SCHWARZ PICK TEOREMİNİN CEBİRSEL KARAKTERİZASYONU

ÖZET

$\bar{D} = \{z \in C : |z| \leq r, r > 0\}$ ve $\bar{U} = \{w \in C : |w| \leq 1\}$ kompleks düzlemde kapalı diskler olmak üzere $B(\bar{D}), \bar{D}$ de tanımlı sınırlı analitik fonksiyonların cebiri olsun. Ayrıca, R bir kompleks cebir iken her bir $\alpha \in C$ için $\Phi(\alpha) = \alpha$ olacak şekilde $\Phi : B(\bar{D}) \to R$ izomorfizminin varlığını kabul edelim. Bu çalışmada, $a \in R$ nin spektrumu kullanılarak Schwarz-Pick teoreminin cebirsel karekterizasyonu verildi.

1. INTRODUCTION

It is well known that from the Riemann mapping theorem that every simple connected region G in the plane (other then the plane itself) is conformally equivalent to the unit disc U. It is proven in 1940 that $G_1 \simeq G_2$ if and only if $B(G_1)$ and $B(G_2)$ are isomorphic. So in algebraic characterization, first and very important step was taken. $G_1 \simeq G_2$ is a property connected with simple-connected regions, in general, any two annuli

$$B_1(r_1; R_1) = \{z \in C : r_1 < |z| < R_1\} \text{ and } B_2(r_2; R_2) = \{z \in C : r_2 < |z| < R_2\}$$

are not conformaly equivalent. $B_1(r_1; R_1) \simeq B_2(r_2; R_2)$ if and only if $\frac{R_1}{r_1} = \frac{R_2}{r_2}$. The algebraic characterization related to this subject was given in [2].

2. ALGEBRAIC CHARACTERIZATIONS

Let Φ be an isomorphism from $B(\bar{D})$ onto R and we will denote elements of $B(\bar{D})$ by $f, g, h, \ldots$ and elements of R by $a, b, c, \ldots$. Let e and 1 be multiplicative identity of R and $B(\bar{D})$, respectivelly . Thus, $1 \in B(\bar{D})$ is the function identically equal to 1 on $\bar{D}$. Since $\Phi : B(\bar{D}) \to R$ is an isomorphism, $\Phi(1) = e$. Furthermore,

$$\Phi(n1) = ne,$$

so that

$$\Phi(\pm\frac{m}{n}.1)) = \pm(\frac{m}{n}).e.$$

$-e$ has two square roots in R, one is the image of $i.1$, the other is the image of $-i.1$. It is algebraically impossible to distinguish between these, since R has an automorphism which takes one into the other (corresponding to the mapping $f \to \bar{f} \in B(\bar{D})$). Thus, we choose one of the root of $-e$ and make it to correspond to $i.1$. We denote it as $i.e$.

Henceforth, we will denote the complex number field by C and the complex rational number field by C_r, where a complex number, both of whose real and imaginary part is a real rational number, is called a complex rational number. Clearly, C_r and C are subrings of $B(\bar{D})$.

Lemma 2.1 : For each $\alpha \in C_r, \phi(\alpha) = \alpha \quad (\text{or } \bar{\alpha})$.

Proof: If $\alpha \in C_r$, there are the rational numbers r_1 and r_2 such that $\alpha = r_1 + ir_2$. Since $\phi(1) = e$ and $\phi(i) = i$ (or $-i$), we get

$$\phi[(r_1 + ir_2)1] = r_1 e + r_2 \text{ ie} \quad (\text{or } r_1 e - r_2 \text{ie}).$$

Lemma 2.2 : For each real number $c, \phi(c1) = ce$.

Proof : If c is a rational number, by the Lemma 2.1, $\phi(c1) = ce$. If c is an irrational number, for each rational number $r, c - r \neq 0$. Thus there exists $(c-r)^{-1} = \frac{1}{c-r}$. Then

$$\phi[(c-r)1] = \phi(c1) - re$$

and

$$\phi[(\frac{1}{c-r})1] = \frac{e}{\phi(c1) - re}.$$

Therefore $\phi(c1) = ce$ [2].

Corallary 2.3 : If $c \in C, \quad \phi(c1) = ce$.

Definition 2.4 : Let R be a ring and $a \in R$. If a has inverse in R, a is called an aritmetic unit, otherwise a is not an aritmetic unit.

Lemma 2.5 : Let $f \in B(\bar{D})$ and let $\bar{R}_f$ be the closed range of f. Then $\lambda \in \bar{R}_f$ iff $f - \lambda 1$ has no inverse in $B(\bar{D})$.

Proof : If $\lambda \in \bar{R}_f$ there is $z_0 \in \bar{D}$ such that $f(z_0) = \lambda$. Then $(f - \lambda 1)(z_0) = 0$. Hence $f - \lambda 1$ has no inverse in $B(\bar{D})$. Now, we suppose that $f - \lambda 1$ has no inverse in $B(\bar{D})$. Then for at least one point $z_0 \in \bar{D}, \quad (f - \lambda 1)(z_0) = 0$. It follow that $f(z_0) = \lambda$, i.e. $\lambda \in \bar{R}_f$.

Lemma 2.6 : $\lambda \in \bar{R}_f$ iff $\Phi(f) - \lambda e$ has no inverse in R.

Proof : If $\lambda \in \bar{R}_f$, $f - \lambda 1$ has no inverse in $B(\bar{D})$ by the Lemma 2.5. Since Φ is an isomorphism, $\Phi(f - \lambda 1) = \Phi(f) - \lambda e$ has no inverse in R [1].

Definition 2.7 : Let f be any function in $B(\bar{D})$. The set

$$\sigma(f) = \{\lambda \in C : f - \lambda 1 \text{ has no inverse in } B(\bar{D})\}$$

is called the spectrum of f.

Definition 2.8 : The spectrum of an element $a \in R$ is the set of all complex number λ such that $a - \lambda e$ is not invertible . We denote the spectrum of a by $\sigma(a)$.

Definition 2.9 : For any $a \in R$, the spectral radius $\rho(a)$ of a is the radius of the smallest closed disc with center at the origin which contains $\sigma(a)$:

$$\rho(a) = \sup\{|\lambda| : \lambda \in \sigma(a)\}.$$

Then $\rho(a)$ is also the maximum modulus (Hereinafter abbreviated MM) of $\Phi^{-1}(a)$.

Now, let us give the algebraic characterization of Schwarz-Pick theorem.

Theorem 2.10 : let $B(\bar{D})$ be an algebra of bounded analytic functions on $\bar{D} = \{z \in C : |z| \leq r\}$, $\Phi : B(\bar{D}) \to R$ be an isomorphism which preserves the constants and R is any algebra. Furthermore, we suppose that there exists only $\lambda_i \in \sigma(a) = \bar{D}$, for each $z_i \in \bar{D}$. Let

$$\Phi^{-1}(a)(z_1) = \lambda_1, \quad \Phi^{-1}(a)(z_2) = \lambda_2$$

for $\lambda_1, \lambda_2 \in \sigma(a)$ such that $|z_1| < r$, $|z_2| < r$ and $\rho(a) = r$. Then

$$(r^2 \frac{\lambda_1 e - \lambda_2 e}{r^2 - \bar{\lambda}_2 e \lambda_1 e}) \leq \rho(r^2 \frac{z_1 e - z_2 e}{r^2 - \bar{z}_2 e z_1 e})$$

inequality holds.

Proof : Let $|z_2| < r$ and z_2 be a fixed point. If $|\lambda_2| = r$, then $\lambda_2 \in \bar{R}_{\Phi^{-1}(a)}$. Hence, according to maximum principle $\Phi^{-1}(a) = f$ is constant.Thus inequalite holds. Now, we suppose that $|\lambda_2| < r$. Let R be any algebra and $d = bc$ for $b, c, d \in R$. If $\Phi^{-1}(b) =$ [illegible], then $\rho(a) = r$ characterizes the Schwarz's [illegible]

Let

$$h(z) = r^2 \frac{z - z_2}{r^2 - \bar{z}_2 z}, \quad m(z) = r^2 \frac{z - \lambda_2}{r_2 - \bar{\lambda}_2 z},$$

where $h : \bar{D} \to \bar{D}$ and $m : \sigma(a) = \bar{D} \to \bar{D}$. We claim that the function $g = m \circ f \circ h^{-1}$ satisfies the conditions of Schwarz's Theorem. $\Phi^{-1}(a) = f$ is a mapping from $\bar{D}$ to $\bar{D}$. First of all g is a mapping from $\bar{D}$ to $\bar{D}$, since $h^{-1}, m \in B(\bar{D})$. Furthermore

$$\begin{aligned} g(0) &= m(f(h^{-1}(0))) \\ &= m(f(z_2)) \\ &= m(\lambda_2) \\ &= 0. \end{aligned}$$

Thus g satisfies the conditions of Schwarz's Theorem. Therefore

$$|g(w)| \leq |w|$$

for $w \in \bar{D}$. From this inequality, we can write

$$|m(f(r^2 \frac{w + z_2}{r^2 + \bar{z}_2 w}))| \leq |w|.$$

If $z_1 \in \bar{D}$ is arbitrary point, $w = r^2 \frac{z_1 - z_2}{r^2 - \bar{z}_2 z_1}$ and thus

$$|m(f(r^2 \frac{r^2 \frac{z_1 - z_2}{r^2 - \bar{z}_2 z_1} + z_2}{r^2 + \bar{z}_2 r^2 \frac{z_1 - z_2}{r^2 - \bar{z}_2 z_1}}))| \leq |r^2 \frac{z_1 - z_2}{r^2 - \bar{z}_2 z_1}|$$

$$\Rightarrow |m(f(r^2 \frac{r^2 z_1 - r^2 z_2 + r^2 z_2 - z_2 \bar{z}_2 z_1}{r^4 - r^2 \bar{z}_2 z_1 + r^2 \bar{z}_2 z_1 - r^2 \bar{z}_2 z_2}))| \leq |r^2 \frac{z_1 - z_2}{r^2 - \bar{z}_2 z_1}|$$

$$\Rightarrow |m(f(r^2 \frac{(r^2 - z_2 \bar{z}_2) z_1}{r^2 (r^2 - z_2 \bar{z}_2)}))| \leq |r^2 \frac{z_1 - z_2}{r^2 - \bar{z}_2 z_1}|$$

$$\Rightarrow |m(f(z_1))| \leq |r^2 \frac{z_1 - z_2}{r^2 - \bar{z}_2 z_1}|$$

$$\Rightarrow |r^2 \frac{\lambda_1 e - \lambda_2 e}{r^2 e - \bar{\lambda}_2 e \lambda_1 e}| \leq |r^2 \frac{z_1 e - z_2 e}{r^2 e - \bar{z}_2 e z_1 e}|.$$

or

$$\Rightarrow [illegible] \leq \rho(r^2 e \frac{z_1 e - z_2 e}{r^2 e - \bar{z}_2 e z_1 e}).$$

If [illegible]

Corallary 2.11 : Let $B(\bar{U})$ be an algebra of bounded analytic functions on $\bar{U} = \{z \in \mathcal{C} : |z| \leq 1\}$, $\Phi : B(\bar{U}) \to R$ be an isomorphism which preserves the constants and R is any algebra. Furthermore, we suppose that there exists only $\lambda_i \in \sigma(a) = \bar{U}$, for each $z_i \in \bar{U}$. On the other hand, let

$$\Phi^{-1}(a)(z_1) = \lambda_1,$$

$$\Phi^{-1}(a)(z_2) = \lambda_2$$

for $\lambda_1, \lambda_2 \in \sigma(a)$ such that $|z_1| < 1$, $|z_2| < 1$ and $\rho(a) = 1$. Then

$$\left(\frac{\lambda_1 e - \lambda_2 e}{1 - \bar{\lambda}_2 e \lambda_1 e}\right) \leq \rho\left(\frac{z_1 e - z_2 e}{1 - \bar{z}_2 e z_1 e}\right).$$

inequality holds.

REFERENCES

[1].Bayraktar, M.,“ Üst yarı düzlemin birim daireye konform tasvirinin cebirsel karekterizasyonu”, Gazi Üniversitesi Fen-Edebiyat Fakültesi Matematik İstatistik Dergisi, 2, 111-116 (1989)

[2].Beck, A., “On Ring On Ring”, Proc. Amer. Math. Soc., 15, 350-353 (1964)

[3].Bers, L., “On Rings of Analytic Functions”, Bull. Amer. Math. Soc., 54, 311-315 (1948)

[4].Helmer, O., “Divisibility Properties of Integral Functions”, Duke Math. J., 6, 346-356 (1940)

[5].Kakutani, S., “Rings of Analytic Functions”, Proc. Michingan Conference on Functions of a Complex Variable, 71-84 (1955)