloor\lfloor \frac{\overline{\alpha}(-z)}{N(\alpha)} \rceil\rceil \\ &= N(\alpha) - \alpha\lfloor\lfloor \frac{\overline{\alpha}z}{N(\alpha)} \rceil\rceil - \alpha\lfloor\lfloor \overline{\alpha} \rceil\rceil + \alpha\lfloor\lfloor \frac{\overline{\alpha}z}{N(\alpha)} \rceil\rceil \\ &= N(\alpha) - \alpha\lfloor\lfloor \overline{\alpha} \rceil\rceil. \end{aligned} \tag{4.84}$$

If $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{Z}$, then, from eq. 4.60 and eq. 4.61,

$$\mu_\alpha(z) + \mu_\alpha(N(\alpha) - z) = 0. \tag{4.85}$$

Therefore, $\mu_\alpha(z)+\mu_\alpha(N(\alpha)-z)\equiv 0 \mod \alpha$. If $\alpha_1,\alpha_2,\alpha_3,\alpha_4\in\mathbb{Z}+\frac{1}{2}$, then, from eq. 4.64 and eq. 4.65,

$$\mu_\alpha(z)+\mu_\alpha(N(\alpha)-z) \quad\equiv\quad 0 \mod \alpha. \tag{4.86}$$

Last of all, $\mu_\alpha(z)+\mu_\alpha(N(\alpha)-z)\equiv 0 \mod \alpha$. This completes the proof. □

5. Examples

Example 5.1. Let $\alpha=\frac{5}{2}+\frac{3}{2}i+\frac{3}{2}j+\frac{3}{2}k$. α is a prime Hurwitz integer because of $N(\alpha)=13$. From eq. 3.4,

$$\mathcal{H}_\alpha^{(1)}=\left\{\begin{array}{l}\mu_\alpha^{(1)}(0)=0,\mu_\alpha^{(1)}(1)=1,\mu_\alpha^{(1)}(2)=2,\mu_\alpha^{(1)}(3)=\frac{1}{2}-\frac{3}{2}i-\frac{3}{2}j-\frac{3}{2}k,\\ \mu_\alpha^{(1)}(4)=\frac{3}{2}-\frac{3}{2}i-\frac{3}{2}j-\frac{3}{2}k,\mu_\alpha^{(1)}(5)=-2+i+j+k,\mu_\alpha^{(1)}(6)=-1+i+j+k,\\ \mu_\alpha^{(1)}(7)=i+j+k,\mu_\alpha^{(1)}(8)=-\frac{3}{2}-\frac{1}{2}i-\frac{1}{2}j-\frac{1}{2}k,\mu_\alpha^{(1)}(9)=-\frac{1}{2}-\frac{1}{2}i-\frac{1}{2}j-\frac{1}{2}k,\\ \mu_\alpha^{(1)}(10)=\frac{1}{2}-\frac{1}{2}i-\frac{1}{2}j-\frac{1}{2}k,\mu_\alpha^{(1)}(11)=\frac{3}{2}-\frac{1}{2}i-\frac{1}{2}j-\frac{1}{2}k,\\ \mu_\alpha^{(1)}(12)=\frac{5}{2}-\frac{1}{2}i-\frac{1}{2}j-\frac{1}{2}k\end{array}\right\}.$$

From eq. 3.5,

$$\mathcal{H}_\alpha^{(2)}=\left\{\begin{array}{l}\mu_\alpha^{(2)}(0)=1-2i-2j-2k,\mu_\alpha^{(2)}(1)=-\frac{5}{2}+\frac{1}{2}i+\frac{1}{2}j+\frac{1}{2}k,\mu_\alpha^{(2)}(2)=-\frac{3}{2}+\frac{1}{2}i+\frac{1}{2}j+\frac{1}{2}k,\\ \mu_\alpha^{(2)}(3)=-\frac{1}{2}+\frac{1}{2}i+\frac{1}{2}j+\frac{1}{2}k,\mu_\alpha^{(2)}(4)=\frac{1}{2}+\frac{1}{2}i+\frac{1}{2}j+\frac{1}{2}k,\mu_\alpha^{(2)}(5)=\frac{3}{2}+\frac{1}{2}i+\frac{1}{2}j+\frac{1}{2}k,\\ \mu_\alpha^{(2)}(6)=-i-j-k,\mu_\alpha^{(2)}(7)=1-i-j-k,\mu_\alpha^{(2)}(8)=2-i-j-k,\\ \mu_\alpha^{(2)}(9)=-\frac{3}{2}+\frac{3}{2}i+\frac{3}{2}j+\frac{3}{2}k,\mu_\alpha^{(2)}(10)=-\frac{1}{2}+\frac{3}{2}i+\frac{3}{2}j+\frac{3}{2}k,\mu_\alpha^{(2)}(11)=-2,\\ \mu_\alpha^{(2)}(12)=-1\end{array}\right\}.$$

With respect to eq. 3.3

$$\mathcal{H}_\alpha=\left\{\begin{array}{l}\mu_\alpha(0)=\mu_\alpha^{(1)}(0)=0,\mu_\alpha(1)=\mu_\alpha^{(1)}(1)=1,\mu_\alpha(2)=\mu_\alpha^{(2)}(2)=-\frac{3}{2}+\frac{1}{2}i+\frac{1}{2}j+\frac{1}{2}k,\\ \mu_\alpha(3)=\mu_\alpha^{(2)}(3)=-\frac{1}{2}+\frac{1}{2}i+\frac{1}{2}j+\frac{1}{2}k,\mu_\alpha(4)=\mu_\alpha^{(2)}(4)=\frac{1}{2}+\frac{1}{2}i+\frac{1}{2}j+\frac{1}{2}k,\\ \mu_\alpha(5)=\mu_\alpha^{(2)}(5)=\frac{3}{2}+\frac{1}{2}i+\frac{1}{2}j+\frac{1}{2}k,\mu_\alpha(6)=\mu_\alpha^{(2)}(6)=-i-j-k,\\ \mu_\alpha(7)=\mu_\alpha^{(1)}(7)=i+j+k,\mu_\alpha(8)=\mu_\alpha^{(1)}(8)=-\frac{3}{2}-\frac{1}{2}i-\frac{1}{2}j-\frac{1}{2}k,\\ \mu_\alpha(9)=\mu_\alpha^{(1)}(9)=-\frac{1}{2}-\frac{1}{2}i-\frac{1}{2}j-\frac{1}{2}k,\mu_\alpha(10)=\mu_\alpha^{(1)}(10)=\frac{1}{2}-\frac{1}{2}i-\frac{1}{2}j-\frac{1}{2}k,\\ \mu_\alpha(11)=\mu_\alpha^{(1)}(11)=\frac{3}{2}-\frac{1}{2}i-\frac{1}{2}j-\frac{1}{2}k,\mu_\alpha(12)=\mu_\alpha^{(2)}(12)=-1\end{array}\right\}.$$

From eq. 3.30, the average energy of $\mathcal{H}_{\frac{5}{2}+\frac{3}{2}i+\frac{3}{2}j+\frac{3}{2}k}$ is

$$\mathcal{E}_{\frac{5}{2}+\frac{3}{2}i+\frac{3}{2}j+\frac{3}{2}k}=\frac{24}{13}=1.8462.$$

Example 5.2. Let $\alpha=3+2i$. α is a prime Hurwitz integer because of $N(\alpha)=13$. From eq. 3.4,

$$\mathcal{H}_\alpha^{(1)}=\left\{\begin{array}{l}\mu_\alpha^{(1)}(0)=0,\mu_\alpha^{(1)}(1)=1,\mu_\alpha^{(1)}(2)=2,\mu_\alpha^{(1)}(3)=-2i,\mu_\alpha^{(1)}(4)=-1+i,\\ \mu_\alpha^{(1)}(5)=i,\mu_\alpha^{(1)}(6)=1+i,\mu_\alpha^{(1)}(7)=-1-i,\mu_\alpha^{(1)}(8)=-i,\mu_\alpha^{(1)}(9)=1-i,\\ \mu_\alpha^{(1)}(10)=2i,\mu_\alpha^{(1)}(11)=-2,\mu_\alpha^{(1)}(12)=-1\end{array}\right\}.$$

From eq. 3.5,

$$\mathcal{H}_\alpha^{(2)} = \left\{ \begin{array}{l} \mu_\alpha^{(2)}(0) = -\frac{1}{2} - \frac{5}{2}i - \frac{1}{2}j - \frac{5}{2}k, \mu_\alpha^{(2)}(1) = -\frac{3}{2} + \frac{1}{2}i - \frac{1}{2}j - \frac{5}{2}k, \mu_\alpha^{(2)}(2) = -\frac{1}{2} + \frac{1}{2}i - \frac{1}{2}j - \frac{5}{2}k, \\ \mu_\alpha^{(2)}(3) = \frac{1}{2} + \frac{1}{2}i - \frac{1}{2}j - \frac{5}{2}k, \mu_\alpha^{(2)}(4) = \frac{3}{2} + \frac{1}{2}i - \frac{1}{2}j - \frac{5}{2}k, \mu_\alpha^{(2)}(5) = -\frac{1}{2} - \frac{3}{2}i - \frac{1}{2}j - \frac{5}{2}k, \\ \mu_\alpha^{(2)}(6) = \frac{1}{2} - \frac{3}{2}i - \frac{1}{2}j - \frac{5}{2}k, \mu_\alpha^{(2)}(7) = -\frac{1}{2} + \frac{3}{2}i - \frac{1}{2}j - \frac{5}{2}k, \mu_\alpha^{(2)}(8) = \frac{1}{2} + \frac{3}{2}i - \frac{1}{2}j - \frac{5}{2}k, \\ \mu_\alpha^{(2)}(9) = -\frac{3}{2} - \frac{1}{2}i - \frac{1}{2}j - \frac{5}{2}k, \mu_\alpha^{(2)}(10) = -\frac{1}{2} - \frac{1}{2}i - \frac{1}{2}j - \frac{5}{2}k, \mu_\alpha^{(2)}(11) = \frac{1}{2} - \frac{1}{2}i - \frac{1}{2}j - \frac{5}{2}k, \\ \mu_\alpha^{(2)}(12) = \frac{3}{2} - \frac{1}{2}i - \frac{1}{2}j - \frac{5}{2}k \end{array} \right\}.$$

With respect to eq. 3.3,

$$\mathcal{H}_\alpha = \mathcal{H}_\alpha^{(1)}.$$

From eq. 3.30, the average energy of $\mathcal{H}_{3+2i}$ is

$$\mathcal{E}_{3+2i} = \frac{28}{13} = 2.1539.$$

The average energy for the transmitted signal, considering the sets of residual class with the same cardinality, the average energy of $\mathcal{H}_{\frac{5}{2}+\frac{3}{2}i+\frac{3}{2}j+\frac{3}{2}k}$ is smaller than the average energy of $\mathcal{H}_{3+2i}$.

Example 5.3. Let $\alpha = 3 + i + j$. α is a prime Hurwitz integer because of $N(\alpha) = 11$. From eq. 3.4,

$$\mathcal{H}_\alpha^{(1)} = \left\{ \begin{array}{l} \mu_\alpha^{(1)}(0) = 0, \mu_\alpha^{(1)}(1) = 1, \mu_\alpha^{(1)}(2) = -1 - i - j, \mu_\alpha^{(1)}(3) = -i - j, \mu_\alpha^{(1)}(4) = 1 - i - j, \\ \mu_\alpha^{(1)}(5) = 2 - i - j, \mu_\alpha^{(1)}(6) = -2 + i + j, \mu_\alpha^{(1)}(7) = -1 + i + j, \mu_\alpha^{(1)}(8) = i + j, \\ \mu_\alpha^{(1)}(9) = 1 + i + j, \mu_\alpha^{(1)}(10) = -1, \end{array} \right\}.$$

From eq. 3.5,

$$\mathcal{H}_\alpha^{(2)} = \left\{ \begin{array}{l} \mu_\alpha^{(2)}(0) = -\frac{1}{2} - \frac{5}{2}i - \frac{3}{2}j - \frac{3}{2}k, \mu_\alpha^{(2)}(1) = -\frac{3}{2} + \frac{1}{2}i + \frac{3}{2}j - \frac{3}{2}k, \mu_\alpha^{(2)}(2) = -\frac{1}{2} + \frac{1}{2}i + \frac{3}{2}j - \frac{3}{2}k, \\ \mu_\alpha^{(2)}(3) = \frac{1}{2} + \frac{1}{2}i + \frac{3}{2}j - \frac{3}{2}k, \mu_\alpha^{(2)}(4) = -\frac{3}{2} - \frac{1}{2}i + \frac{1}{2}j - \frac{3}{2}k, \mu_\alpha^{(2)}(5) = -\frac{1}{2} - \frac{1}{2}i + \frac{1}{2}j - \frac{3}{2}k, \\ \mu_\alpha^{(2)}(6) = \frac{1}{2} - \frac{1}{2}i + \frac{1}{2}j - \frac{3}{2}k, \mu_\alpha^{(2)}(7) = \frac{3}{2} - \frac{1}{2}i + \frac{1}{2}j - \frac{3}{2}k, \mu_\alpha^{(2)}(8) = -\frac{1}{2} - \frac{3}{2}i - \frac{1}{2}j - \frac{3}{2}k, \\ \mu_\alpha^{(2)}(9) = \frac{1}{2} - \frac{3}{2}i - \frac{1}{2}j - \frac{3}{2}k, \mu_\alpha^{(2)}(10) = \frac{3}{2} - \frac{3}{2}i - \frac{1}{2}j - \frac{3}{2}k, \end{array} \right\}.$$

With respect to eq. 3.3

$$\mathcal{H}_\alpha = \left\{ \begin{array}{l} \mu_\alpha(0) = \mu_\alpha^{(1)}(0) = 0, \mu_\alpha(1) = \mu_\alpha^{(1)}(1) = 1, \mu_\alpha(2) = \mu_\alpha^{(1)}(2) = -1 - i - j, \\ \mu_\alpha(3) = \mu_\alpha^{(1)}(3) = -i - j, \mu_\alpha(4) = \mu_\alpha^{(1)}(4) = 1 - i - j, \mu_\alpha(5) = \mu_\alpha^{(2)}(5) = -\frac{1}{2} - \frac{1}{2}i + \frac{1}{2}j - \frac{3}{2}k, \\ \mu_\alpha(6) = \mu_\alpha^{(2)}(6) = \frac{1}{2} - \frac{1}{2}i + \frac{1}{2}j - \frac{3}{2}k, \mu_\alpha(7) = \mu_\alpha^{(1)}(7) = -1 + i + j, \mu_\alpha(8) = \mu_\alpha^{(1)}(8) = i + j, \\ \mu_\alpha(9) = \mu_\alpha^{(1)}(9) = 1 + i + j, \mu_\alpha(10) = \mu_\alpha^{(1)}(10) = -1, \end{array} \right\}.$$

From eq. 3.30, the average energy of $\mathcal{H}_{3+i+j}$ is

$$\mathcal{E}_{3+i+j} = \frac{24}{11} = 2.1818.$$

Table I is presented the average energies of prime Hurwitz integers such that $N = 6k + 1 \leq 50$, where $k \in \mathbb{Z}^+$.

6. The code rate for codes over Hurwitz integers

Let β be a Hurwitz integer. β is a unit if and only if $N(\beta) = 1$. There are precisely 24 units in $\mathcal{H}$. The set of units in $\mathcal{H}$ that denoted by $\mathcal{U}_\mathcal{H}$ is shown by

$$\mathcal{U}_\mathcal{H} = \left\{ \pm 1, \pm i, \pm j, \pm k, \pm 1 \pm i \pm j \pm k, \pm\frac{1}{2} \pm \frac{1}{2}i \pm \frac{1}{2}j \pm \frac{1}{2}k \right\}. \quad (6.1)$$

N	Hurwitz Integers ($\mathbb{Z}$)	Hurwitz Integers ($\mathbb{Z}+\frac{1}{2}$)	Average Energy
7	$2+i+j+k$	$\frac{3}{2}+\frac{3}{2}i+\frac{3}{2}j+\frac{1}{2}k$	0.8571
		$\frac{5}{2}+\frac{1}{2}i+\frac{1}{2}j+\frac{1}{2}k$	
13	$2+2i+2j+k$	$\frac{5}{2}+\frac{3}{2}i+\frac{3}{2}j+\frac{3}{2}k$	1.8462
		$\frac{7}{2}+\frac{1}{2}i+\frac{1}{2}j+\frac{1}{2}k$	
19	$4+i+j+k$	$\frac{5}{2}+\frac{5}{2}i+\frac{5}{2}j+\frac{1}{2}k$	2.5263
		$\frac{7}{2}+\frac{3}{2}i+\frac{3}{2}j+\frac{3}{2}k$	
31	$3+3i+3j+2k$	$\frac{7}{2}+\frac{5}{2}i+\frac{5}{2}j+\frac{5}{2}k$	4.2581
		$\frac{11}{2}+\frac{1}{2}i+\frac{1}{2}j+\frac{1}{2}k$	
37	$5+2i+2j+2k$	$\frac{7}{2}+\frac{7}{2}i+\frac{7}{2}j+\frac{1}{2}k$	5.0270
		$\frac{11}{2}+\frac{3}{2}i+\frac{3}{2}j+\frac{3}{2}k$	
43	$4+3i+3j+3k$	$\frac{7}{2}+\frac{7}{2}i+\frac{7}{2}j+\frac{5}{2}k$	6.0000
		$\frac{13}{2}+\frac{1}{2}i+\frac{1}{2}j+\frac{1}{2}k$	

Table 1. For $N \leq 50$, the Average Energies of Prime Hurwitz Integers such that $N = 6k+1$, where $k \in \mathbb{Z}^+$.

In this section, we give the code rate of a code C over $\mathcal{H}$. Note that $\mathcal{H}_\alpha$ has $N(\alpha)$ elements. Let $N(\alpha) = p$ be a prime integer such that $p \equiv 1 \mod 24$. A code C over $\mathcal{H}$ has length $n = \frac{p^k-1}{24}$. Here k is the dimension of a code C over $\mathcal{H}$. Let $|C| = |\mathcal{H}_\alpha| = N(\alpha) = p$. The coding rate of a code C over $\mathcal{H}$ is computed by

$$R = \frac{k}{n} = \frac{24k}{p^k - 1}. \tag{6.2}$$

In this study, we consider $k = 1$ because of $k = \log_p |C| = \log_p p = 1$, where $|C| = N(\alpha) = p$.

Example 6.1. For $p \leq 100$, $p = 73$ and $p = 97$ are prime integers such that $p \equiv 1 \mod 24$.

i. For $N(\alpha) = p = 73$, α is a prime Hurwitz integer. The length of a code C over $\mathcal{H}$ is $n = \frac{72}{24} = 3$. The rate of a code C over $\mathcal{H}$ is $R = \frac{24}{72} = \frac{1}{3}$. C is a $(3,1)-$ code over $\mathcal{H}$ because of $n = 3$ and $k = 1$.
ii. For $N(\alpha) = p = 97$, α is a prime Hurwitz integer. The length of a code C over $\mathcal{H}$ is $n = \frac{96}{24} = 4$. The rate of a code C over $\mathcal{H}$ is $R = \frac{24}{96} = \frac{1}{4}$. C is a $(4,1)-$ code over $\mathcal{H}$ because of $n = 4$ and $k = 1$.

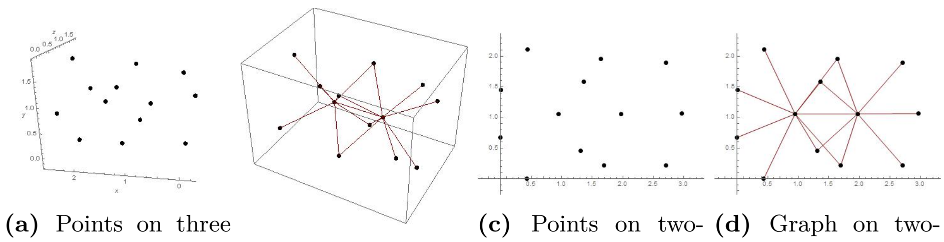
7. Graph layout methods

Graphs are used to show the relationship between items, in general. Graph drawing enables visualization of these relationships. The usefulness of the visual representation depends upon whether the drawing is aesthetic. While there are no strict criteria for aesthetic drawing, it is generally agreed that such a drawing has minimal edge crossing and even spacing between vertices. Two popular straight-edge drawing algorithms, the spring embedding, and spring-electrical embedding work by minimizing the energy of physical models of the graph. The high-dimensional embedding method, on the other hand, embeds a graph in high-dimensional space and then projects it back to two or three-dimensional space. Random, circular, and spiral embedding do not utilize connectivity information for laying out a graph. In this study, we do not consider random embedding, and circular embedding. We consider spring, the high-dimensional, and spiral embedding methods, in this study. The spring embedding algorithm assigns force between each pair of nodes. When two nodes are too close together, a repelling force comes into effect. When two nodes are too far apart, they are subject to an attractive force. This scenario can be illustrated by linking the vertices with springs, hence the name "spring embedding". In the high-dimensional embedding method, a graph is embedded in high-dimensional space and then projected back to two or three-dimensional space. The high-dimensional embedding method tends to be very fast but its results are often of lower quality than force-directed algorithms. We use the Wolfram Mathematica 10.2 program, which has algorithms used for layered/hierarchical drawing of directed graphs, and for drawing trees. These algorithms are implemented via four functions: "GraphPlot," "GraphPlot3D," "LayeredGraphPlot," and "TreePlot". In this study, we use two of these algorithms for graph drawing, i.e., "GraphPlot," and "GraphPlot3D". You can find more details on the wolfram.com.

Definition 7.1. Spring Embedding The spring embedding is a graph-drawing technique to position vertices of a graph so that they minimize the mechanical energy when each edge corresponds to a spring. The spring embedding is typically used to lay out regular structured graphs. You can find more details on the wolfram.com.

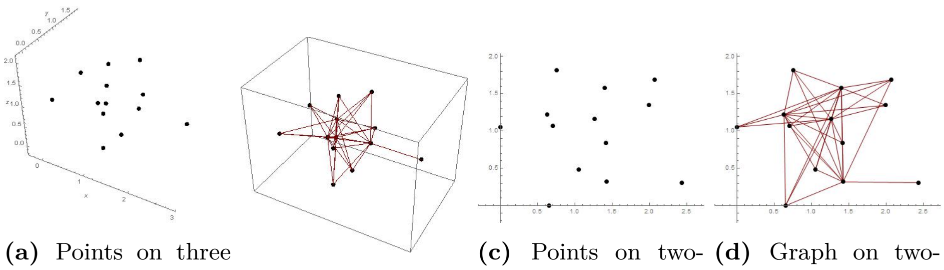
Definition 7.2. High Dimensional Embedding The high-dimensional embedding is a graph-drawing technique to position vertices of a graph in a high-dimensional space, and then project back to two- or three-dimensional space. The high-dimensional embedding is typically used for fast layout of graphs. You can find more details on the wolfram.com.

Definition 7.3. Spiral Embedding The spiral embedding is a graph-drawing technique to position vertices of a graph on a 3D spiral projected to 2D. The spiral embedding is typically used to lay out path graphs. You can find more details on the wolfram.com.



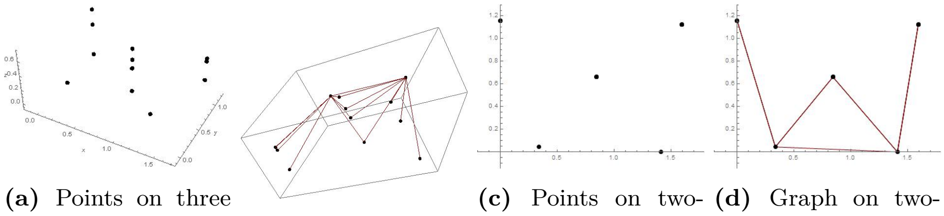
(a) Points on three dimensional coordinate system **(b)** Graph in a three-dimensional box **(c)** Points on two-dimensional coordinate system **(d)** Graph on two-dimensional coordinate system

Figure 1. The Spring Embedding of $3 + 2i$ on Two or Three-Dimensional Coordinate System.



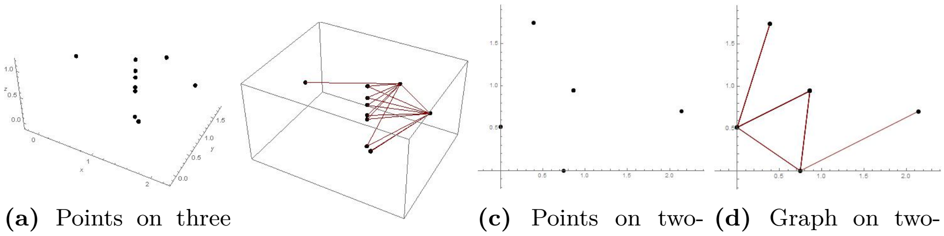
(a) Points on three dimensional coordinate system **(b)** Graph in a three-dimensional box **(c)** Points on two-dimensional coordinate system **(d)** Graph on two-dimensional coordinate system

Figure 2. The Spring Embedding of $\frac{5}{2}+\frac{3}{2}i+\frac{3}{2}j+\frac{3}{2}k$ on Two or Three-Dimensional Coordinate System.



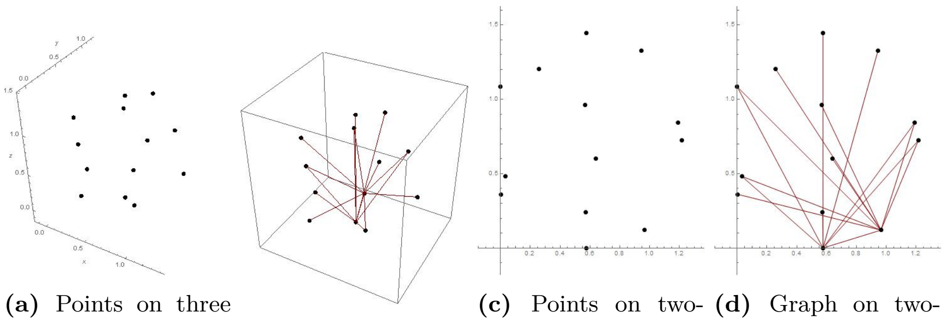
(a) Points on three dimensional coordinate system
(b) Graph in a three-dimensional box
(c) Points on two-dimensional coordinate system
(d) Graph on two-dimensional coordinate system

Figure 3. The High-Dimensional Embedding of $3 + 2i$ on Two or Three-Dimensional Coordinate System.



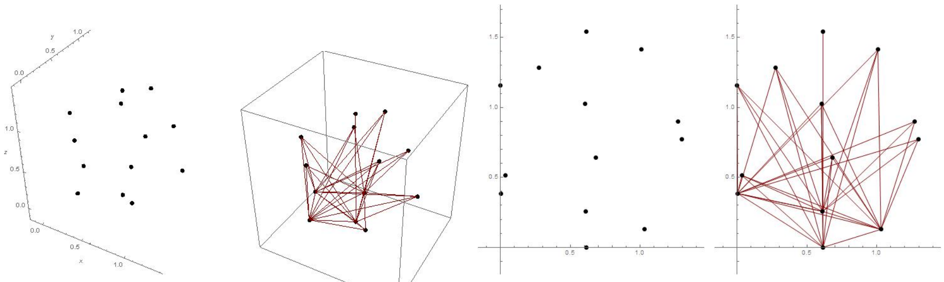
(a) Points on three dimensional coordinate system
(b) Graph in a three-dimensional box
(c) Points on two-dimensional coordinate system
(d) Graph on two-dimensional coordinate system

Figure 4. The High-Dimensional Embedding of $\frac{5}{2} + \frac{3}{2}i + \frac{3}{2}j + \frac{3}{2}k$ on Two or Three-Dimensional Coordinate System.



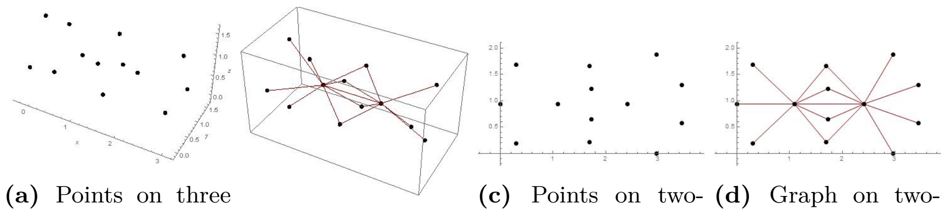
(a) Points on three dimensional coordinate system
(b) Graph in a three-dimensional box
(c) Points on two-dimensional coordinate system
(d) Graph on two-dimensional coordinate system

Figure 5. The Spiral Embedding of $3 + 2i$ on Two or Three-Dimensional Coordinate System.



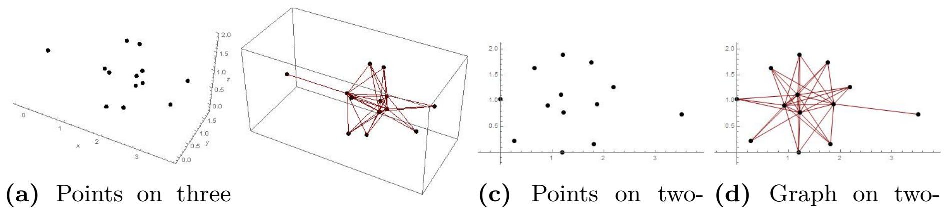
(a) Points on three dimensional coordinate system **(b)** Graph in a three-dimensional box **(c)** Points on two-dimensional coordinate system **(d)** Graph on two-dimensional coordinate system

Figure 6. The Spiral Embedding of $\frac{5}{2}+\frac{3}{2}i+\frac{3}{2}j+\frac{3}{2}k$ on Two or Three-Dimensional Coordinate System.



(a) Points on three dimensional coordinate system **(b)** Graph in a three-dimensional box **(c)** Points on two-dimensional coordinate system **(d)** Graph on two-dimensional coordinate system

Figure 7. Points and Graph of $3+2i$ on Two or Three-Dimensional Coordinate System.



(a) Points on three dimensional coordinate system **(b)** Graph in a three-dimensional box **(c)** Points on two-dimensional coordinate system **(d)** Graph on two-dimensional coordinate system

Figure 8. Points and Graph of $\frac{5}{2}+\frac{3}{2}i+\frac{3}{2}j+\frac{3}{2}k$ on Two or Three-Dimensional Coordinate System.

8. Conclusion

In this study, we presented a new algebraic construction technique used for codes over Hurwitz integers using prime Hurwitz integers, unlike Rohweder et al. (see [6]). In [6], they consider primitive Lipschitz integers. In this study, we consider prime Hurwitz integer. The algebraic construction technique in this study can also use to primitive Hurwitz integers. It is to construct the residue class rings of Hurwitz integers which is the norm of each element is less than or equal to $\frac{N(\alpha)}{2}$. Therefore, it is a suitable algebraic construction technique to obtained the minimum energy codes over higher-dimensional algebraic structures. Also, we presented some results for two mathematical notations, and for the algebraic construction technique in this study. In addition, we obtained some new block codes over Hurwitz integers with respect to the algebraic construction technique that presented in this study. You can see these block codes over Hurwitz integers in Section 5.

Moreover, we obtained $(3, 1)-$code, and $(4, 1)-$code (See Section 6). Lastly, we presented graphs of the residue class obtained with respect to related algebraic construction technique in the ring of Hurwitz integers, particularly prime Hurwitz integers.

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