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ON THE PRINCIPAL CURVATURES OF PARALLEL HYPERSURFACES IN E^n

A. KILIÇ

Fırat University, Faculty of Art and Sciences, Department of Mathematics
23119 ELAZIĞ / TURKEY

SUMMARY

In this paper, it has been showed that the parallel image of a hypersurface in E^n ($n>3$) under the natural map which preserves the connection has constant principal curvatures.

Preliminaries

Let M be a hypersurface of n-dimensional Euclidean space E^n with connection D, X(M) be the space of vector fields on M and $X_1, X_2, \ldots, X_{n-1}$ be principal orthogonal unit vectors in X(M). Then we have

$\langle D_{X_i} X_i, X_i \rangle = 0$

and for $i \neq j$,

$\langle D_{X_j} X_i, X_i \rangle = 0.$

Thus we conclude that the components of the vector fields $D_{X_i} X_i$ and $D_{X_j} X_i$ in the direction of X_i are zero. So,

$$D_{X_i} X_i = \sum_{s=1}^{n-1} a^i_{is} X_s \quad , \quad a^i_{ii} = 0$$

$$D_{X_j} X_i = \sum_{s=1}^{n-1} a^j_{is} X_s \quad , \quad a^j_{ii} = 0. \qquad (1)$$

Now we will find a relation between the coefficients a^j_{ik} as follows:

We know that,

$$\langle D_{X_i} X_j, X_k \rangle + \langle X_j, D_{X_i} X_k \rangle = 0. \qquad (2)$$

Considering the equation (1) and (2) together

$$a^i_{jk} = -a^i_{kj}$$

is obtained.

This shows that coefficients a^i_{jk} are anti-symmetric.

Now, the Codazzi-Mainardi Equation can be written as follows:

$$D_{X_i} L X_j - D_{X_j} L X_i = L([X_i, X_j]), \text{ for } i \neq j,$$

$$D_{X_i} k_j X_j - D_{X_j} k_i X_i = L(D_{X_i} X_j - D_{X_j} X_i).$$

Thus,

$$X_i[k_j]X_j + k_j a^i_{ji} X_i + k_j \sum_{j\neq s\neq i}^{n-1} a^i_{js} X_s - X_j[k_i]X_i - k_i a^j_{ij} X_j - k_i \sum_{j\neq s\neq i}^{n-1} a^j_{is} X_s - k_i a^i_{ji} X_i$$

$$- \sum_{j\neq s\neq i}^{n-1} k_s a^i_{js} X_s + k_j a^j_{ij} X_j + \sum_{j\neq s\neq i}^{n-1} k_s a^j_{is} X_s = 0.$$

Since $i \neq j \neq s$, then,

$$\{(k_j - k_i) a^i_{ji} - X_j[k_i]\} X_i + \{a^j_{ij}(k_j - k_i) + X_i[k_j]\} X_j$$

$$+ \sum_{j\neq s\neq i}^{n-1} \{a^i_{js}(k_j - k_s) + a^j_{is}(k_s - k_i)\} X_s = 0. \qquad (3)$$

On the other hand, it is known that X_m's are the linearly independent, for $1 \leq m \leq n-1$, by the equation (3),

$$a^j_{ij} = \frac{X_i[k_j]}{k_i - k_j} \qquad (4)$$

and

$$a^i_{js} = \frac{k_i - k_s}{k_j - k_s} a^j_{is}.$$

is obtained.

NATURAL MAP AND PRINCIPAL CURVATURES

Let M be a hypersurface of n-dimensional Euclidean space E^n with unit normal $N = (a_1, a_2, \ldots, a_n)$, where each a_i is a C^∞ on M. Let $M_r = \{p + rN_p : p \text{ in } M\}$, for $r \neq 0$, that is, if $p = (p_1, p_2, \ldots, p_n)$ is in M, then,

$$f(p) = p + rN_p = (p_1 + ra_1(p), \ldots, p_n + ra_n(p)) \text{ is in } M_r.$$

The map f is called the natural map of M into M_r, and if f is univalent, then M_r is a parallel hypersurface of M [1].

THEOREM 1: Let $f: M \to M_r$ be a natural map. Then, $f_*(X) = X + rLX$ and $L_r^* f_* X = LX$, for X in M_p and f preserves principal directions of curvature. Where M_p is tangent space at p in M, L and L_r are Weingarten maps on M and M_r, respectively [1].

If M and M_r are hypersurfaces of n-dimensional Euclidean space with connections D' and D', respectively, then a map $f:M \to M_r$ is connection preserving if $f_*D_XY = D'_{f_*X} f_*Y$ for all vectors X and vector fields Y [1].

Now, let X be a unit vector at m in M with $LX = kX$, so $L_r(f_*X) = L(X) = kX$ and $f_*X = (1+rk)X$. If $1+rk=0$, then $f_*X = 0$ and $L_r(f_*X) = kX = 0$, so $k = 0$ and $1 = 0$ thus $1+rk \neq 0$ if M_r is a hypersurface [1].

THEOREM 2: Let M be a complete and connected hypersurface in E^n. If natural map f is a connection preserving map then each k_i is constant, for $1 \leq i \leq n-1$.

PROOF: Let X_i and X_j be principal local vector fields on M and L be Weingarten map on M_p. Thus from Theorem 1,

$$f_*D_{X_i}X_j = D_{X_i}X_j + r\,L\,D_{X_i}X_j = D_{f_*X_i} f_*X_j = D_{X_i+rLX_i}(X_j + rLX_j)$$

or

$$D_{LX_i}X_j + D_{X_i}LX_j + rD_{LX_i}LX_j - LD_{X_i}X_j = 0 \qquad (5).$$

is expressed.

Since $LX_i = k_iX_i$, $1 \leq i, j, \leq n-1$, then by the equation (5),

$$D_{k_iX_i}X_j + D_{X_i}k_jX_j + rD_{k_iX_i}k_jX_j - L\Big(\sum_{s\neq j}^{n-1} a^i_{js}X_s\Big) = 0$$

or

$$k_i \sum_{s\neq j}^{n-1} a^i_{js}X_s + X_i[k_j]X_j + k_j \sum_{s\neq j}^{n-1} a^i_{js}X_s + rk_iX_i[k_j]X_j$$

$$+ rk_ik_j \sum_{s\neq j}^{n-1} a^i_{js}X_s - \sum_{s\neq j}^{n-1} k_s a^i_{js}X_s = 0$$

is obtained.

Finally, we get

$$\sum_{j\neq s\neq i}^{n-1} a^i_{js}(k_i + k_j + rk_ik_j - k_s)X_s + k_j(1+rk_i)a^i_{ji}X_i + X_i[k_j](1+rk_i)X_j = 0.$$

Since X_i, $1 \leq i \leq n-1$, are linearly independent, then we have

$$X_i[k_j](1+rk_i) = 0 \qquad (6)$$

and

$$k_j\,(1+rk_i)a^i_{ji}=0$$

or

$$a^i_{j3}\,(k_i+k_j+rk_ik_j-k_3)=0. \qquad (7)$$

Since $1+rk_i \neq 0$, by the equation(6) we obtain

$$X_i\,[k_j]=0 \qquad (8)$$

So, by the equation(4) we have

$$a^j_{ij}=0. \qquad (9)$$

Furthermore, let m be a non-umbilic point in the connected neighborhood A of m. Since $X_i, 1 \leq i \leq n-1$, are orthonormal then, $LX_i = k_iX_i$. For i=j, the equation (5) can be expressed as follows:

$$D_{kixi}X_i + D_{Xi}k_iX_i = rD_{kixi}k_iX_i - LD_{Xi}X_i = 0$$

or

$$k_iD_{Xi}X_i + X_i\,[k_i]X_i + k_i\,D_{Xi}X_i + rk_iX_i[k_i]X_i rk^2{}_i\,D_{Xi}X_i - LD_{Xi}X_i = 0.$$

On the other hand, the equation (9) implies that $D_{Xi}X_i = 0$, so

$$X_i[k_i]\,X_i + rk_iX_i[k_i]\,X_i = 0$$

or

$$X_i\,[k_i]\,(1+rk_i)X_i = 0.$$

Therefore,

$$X_i[k_i] = 0. \qquad (10)$$

Thus the equations (8) and (10) imply that k_i are constant on A.

Corollary1: Let $\bar{k}_i, 1 \leq i \leq n-1$, be denoted principal curvature of M_r. If k_i's are constant then $\bar{k}_i$'s are constant.

Proof: It is clear that from

$$\bar{k}_i = \frac{k_i}{1+rk_i}.$$

Corollary2: If the natural map of M onto M_r is a connection preserving, then the hypersurfaces M and M_r have the constant principal curvatures

E^n DE PARALEL HİPERYÜZEYLERİN ASLİ EĞRİLİKLERİ ÜZERİNE

ÖZET

Bu makalede E^n $(n > 3)$'de bir hiperyüzeyin konneksiyonu koruyan doğal dönüşüm altındaki bir paralel görüntüsünün asli eğriliklerinin sabit olduğu gösterildi.

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