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On a 2-Form Derived by Riemannian Metric in the Tangent Bundle

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ABSTRACT

In a recent paper [1] the authors have investigated the curious fact that the canonıcal symplectic structure $dp = dp_i \wedge dx^i$ on cotangent bundle may be given by the introduction of symplectic isomorphism between tangent and cotangent bundles. Our analysis began with the observation that the complete lift of the symplectic structure from the base manifold to its tangent bundle is being a closed 2-form and consequently we proved that its image by the simplectic isomorphism is the natural 2-form dp. We apply this construction in the case where the basic manifold of bundles is a Riemannian manifold with metric g and consider a new 1-form $\omega = g_{ij}y^j dx^i$ and its exterior differential on the tangent bundle, from which the symplectic structure is derived.

Keywords: *Tangent bundle, 2-form, symplectic manifold, complete lift, musical isomorphism.*
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1. Introduction

Let M be an n-dimensional C^∞-manifold and $T_P(M)$ $(T_P^*(M))$ the tangent (cotangent) vector space at a point $P \in M$. Then the set $T(M) = \bigcup\limits_{P\in M} T_P(M)$ $\left(T^*(M) = \bigcup\limits_{P\in M} T_P^*(M)\right)$ is by definition the tangent (cotangent) bundle over the manifold M. For any point $\tilde{P}$ of $T_P(M)$ $(T_P^*(M))$ such that $\tilde{P} \in T_P(M)$ $(\tilde{P} \in T_P^*(M))$ the correspondence $\tilde{P} \to P$ determines the natural tensor bundle projection $\pi : T(M) \to M$ $(\pi : T^*(M) \to M)$, that is $\pi(\tilde{P}) = P$.

Suppose that the base space M is covered by a system of coordinate neighborhoods (U, x^i), where x^i, $i = 1, ..., n$ are local coordinates in the neighborhood U. The open set $\pi^{-1}(U) \subset T(M)$ $(\pi^{-1}(U) \subset T^*(M))$ is naturally diffeomorphic to the direct product $U \times \mathbb{R}^n$ in such a way that a point $\tilde{P} \in T_P(M)$ $(\tilde{P} \in T_P^*(M))$ is represented by an ordered pair (P, y) $((P, p))$ of the point $P \in M$ and a vector (covector) $y \in \mathbb{R}^n$ $(p \in \mathbb{R}^n)$ whose components are given by y^i (p_i) of $\tilde{P}$ in $T_P(M)$ $(T_P^*(M))$ with respect to the frame (coframe) ∂_i (dx^i). Denoting by (x^i) the coordinates of $P = \pi(\tilde{P})$ in U and establishing the correspondence $(x^i, y^i) \to \tilde{P} \in \pi^{-1}(U)$ $((x^i, p_i) \to \tilde{P} \in \pi^{-1}(U))$, we can introduce a system of local coordinates $(x^i, x^{\bar{i}}) = (x^i, y^i)$ $((x^i, \tilde{x}^{\bar{i}}) = (x^i, p_i))$, $\bar{i} = n+1, ..., 2n$ in the open set $\pi^{-1}(U) \subset T(M)$ $(\pi^{-1}(U) \subset T^*(M))$. We call $(x^i, x^{\bar{i}}) = (x^J)$, $J = 1, ..., 2n$ $((x^i, \tilde{x}^{\bar{i}}) = (\tilde{x}^J)$, $J = 1, ..., 2n)$ the induced coordinates in $\pi^{-1}(U) \subset T(M)$ $(\pi^{-1}(U) \subset T^*(M))$.

A manifold M is symplectic if it possesses a nondegenerate 2-form ε which is closed (i.e. $d\varepsilon = 0$). For any manifold M of dimension n, the cotangent bundle $T^*(M)$ is a symplectic $2n$-manifold with symplectic 2-form $\varepsilon = -dp = dx^i \wedge dp_i$, where $p = p_i dx^i$ is the Liouville form (basic 1-form) in $T^*(M)$.

In Riemannian geometry, the musical isomorphism (or canonical isomorphism) is an isomorphism between the tangent and cotangent bundles of a Riemannian manifold given by its metric. Let now (M, g) be a Riemannian manifold. Then the musical isomorphisms $g^b : TM \to T^*M$ and $g^\sharp : T^*M \to TM$ are given by

$$g^b : x^I = (x^i, x^{\bar{i}}) = (x^i, v^i) \to \tilde{x}^K = (x^k, \tilde{x}^{\bar{k}}) = (x^k = \delta_i^k x^i, p_k = g_{ki}v^i)$$

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and

$$g^{\sharp} : \tilde{x}^K = (x^k, \tilde{x}^{\bar{k}}) = (x^k, p_k) \to x^I = (x^i, x^{\bar{i}}) = (x^i = \delta^i_k x^k, v^i = g^{ik} p_k) \ ,$$

where $g^{ik} g_{kj} = \delta^i_j$, δ^i_j is the Kronecker symbol. The Jacobian matrices of g^b and $g^{\sharp}$ are given respectively by

$$(g^b)_* = \tilde{A} = (\tilde{A}^K_I) = \begin{pmatrix} \tilde{A}^k_i & \tilde{A}^k_{\bar{i}} \\ \tilde{A}^{\bar{k}}_i & \tilde{A}^{\bar{k}}_{\bar{i}} \end{pmatrix} = \left(\frac{\partial \tilde{x}^K}{\partial x^I} \right) = \begin{pmatrix} \delta^k_i & 0 \\ v^s \partial_i g_{ks} & g_{ki} \end{pmatrix}$$

and

$$(g^{\sharp})_* = A = (A^I_K) = \begin{pmatrix} A^i_k & A^i_{\bar{k}} \\ A^{\bar{i}}_k & A^{\bar{i}}_{\bar{k}} \end{pmatrix} = \left(\frac{\partial x^I}{\partial \tilde{x}^K} \right) = \begin{pmatrix} \delta^i_k & 0 \\ p_s \partial_k g^{is} & g^{ik} \end{pmatrix} . \tag{1.1}$$

As a prerequisite for our analysis, we need various techniques for lifting objects from a manifold to its tangent and cotangent bundles. Most of these have been described by Yano and Ishihara [3] (also, see [2]). The main purpose of this paper is to study the properties of symplectic 2-form $d\omega$, where $\omega = g_{ij} y^j dx^i$ is the Liouville type 1-form in the tangent bundle.

2. Symplectic 2-form

Let now (M, g) be a Riemannian (or a pseudo-Riemannian) manifold. Then

$$\omega = \imath g = g_{ji} y^j dx^i$$

is a new 1-form in TM. Consequently its exterior differential, i.e. 2-form $d\omega$ has the following expression

$$\begin{aligned} d\omega &= d(g_{js} y^s) \wedge dx^j = (dg_{js}) y^s \wedge dx^j + g_{js} dy^s \wedge dx^j = (\partial_k g_{js}) y^s dx^k \wedge dx^j + g_{js} dy^s \wedge dx^j \\ &= \frac{1}{2} (\partial_k g_{js}) y^s dx^k \wedge dx^j + \frac{1}{2} (\partial_k g_{js}) y^s dx^k \wedge dx^j + \frac{1}{2} g_{js} dy^s \wedge dx^j + \frac{1}{2} g_{js} dy^s \wedge dx^j \\ &= \frac{1}{2} (\partial_i g_{js}) y^s dx^i \wedge dx^j + \frac{1}{2} (\partial_j g_{is}) y^s dx^j \wedge dx^i + \frac{1}{2} g_{ij} dy^j \wedge dx^i + \frac{1}{2} g_{ij} dy^j \wedge dx^i \\ &= -\frac{1}{2} (\partial_i g_{js}) y^s dx^j \wedge dx^i + \frac{1}{2} (\partial_j g_{is}) y^s dx^j \wedge dx^i + \frac{1}{2} g_{ij} dy^j \wedge dx^i - \frac{1}{2} g_{ji} dx^j \wedge dy^i \\ &= \frac{1}{2} (\partial_j g_{is} - \partial_i g_{js}) y^s dx^j \wedge dx^i + \frac{1}{2} g_{ji} dx^{\bar{j}} \wedge dx^i - \frac{1}{2} g_{ji} dx^j \wedge dx^{\bar{i}} . \end{aligned}$$

Since the 2-form $d\omega$ in TM has the following general expression

$$\begin{aligned} d\omega = \frac{1}{2} (d\omega)_{JI} dx^J \wedge dx^I &= \frac{1}{2} (d\omega)_{ji} dx^j \wedge dx^i + \frac{1}{2} (d\omega)_{\bar{j}i} dx^{\bar{j}} \wedge dx^i + \frac{1}{2} (d\omega)_{j\bar{i}} dx^j \wedge dx^{\bar{i}} \\ &+ \frac{1}{2} d\omega_{\bar{j}\,\bar{i}} dx^{\bar{j}} \wedge dx^{\bar{i}}, \end{aligned}$$

therefore we have

$$(d\omega)_{ji} = (\partial_j g_{is} - \partial_i g_{js}) y^s, \quad (d\omega)_{\bar{j}i} = g_{ji}, \quad (d\omega)_{j\bar{i}} = -g_{ji}, \quad (d\omega)_{\bar{j}\,\bar{i}} = 0 \, ,$$

i.e. an anti-symmetric (0,2)-tensor $d\omega$ has the following components

$$d\omega = ((d\omega)_{JI}) = \begin{pmatrix} (d\omega)_{ji} & (d\omega)_{j\bar{i}} \\ (d\omega)_{\bar{j}i} & (d\omega)_{\bar{j}\,\bar{i}} \end{pmatrix} = \begin{pmatrix} (\partial_j g_{is} - \partial_i g_{js}) y^s & -g_{ji} \\ g_{ji} & 0 \end{pmatrix} . \tag{2.1}$$

On the other hand $d(d\omega) = d^2\omega = 0$, i.e. $d\omega$ defines a symplectic 2-form in the tangent bundle TM.
Let now $\tilde{X}$ be any vector field on TM. Using the Cartan formula ($L_{\tilde{X}} \Omega = d\imath_{\tilde{X}} \Omega + \imath_{\tilde{X}} d\Omega$), we obtain

$$L_{\tilde{X}} d\omega = d(\imath_{\tilde{X}} d\omega) + \imath_{\tilde{X}} d^2\omega = d(\imath_{\tilde{X}} d\omega),$$

where $\imath_{\tilde{X}}$ is the interior product. Thus we have

Theorem 2.1. *A vector field $\tilde{X}$ on TM is a symplectic vector field with respect to $d\omega$, if*

$$d(\imath_{\tilde{X}}d\omega)=0,$$

i.e. if the interior product $\imath_{\tilde{X}}d\omega$ is closed.

Let now $\tilde{X}={}^CX$, where ${}^CX=X^i\partial_i+y^s\partial_sX^i\partial_{\bar{i}}$ is the complete lift of vector field $X=X^i\partial_i$ from M to TM [3, p.15]. Then using (2.1) we obtain

$$\begin{aligned}\imath_{{}^CX}d\omega={}^CX^J(d\omega)_{JI}&=(X^j(d\omega)_{ji}+y^s\partial_sX^j(d\omega)_{\bar{j}i},X^j(d\omega)_{j\bar{i}}+y^s\partial_sX^j(d\omega)_{\bar{j}\bar{i}})\\&=(X^jy^s(\partial_jg_{is}-\partial_ig_{js})+y^s(\partial_sX^j)g_{ji},-X^jg_{ji})=(X^jy^s(\Gamma^t_{js}g_{it}-\Gamma^t_{is}g_{jt})\\&+y^s(\nabla_sX^j-\Gamma^j_{st}X^t)g_{ji},-X^jg_{ji})=(y^s(\nabla_sX^j)g_{ji}-X^jy^s\Gamma^t_{is}g_{jt},-X^jg_{ji})\\&=(y^s\nabla_s(g_X)_i,0)-(y^s\Gamma^t_{is}(g_X)_t,(g_X)_i)=\imath(\nabla g_X)-{}^Hg_X\,,\end{aligned}$$

where $\imath(\nabla g_X)=(y^s\nabla_s(g_X)_i,0)$ and ${}^Hg_X=(y^s\Gamma^t_{is}(g_X)_t,(g_X)_i)$ are the vertical and horizontal lifts of $g_X=X^jg_{ji}dx^i$ to TM, respectively. Let now $\nabla X=0$. It is clear that, then $\imath(\nabla g_X)=0$. Therefore, we have

$$L_{{}^CX}d\omega=d(\imath_{{}^CX}d\omega)=-d({}^Hg_X).$$

On the other hand using $d({}^Hg_X)={}^H(dg_X)$[3, p.24] and

$$dg_X=((\partial_jg_X)_i-(\partial_ig_X)_j)dx^j\wedge dx^i=((\nabla_jg_X)_i-(\nabla_ig_X)_j)dx^j\wedge dx^i=0$$

we find $L_{{}^CX}d\omega=0$. Thus we have

Theorem 2.2. $\nabla X=0$, *then the complete lift ${}^CX$ is a symplectic vector field with respect to the symplectic 2-form $d\omega$.*

Also for any vector fields $X=X^i\partial_i$ and $Y=Y^j\partial_j$, we have

$$\begin{aligned}(d\omega)({}^CX,{}^CY)&=(d\omega)_{JI}{}^CX^J{}^CY^I=(d\omega)_{ji}X^jY^i+(d\omega)_{\bar{j}i}y^s(\partial_sX^j)Y^i\\&+(d\omega)_{j\bar{i}}X^jy^s\partial_sY^i=y^s(\partial_jg_{is}-\partial_ig_{js})X^jY^i+g_{ji}y^s(\partial_sX^j)Y^i-g_{ji}X^jy^s\partial_sY^i\\&=y^s(\Gamma^t_{js}g_{it}-\Gamma^t_{is}g_{tj})X^jY^i+g_{ji}y^s(\partial_sX^j)Y^i-g_{ji}X^jy^s\partial_sY^i\\&=y^s\Gamma^j_{ts}g_{ij}X^tY^i+g_{ji}y^s(\partial_sX^j)Y^i-(y^s\Gamma^i_{ts}g_{ij}X^jY^t+g_{ji}X^jy^s\partial_sY^i)\\&=Y^ig_{ij}y^s(\partial_sX^j+\Gamma^j_{st}X^t)-X^jg_{ji}y^s(\partial_sY^i+\Gamma^i_{st}Y^t)\\&=(\imath_Yg)(\imath(\nabla X)-(\imath_Xg)(\imath(\nabla Y).\end{aligned}$$

Thus

$$(d\omega)({}^CX,{}^CY)=(\imath_Yg)(\imath(\nabla X)-(\imath_Xg)(\imath(\nabla Y)\,. \tag{2.2}$$

Since if S and T are 2-forms in the tangent bundle, such that

$$S({}^CX,{}^CY)=T({}^CX,{}^CY)\,,$$

then $S=T$ [3, p. 33]. From here we have the symplectic 2-form $d\omega$ is completely determined by (2.2).

3. Pullback of $d\omega$

It is well known that in the cotangent bundle $T^*(M)$ there exists a closed 2-form $\varepsilon=dp=dp_i\wedge dx^i$, where $p=p_idx^i$, i.e. $T^*(M)$ is naturally a symplectic manifold. If we write $\varepsilon=\frac{1}{2}\varepsilon_{KL}dx^K\wedge dx^L$, then we have

$$\varepsilon=(\varepsilon_{KL})=\begin{pmatrix}0&\delta^l_k\\-\delta^k_l&0\end{pmatrix}\,.$$

From (1.1) and (2.1) we see that the pullback of $d\omega$ by $g^\sharp$ is a 2-form $(g^\sharp)^*\,d\omega$ on T^*M and has components

$$((g^\sharp)^*\,d\omega)_{kl}=A^I_kA^J_l(d\omega)_{IJ}=A^i_kA^j_l(d\omega)_{i\,j}+A^{\bar{i}}_kA^j_l(d\omega)_{\bar{i}\,j}+A^i_kA^{\bar{j}}_l(d\omega)_{i\,\bar{j}}$$

$$= y^s(\partial_k g_{ls} - \partial_l g_{ks}) + p_s(\partial_k g^{is})g_{i\,l} - p_s(\partial_l g^{js})g_{kj} = p_t g^{ts}(\partial_k g_{ls} - \partial_l g_{ks}) - p_t((\partial_k g_{i\,l})g^{it}$$
$$-(\partial_l g_{kj})g^{jt}) = 0$$
$$((g^\sharp)^*\,d\omega)_{k\,\bar{l}} = A^i_k A^{\bar{j}}_{\bar{l}}(d\omega)_{i\,\bar{j}} = -\delta^i_k g^{jl} g_{i\,j} = -\delta^i_k \delta^l_i = -\delta^l_k\ ,$$
$$((g^\sharp)^*\,d\omega)_{\bar{k}\,l} = A^{\bar{i}}_{\bar{k}} A^j_l (d\omega)_{\bar{i}\,j} = g^{ik}\delta^j_l g_{i\,j} = \delta^k_j \delta^j_l = \delta^k_l\ ,$$
$$((g^\sharp)^*\,d\omega)_{\bar{k}\,\bar{l}} = 0$$

or

$$(g^\sharp)^*\,d\omega = (((g^\sharp)^*\,d\omega)_{KL}) = \begin{pmatrix} 0 & -\delta^l_k \\ \delta^k_l & 0 \end{pmatrix}.$$

From here follows that the pullback $(g^\sharp)^*\,d\omega$ coincides with the symplectic form $\tilde{\omega} = -dp = -dp_i \wedge dx^i = dx^i \wedge dp_i$. Thus we have

Theorem 3.1. *Let (M, g) be a Riemanian (or a pseudo-Riemannian) manifold. The natural symplectic structure $-dp = dx^i \wedge dp_i$ on cotangent bundle T^*M is a pullback by $g^\sharp$ of exterior derivative $d\omega$, where $\omega = y^i g_{ij} dx^j$, i.e. $(g^\sharp)^*\,d\omega = -dp$.*

A diffeomorphism between any two symplectic manifods $f : (M, \omega) \to (N, \omega')$ is called symplectomorphism if $f^*\omega' = \omega$, where f^* is the pullback of f. Since $d(d\omega) = 0$ and $d(dp) = 0$, from Theorem 3 we have

Corollary 3.1. *The musical isomorphism $g^\sharp : (T^*M, -dp) \to (TM, d\omega)$ is a symplectomorphism.*

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Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

References

[1] Salimov, A., Asl, M.B., Kazimova, S.: *Problems of lifts in symplectic geometry.* Chin Ann. Math. Ser. B. **40**(3), (2019), 321-330.
[2] Salimov, A.: *Tensor operators and their applications.* Nova Science Publishers Inc., New York, (2013).
[3] Yano, K., Ishihara, S.:*Tangent and cotangent bundles.* Marcel Dekker Inc., New York, (1973).

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