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A NOTE ON THE INVERSES OF COMPANION MATRICES

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ABSTRACT

For the companion matrix $B=A-ab^t$, it was given a statement depend on the component of b-vector of the matrix A and it was searched the structure of A^{-1}.

ÖZET

Bir $B=A-ab^t$ companion matrisi için A matrisinin, b-vektör bileşenine bağlı bir ifadesi verildi ve A matrisinin tersinin yapısı araştırıldı.

1. INTRODUCTION

In [1] W. Barrett and in [2] W. Barrett and P. J. Feinsilver gave a proof of a theorem which characterized the inverses of tridiagonal matrices. In these studies, it was given a definition called the triangle property. Main theorems were established by using this property. It was profited from 2-minors of matrix while it was made the proof of the theorems. In this paper we considered a companion matrix B. i.e. $B=A-ab^t$ where $a=(a_0,\cdots,a_{n-1})^t\in\mathbb{R}^n$ and $b=(b_0,\cdots,b_{n-1})^t\in\mathbb{R}^n$ such that $a_0=1$ and $b_0\neq 0$. It was given a statement depend on the component of b-vector of the matrix A and it was searched the structure of A^{-1}.

2. MAIN RESULTS

Definition 1. A_{ij} is the minor that it is obtained deleting i-th row and j-th column of matrix A.

Definition 2. A matrix $H=(h_{ij})_{nxn}$ is called a lower (upper) Hessenberg matrix if $h_{ij}=0$ for all pairs (i,j) such that $i+1<j$ $(j+1<i)$.

Definition 3. Let

$$B=\begin{bmatrix} 0 & 0 & . & . & . & 0 & -b_0 \\ 1 & 0 & . & . & . & 0 & -b_1 \\ 0 & 1 & . & . & . & 0 & -b_2 \\ . & . & . & . & . & . & . \\ 0 & 0 & . & . & . & 1 & -b_{n-1} \end{bmatrix}. \tag{1}$$

The matrix B is said to be a companion matrix. It can be written the following form:

$$B=A-ab^t \tag{2}$$

where A is nxn matrix, $a=(a_0,\cdots,a_{n-1})^t$ and $b=(b_0,\cdots,b_{n-1})^t$.

Lemma 1. Let B be a companion matrix such that $a=(a_0,\cdots,a_{n-1})^t$ and $b=(b_0,\cdots,b_{n-1})^t$ where $a_0=1$ and $b_0\neq 0$. If the matrix A can be written as

$$A = \begin{cases} a_{1i}=b_{i-1}, & for\ \ i=1,\dots,n-1 \\ a_{i+1,i}=1+\dfrac{b_{i-1}b_i}{b_0} & for\ \ i=1,\dots,n-2 \\ a_{i,1}=b_{i-1} & for\ \ i=3,\dots,n \\ a_{1n}=b_{n-1}-b_0 & \\ a_{i,n}=\dfrac{b_{i-1}}{b_0}a_{1n} & for\ \ i=2,\dots,n \\ a_{n,n-1}=1+\dfrac{b_{n-1}^2}{b_0} & \\ a_{ij}=\dfrac{b_{i-1}b_{j-1}}{b_0} & for\ \ i,j=2,\dots,n-1;\ \ i\le j \\ a_{ij}=\dfrac{b_{i-1}b_{j-1}}{b_0} & for\ \ i=4,\dots,n,\ j=2,\dots,n-2;\ i-1>j, \end{cases} \qquad (3)$$

then

$$detA=(-1)^{n-1}(b_{n-1}-b_0) \qquad (4)$$

where all b_i are the components of b-vector.

Proof. Let the matrix A be as (3). Hence $b_{n-1}-b_0$ is common factor of the last column of A. Let us take this common factor in outside of the determinant. If we subtract from (n-1)-th and (n-2)-th column multiple b_{n-2} and b_{n-3} of n-th column, respectively, and we continue this operation, then we obtain

$$detA=(-1)^{n-1}(b_{n-1}-b_0)\begin{vmatrix} 0 & 0 & . & . & . & 0 & 1 \\ 1 & 0 & . & . & . & 0 & \dfrac{b_1}{b_0} \\ . & . & . & . & . & . & . \\ 0 & 0 & . & . & . & 0 & \dfrac{b_{n-2}}{b_0} \\ 0 & 0 & . & . & . & 1 & \dfrac{b_{n-1}}{b_0} \end{vmatrix}. \qquad (5)$$

Hence

$$detA=(-1)^{n-1}(b_{n-1}-b_0). \tag{6}$$

Thus the proof is completed.

Theorem 1. Let B be a companion matrix such that $a=(a_0,\cdots,a_{n-1})^t$ and $b=(b_0,\cdots,b_{n-1})^t$ where $a_0=1$ and $b_0\neq0$. If the matrix A is defined as (3) and it is regular, then the inverse A^{-1} of A is lower Hessenberg matrix such that $a^{-1}_{i,i+1}=1$ for i=1,...,n-2 and $a^{-1}_{ij}=0$ for i,j=2,...,n-1; $i\geq j$, where all a^{-1}_{ij} are elements of A^{-1} for i,j=1,...,n.

Proof: Let's make at three step the proof.

First of all, we show that A^{-1} is a lower Hessenberg matrix. Since the matrix A is defined as in (3), the minors $A\binom{i,\,i+1}{i+1,\,j}$ of A are zero for i=1,...,n-2 and j=i+2,...,n. Therefore the minors A_{ij} of A are zero for j-i>1 (see [1], Theorem 1). Then the matrix A^{-1} is a lower Hessenberg matrix.

In the second step we will show that $a^{-1}_{i,i+1}=1$ for i=1,..., n-1. From the definition of inverse of a matrix we have

$$a^{-1}_{i,i+1}=\frac{A_{i+1,i}}{detA}. \tag{7}$$

Now we show that $A_{i+1,i}=detA$ for i=1,...,n-2. Since $b_{n-1}-b_0$ is common factor of the last column of $A_{i+1,i}$, we can take this common factor in outside of the determinant. If we apply elementary operations, we obtain

$$A_{i+1,i}=(-1)^{n-1}(b_{n-1}-b_0)\,|I_{n-1}|, \tag{8}$$

where I_{n-1} is (n-1)x(n-1) identity matrix. Thus

$$A_{i+1,i}=detA. \tag{9}$$

Because of (9) we write $a^{-1}_{i,i+1}=1$.

In the third step let's show that $a^{-1}_{ij}=0$ for i,j=2,...,n-1 and $i\geq j$, i.e. $A_{ji}=0$. Since j-th row and i-th column of A not found in A_{ji}, the first and (j-1)-th rows of this minors are linear dependent. Therefore

$$A_{ji}=0 \quad for \;\; i,j=2,\ldots,n-1; \;\; i\geq j. \tag{10}$$

Consequently

$$a^{-1}_{ij}=0 \quad for \;\; i,j=2,\ldots,n-1; \;\; i\geq j. \tag{11}$$

Thus the proof is completed.

Corollary 1. Let the matrix A be as in (3). Then the a-vector in B is in the following form:

$$a=\left(a_0, \frac{b_1}{b_0}, \frac{b_2}{b_0}, \ldots, \frac{b_{n-1}}{b_0}\right) \tag{12}$$

where $a_0=1$ and $b_0\neq0$.

Proof: Since the matrix A is as in (3), the minors $A\binom{i,i+1}{i+1,j}$ of A are zero for i=1,...,n-2 and j=i+2,...,n. Clearly, we have $a_i=\frac{b_i}{b_0}$ for i=1,...,n-1.

REFERENCES

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