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THE λ - TRANSLATION AND THE CONVEXITY OF A EUCLIDEAN HYPERSURFACE

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SUMMARY

In this paper, we studied some properties of the first and second fundamental tensors of a closed convex hypersurface in a Euclidean space and worked out that the λ- translation preserves the convexity of the hypersurface provided that $\lambda > 0$.

ÖKLİD UZAYINDA BİR HİPERYÜZEYİN KONVEKSLİĞİ VE λ - ÖTELENMESİ

ÖZET

Bu makalede bir öklid uzayındaki kapalı konveks bir hiperyüzeyin birinci ve ikinci temel tensörlerinin bazı özelliklerini çalışıp bir λ - ötelenmesinin, $\lambda > 0$ olmak şartı ile hiperyüzeyin konveksliğini koruduğunu elde ettik.

I. INTRODUCTION

Let V_{n-1} be a hypersurface imbedded in a Euclidean space $\mathfrak{R}^n$ of n-dimension. We shall suppose that the hypersurface is represented parametrically by the equations

$$x^j = x^j(u^\alpha) , \quad \alpha = 1,2,\ldots,n-1 , \qquad (1.1)$$

in which the $n-1 = m$ -variables u^α represent the parameters of V_{n-1} and the x^j's are the local coordinates in $\mathfrak{R}^n$. It will be assumed that the functions of (1.1) are at least of class C^3 in the u^α. Furthermore, we will assume that the matrix $[\partial x^j / \partial u^\alpha]$ has the maximal rank m. Throughout this paper Greek indices take the values 1 to n-1 and Latin indices the values 1 to n unless stated otherwise. we use the convention that repeated indices imply summation.

By using the Christoffel symbols of V_{n-1} and $\mathfrak{R}^n$ we can define the mixed covariant vectors. $\partial x^j / \partial u^\beta$ is a covariant vector field and we can define the mixed covariant derivation of it (with respect to u^γ) as ([1],[2])

$$\frac{\partial^2 x^j}{\partial u^\beta ,u^\gamma} = \frac{\partial^2 x^j}{\partial u^\beta \partial u} - \frac{\partial x^j}{\partial u^\lambda} \Gamma^{\lambda}_{\beta\gamma} + \Gamma^{j}_{hk} \frac{\partial x^h}{\partial u^\beta} \frac{\partial x^k}{\partial u^\gamma} . \qquad (1.2)$$

It will be used a comma to distinguish the mixed covariant derivatives from the usual ones. So it is a scalar multiply of the normal vector field of V_{n-1}, that is

$$\frac{\partial^2 x^j}{\partial u^\beta ,u^\gamma} = \Omega_{\beta\gamma} N^j . \qquad (1.3)$$

II. SOME RELATIONS BETWEEN THE FUNDAMENTAL FORMS

If $g_{hj}(x^k)$ denotes the metric tensor of $\mathfrak{R}^n$, from $g_{hj} N^h N^j = 1$ and Ricci's lemma, it follows that $g_{hj} N^h \partial N^j / \partial u^\gamma = 0$, so that $\partial N^j / \partial u^\gamma$ is in the tangent space of V_{n-1}. Thus we may write

$$(\partial N^j / \partial u^\gamma) = \eta^\alpha_{\ \gamma} (\partial x^j / \partial u^\alpha) \qquad (II.1)$$

When we differentiate $g_{hj} (\partial x^h / \partial u^\beta) N^j$ with respect to u^γ and noting (1.2), (1.3) and (II.1) we get that

$$g_{hj}\Omega_{\beta\gamma}N^h N^j + g_{hj}(\partial x^h/\partial u^\beta)\cdot(\partial x^j/\partial u^\alpha)\eta^\alpha{}_\gamma = 0 \text{ and that}$$

$$g_{\beta\alpha}\eta^\alpha{}_\gamma = -\Omega_{\beta\gamma} \qquad (\text{II}.2)$$

Let us define a new symbol $\Omega^\beta{}_\alpha = g^{\beta\gamma}\,\Omega_{\gamma\alpha}$, then it is possible to write that

$$(\partial N^j/\partial u^\alpha) = (\partial x^j/\partial u^\beta)\,\Omega^\beta{}_\alpha \qquad (\text{II}.3)$$

A displacement at a point $M(x^1(u^1,\dots,u^{n-1}),\dots,x^n(u^1,\dots,u^{n-1}))$ of V_{n-1} is given in the tangent space $T_n(M)$ by the equations

$$dx^1 = \sum_{\alpha=1}^{n-1}(\partial x^1/\partial u^\alpha)\,du^\alpha\ ,\dots,\quad dx^n = \sum_{\alpha=1}^{n-1}(\partial x^n/\partial u^\alpha)\,du^\alpha$$

This imply a displacement of normal vector at point M, by using (II.3) we have it as follows

$$\tilde{d}N^j = (\partial N^j/,\partial u^\alpha)\,du^\alpha \text{ or } \tilde{d}N^j = -(\partial x^j/\partial u^\beta)\,\Omega_\alpha{}^\beta\,du^\alpha \qquad (\text{II}.4)$$

Thus we may write the second fundamental tensor as follows

$$h_{\alpha\beta} = g_{jh}\frac{\partial x^j}{\partial u^\alpha,\partial u^\beta}N^h = -g_{jh}\frac{\partial x^j}{\partial u^\alpha}\,\frac{\partial N^h}{,\partial u^\beta} = g_{jh}\frac{\partial x^j}{\partial u^\alpha}\,\frac{\partial x^h}{\partial u^\gamma}\,\Omega_\beta{}^\gamma$$

or

$$h_{\alpha\beta} = g_{\alpha\gamma}\,\Omega_\beta{}^\gamma \qquad (\text{II}.6)$$

As to third fundamental tensor, from (II.3), (II.5) and its classical definition we have that

$$e_{\alpha\beta} = g_{jh}\frac{\partial N^h}{,\partial u^\alpha}\,\frac{\partial N^h}{,\partial u^\beta} = g_{jh}\frac{\partial x^j}{\partial u^\delta}\,\frac{\partial x^h}{\partial u^\varepsilon}\,\Omega_\alpha{}^\delta\Omega_\beta{}^\varepsilon \qquad (\text{II}.7)$$

or by taking into that

$$g_{\alpha\beta} = g_{hj}\frac{\partial x^j}{\partial u^\alpha}\,\frac{\partial x^h}{\partial u^\beta}$$

and using that $\Omega^{\beta}{}_{\alpha} = g^{\beta\gamma}\,\Omega_{\gamma\alpha}$ we get

$$e_{\alpha\beta} = h_{\alpha\gamma}\;\Omega^{\gamma}{}_{\beta} \qquad \text{(II.8)}$$

Finally from (II.6) we find that

$$e_{\alpha\beta} = g_{\delta\varepsilon}\;\Omega^{\delta}{}_{\alpha}\,\Omega^{\varepsilon}{}_{\beta}\,. \qquad \text{(II.9)}$$

III. λ-TRANSLATION AND THE CONVEXITY OF A HYPERSURFACE

THEOREM III.1. A smooth closed hypersurface of $\mathfrak{R}^n$ which is locally diffeomorphic to a hypersphere is convex if and only if its second fundamental form is semi-definite, i.e. $h_{\alpha\beta}\,x^{\alpha}x^{\beta} \le 0$ $(g_{\alpha\beta}\,x^{\alpha}x^{\beta} \ne 0)$ holds at its each point.

PROOF: We assume that the normal of hypersurface V_{n-1} is directed outwards. Suppose that $h_{\alpha\beta}\,x^{\alpha}x^{\beta} \le 0$ holds ewerywhere and that V_{n-1} is not convex. Now, let $r(u^{\alpha}) = r(u^1,\dots,u^{n-1})$ be a parametrization in P, with $P = r(0,\dots,0)$. the distance from a point $Q = r(u^1,\dots,u^{n-1})$ to the tangent plane $T_p(V_{n-1})$ is given by $N_p.\overrightarrow{PQ}$. Since $r(u^{\alpha})$ is differentiable, we have Taylor's formula:

$$r(u^1,\dots,u^{n-1}) = r(0,\dots,0) + \sum_{\alpha=1}^{n-1} (\partial r/\partial u^{\alpha})\, u^{\alpha} + (1/2)\left[\sum_{\alpha=1}^{n-1} (\partial^2 r/\partial u^{\alpha}\partial u^{\alpha})\,(u^{\alpha})^2 + 2\sum_{\substack{\alpha<\beta=1\\ \alpha\ne\beta}}^{n-1} (\partial^2 r/\partial u^{\alpha}\partial u^{\beta})\, u^{\alpha}u^{\beta}\right] + \overline{R}\,,$$

where the derivatives are taken at $(0,\dots,0)$ and the remainder $\overline{R}$ satisfies the condition

$$\lim_{(u^1,\dots,u^{n-1})\to(0,\dots,0)} \left(\overline{R}\Big/\sum_{\alpha=1}^{n-1}(u^{\alpha})^2\right) = 0\,.$$

Hence it follows that

$$N_p\,\overrightarrow{PQ} = N_p\big(r(u^1,\dots,u^{n-1}) - r(0,\dots,0)\big) =$$

$$(1/2)\left[\sum_{\alpha=1}^{n-1} N_p.\,(\partial^2 r/\partial u^{\alpha}\partial u^{\alpha})\,(u^{\alpha})^2 + 2\sum_{\substack{\alpha<\beta=1\\ \alpha\ne\beta}}^{n-1} (\partial^2 r/\partial u^{\alpha}\partial u^{\beta})\, u^{\alpha}u^{\beta}\right] + R\,,$$

where $\bar{R} = N_p.R$, or

$$N_p.PQ = 1/2\,(h_{\alpha\beta}\,du^{\alpha}du^{\beta}) + R \; . \qquad (III.1)$$

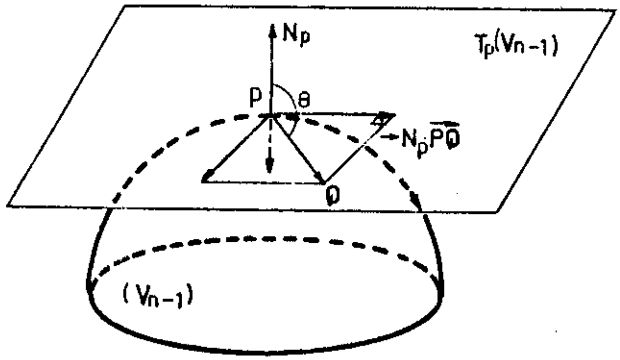

Figure III.I

Since V_{n-1} is connected and locally diffeomorphic to a sphere by the Gauss map , it has a positive definite Gaussian curvature and , therefore , it is bending away from its tangent space in all tangent directions at p. Thus, we can find a line segment PQ such that P,Q exist on V_{n-1} and the lenght of PQ is so short that

$$N.PQ = 1/2\,(h_{\alpha\beta}\,du^{\alpha}du^{\beta}) + ... \leq 0 \; , \qquad (III.2)$$

where N denotes the unit normal at point P. From (III.2) we write that

$$N.PQ = \|N\| \; \|PQ\| \; \cos\theta \leq 0 \text{ or } \cos\theta \leq 0 \text{ , where } \theta = \angle (N,PQ).$$

So we get $\pi/2 \leq 3\,\pi/2$, thus we see that the line segment PQ does not lie on the outside of V_{n-1}. This contradicts the assumption. Hence V_{n-1} is convex. Next suppose that V_{n-1} is convex and at the point P of V_{n-1} , $(h_{\alpha\beta})_p X^{\alpha}X^{\beta} > 0$, $(g_{\alpha\beta})_p X^{\alpha}X^{\beta} \neq 0$ holds. Then we have that

$$r.N_p = r_p.N_p + 1/2\,(h_{\alpha\beta})_p\,du^{\alpha}du^{\beta} + \varepsilon \; , \qquad (III.3)$$

where $r = OA$, $r_p = OP$, and A is a neighboring point of P on V_{n-1} and $|\varepsilon|$ is an infinitesimal quantity whose degree is higher than that of $(g_{\alpha\beta})_p\,du^{\alpha}du^{\beta}$. Putting $du^{\alpha} = X^{\alpha}t$ we have from (III.3)

$$(\vec{r} - r_p)N_p = 1/2\,(h_{\alpha\beta})_p X^{\alpha}X^{\beta}t^2 + \varepsilon \; . \qquad (III.4)$$

If we put $OP = r(u^{\alpha})$, $OP_1 = r(u^{\alpha} + t\,X^{\alpha})$, $OP_2 = r(u^{\alpha} - t\,X^{\alpha})$,
then we find from (III.4) that the line segment P_1P_2 lies on the outside of V_{n-1} if $|t|$ is small enough.

A slightly different version of the proof of this theorem can be found in [3].

DEFINITION III.1: Let V_{n-1} be a compact and orientable hypersurface which is imbedded in a Euclidean space $\mathbb{R}^n$. If $r = (X^1(u^\alpha), ..., X^n(u^\alpha))$, $1 \leq \alpha \leq n-1$, is any point on V_{n-1} and $N = N(u^\alpha)$ is C^∞ unit normal vector field on V_{n-1}, then we may consider an imbedding $X_\lambda : V_{n-1} \to \mathbb{R}^n$ defined by

$$X_\lambda(r) = r(\lambda) = r + \lambda N \ , \lambda \in \mathbb{R} \ , \tag{III.5}$$

where, u^α's are the parameters of V_{n-1} and X^i's are the local coordinates in V_{n-1}. Then this imbedding may be called the λ- translation of V_{n-1}.

Since $N(u^\alpha) . \partial r / \partial u^\alpha = 0$, differentiating (III.5) along V_{n-1} we may write that

$$\frac{\partial r(\lambda)}{\partial u^\alpha} = \frac{\partial r}{\partial u^\alpha} + \lambda \frac{\partial N}{\partial u^\alpha}, \quad N . \frac{\partial r(\lambda)}{\partial u^\alpha} = 0 \ , N(\lambda) = N.$$

Hence we can get the fundamental relations of $V_{n-1}(\lambda)$ by the following equations

$$g_{\alpha\beta}(\lambda) = g_{\alpha\beta} - 2\lambda h_{\alpha\beta} + \lambda^2 e_{\alpha\beta} = (g_{\alpha\gamma} - \lambda h_{\alpha\gamma})(g_{\beta\delta} - \lambda h_{\beta\delta}) g^{\gamma\delta} \tag{III.6}$$

$$h_{\alpha\beta}(\lambda) = N . \frac{\partial^2 r(\lambda)}{\partial u^\alpha \partial u^\beta} = h_{\alpha\beta} + \lambda \left(N . \frac{\partial^2 N}{\partial u^\alpha \partial u^\beta}\right) = h_{\alpha\beta} - \lambda e_{\alpha\beta} \tag{III.7}$$

LEMMA III.1: For a convex hypersurface V_{n-1}, when $\lambda > 0$ the quadratic form $g_{\alpha\beta}(\lambda) X^\alpha X^\beta$ is positive definite.

PROOF: We have from (III.6)
$g_{\alpha\beta}(\lambda) X^\alpha X^\beta = (g_{\alpha\gamma} - \lambda h_{\alpha\gamma}) X^\alpha (g_{\beta\delta} - \lambda h_{\beta\delta}) X^\beta g^{\gamma\delta} \geq 0$. $g_{\alpha\beta}(\lambda) X^\alpha X^\beta$ becomes zero if and only if $(g_{\alpha\gamma} - \lambda h_{\alpha\gamma}) X^\gamma = 0$. Hence we get $g_{\alpha\beta} X^\alpha X^\beta = \lambda h_{\alpha\beta} X^\alpha X^\beta$. Now considering Theorem (III.1), if the last equality holds, then $g_{\alpha\beta} X^\alpha X^\beta \leq 0$ and so $X^\alpha = 0$.

LEMMA III.2: For a convex hypersurface V_{n-1}, when $\lambda > 0$ the quadratic form $h_{\alpha\beta}(\lambda) X^\alpha X^\beta$ is negative semidefinite.

PROOF: From (II.6) - (II.9), (III.7) and the Theorem III.1 we can write

$$h_{\alpha\beta}(\lambda) X^\alpha X^\beta = h_{\alpha\beta}(\lambda) X^\alpha X^\beta - \lambda e_{\alpha\gamma} X^\alpha X^\beta = h_{\alpha\beta} X^\alpha X^\beta - \lambda h_{\alpha\gamma} h_{\beta\delta} g^{\gamma\delta} X^\alpha X^\beta$$
$$= h_{\alpha\beta} X^\alpha X^\beta - \lambda (h_{\alpha\gamma} X^\alpha)(h_{\beta\delta} X^\beta) g^{\gamma\delta} \leq 0 \ .$$

From Lemma III.1, Lemma III.2 and the Theorem III.1 we have easily the following

THEOREM III.2: Let V_{n-1} be a closed convex hypersurface of V_{n-1}, which is locally diffeomorphic to a sphere. When $\lambda > 0$, the λ- translation preserves the convexity of the hypersurface.

IV. λ-TRANSLATION AND THE PRINCIPAL CURVATURES OF A HYPERSURFACE

Let $k = k(u^\alpha)$ be any principal curvature function on V_{n-1}. Then from

$$\partial r(\lambda) / \partial u^\alpha = \partial r / \partial u^\alpha + \lambda (\partial N / \partial u^\alpha)$$

and the equation of Olinde - Rodriques in the form

$$(\partial N / \partial u^\alpha) + k(\partial r / \partial u^\alpha) = 0$$

it follows that

$$\partial r(\lambda) / \partial u^\alpha = (\partial N / \partial u^\alpha)(-1/k + \lambda), k \equiv 0 \qquad \text{(IV.1)}$$

So, from (IV.1), the principal curvature function on $V_{n-1}(\lambda)$ is given by

$$k(\lambda) = (k/1 - \lambda k). \qquad \text{(IV.2)}$$

DEFINITION IV.1: A point P of a hypersurface V_{n-1} in $\mathbb{R}^n$ of dimension $n \geq 3$ is umbilical if $k_1 = k_2 = ... = k_{n-1}$ at the point P, where k_i's are the principal curvatures at the point P of the hypersurface.

By the equation of Olinde - Rodriques the fact that all points of V_{n-1} are umbilical points can be expressed in the equations [2]

$$\partial N / \partial u^\alpha + k(\partial r / \partial u^\alpha) = 0, \quad (\text{or } h_{\alpha\beta} = k g_{\alpha\beta}) \qquad \text{(IV.3)}$$

If all points of V_{n-1} are umbilical points, then (IV.3) is equivalent to $N + r k = p$, where p is a constant vector.

LEMMA IV.1: To be $\lambda > 0$ or $\lambda < 0$, the λ - translation of a sphere is again sphere.

PROOF: Since all points of a sphere are umbilical points, from $N + rk = p$ we have that

$$N + (r + \lambda N)(k/1 - \lambda k) = p + \lambda \text{ or } N = -rk + (p + \lambda)(1 - \lambda k).$$

Thus, since $N.N = 1$, we obtain that

$$(r - a(\lambda))(r - a(\lambda)) = 1/k^2, \text{ where } a(\lambda) = p + /k(\lambda).$$

This lemma shows us that λ - translation ($\lambda > 0$ or $\lambda < 0$) preserves the convexity of a sphere , but when $\lambda < 0$ this is not always true for any hypersurface. For instance, let us consider an ellipsoid instead of a sphere. Assuming that the normal of it is again directed outwards, in the case $\lambda < -b < 0$ (where b is the length of one of the minor axes of the ellipsoid) we see that the λ - translation of it is not always convex. For an ellipse in a plane Fig IV.1.(a)-(b) tells us that how the image of the λ - translation of an ellipse changes according to λ ($\lambda < 0$). Fig.IV.1.(a) illustrates the case $-b < \lambda < 0$ and Fig IV.1.(b) illusrates the case $\lambda < -b < 0$. In the first case the principal curvatures on the λ - translation of the ellipse do not have different signs. Whereas in the latter case the principal curvatures on the λ - translation of the ellipse have different signs, therefore, in this case the λ - translation is not convex.

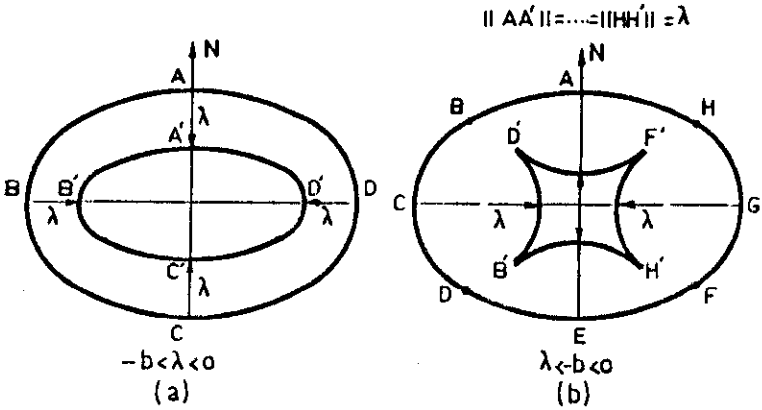

Figure IV.1

Finally, we obtain the following conclusions.

CONCLUSIONS

1) Let k be a principal curvature function on V_{n-1}. From (IV.2), if $\lambda = 1/k$ the λ- translation of V_{n-1} does not represent a hypersurface.

2) The λ- translation ($\lambda > 0$ or $\lambda < 0$) of a connected umbilic hypersurface of $\mathfrak{R}^n$ is still convex.

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