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ON COMMUTATOR RACK SPECIES AND MATRIX RACK SPECIES

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ABSTRACT

In this paper, I define a free rack species, a commutator rack species, a free commutator rack species, and a semi- free commutator rack species like the racks and the rack species that introduced in [FR] in 1993. In addition to these definitions, I define a knot rack species. Also, using this definition I demonstrate some corollaries and some lemmas. In the last section, I define a matrix rack and a matrix rack species and provide some lemmas.

ÖZET

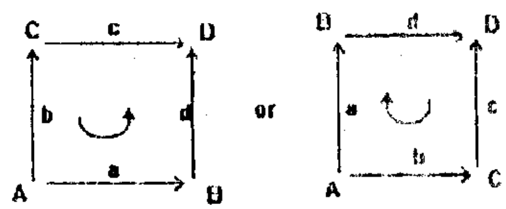
Bu makalede, [FR] nin 1993 de verdiği rack ve rack species tanımlarına benzer free rack species, komutatör rack species, serbest ve yarı serbest komutatör rack species' leri tanımladım. Bu tanımlara ek olarak, düğüm rack species'i tanımladım. Aynı zamanda bu tanımları kullanarak bazı özellikler ve bazı lemmaları gösterdim. Son kısımda, matrix rack ve matrix rack species'leri tanımladım ve bazı lemmaları ispatladım.

Key words: Rack , rack species.

INTRODUCTION

1.SPECIES

An oriented square in a directed graph is a diagram of edges a,b,c,d which can be represented in two ways.



The orientation being represented by the symbol ⌣ or ⌣. It shall be referred to the edge a in eighter of the above representations as the base of the square.

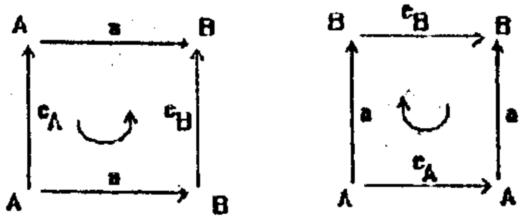
Definition 1.1: A species in its most primitive form consists of:

S1. A directed graph Γ

S2. A collection of oriented squares in Γ called preferred squares.

In addition a species may satisfy any or all of the extra axioms.

*S3.Identity squares:*Any loop (i.e. edge with the same start and end-point.) in Γ may be designated as an identity loop. We will never allow two identity loops based at the some vertex.We shall usually use the notation e_A for the identity loop at the vertex A. If identity loops exist, then squares such as those illustrated below are preferred.

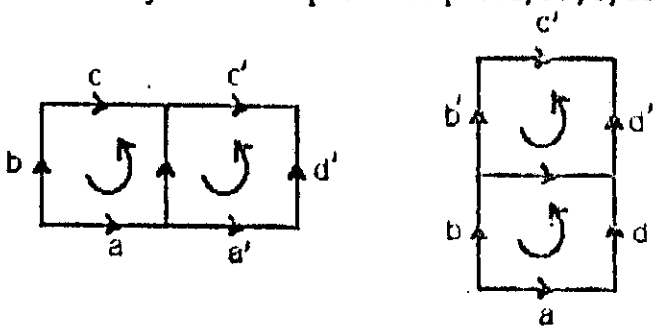


Definition 1.2 : A species with identities is a species equipped with identity loops at each vertex.

S4.Composition laws: Suppose that Γ is infact a category, so that edges (morphisms) can be composed, then there are two composition laws which may be satisfied by a species.

i) Horizontal composition : Preferred squares a, b, c, d, and a′, b′, c′, d′ with d = b′ may be composed horizontally to form the preferred square aa′, b, cc′ , d′ with base aa′

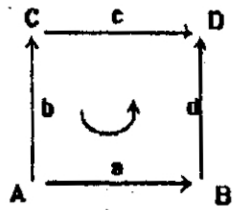
ii) Vertical composition : Preferred squares a, b, c, d, and a′, b′, c′, d′ with c=a′ may be composed vertically to form the preferred square a, bb′, c′, dd′ with base a.



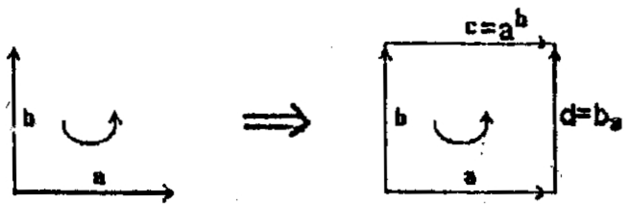
Composing squares

S.5. The Vertebrate laws :

Given edges a: A→B and b: A→C then there are unique c, d, D, so that



is preferred . Notice that the other edges are determined by two binary operations $c = a^b$, $d = b_a$



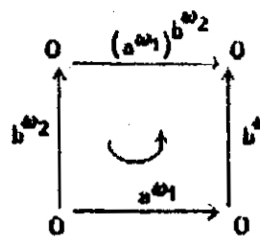
Lemma 1.1 : The binary operations of a vertebrate species satisfy the axioms :

a) $a^{bc_b} = a^{cb^c}$, b) $a_{bc_b} = a_{cb^c}$, c) ${a_b}^{c_b} = a^{c}_{b^c}$

Proof /1/.

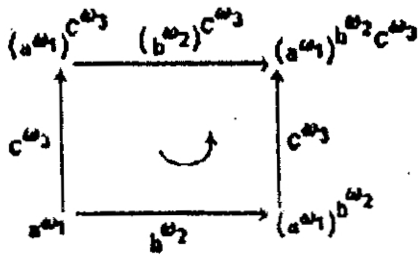
2. FREE RACK SPECIES.

1) A Free rack FR(X) defines a species SFR(X) with a single vertex and with FR(X) the set of edges. The preferred squares are of the following type



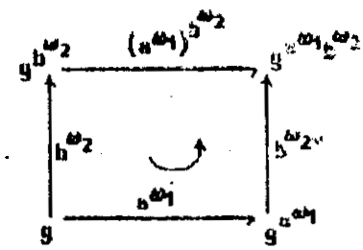
$$a^{\omega_1}.b^{\omega_2}((a^{\omega_1})^{b^{\omega_2}})^{-1}(b^{\omega_2})^{-1}= 1$$

2) A free rack FR(X) determines a second species $S_{FR(X)}$ (FR(X)) by taking as vertices the set FR(X), and edges $a^{w_1} \xrightarrow{b^{\omega_2}} (a^{w_1})^{b^{w_2}.c^{w_3}}$, and preferred squares of the following type

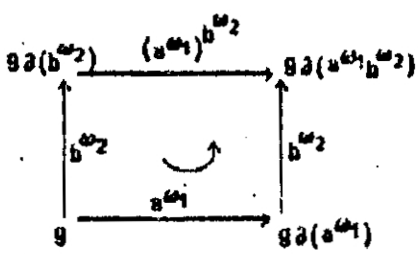


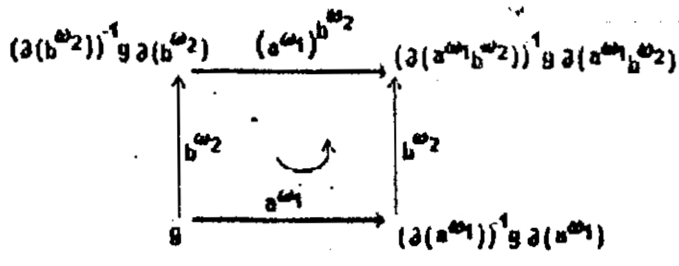
The species $S_{FR(X)}$ (FR(X)) covers the species S(FR(X)).

3) A G - rack ∂ : FR(X) →G determines a species S_G(FR(X)), by taking as vertices the set G, and edges $g \xrightarrow{a^{\omega}} g^{a^{\omega}}$, for $a^{\omega} \in$FR(X), Where $g^{a^{\omega}}$ is defined to be $(\partial(a^{\omega}))^{-1}.g$. The preferred squares are of the following type



4) There are two similar species associated to a G - rack defined by redefining $\partial(a^{\omega_1}.b^{\omega_2}) = \partial(a^{\omega_1}).\partial(b^{\omega_2})$ to be $g.\partial(a^{\omega})$ and $(d(a^{\omega}))^{-1}.g.\partial(a^{\omega})$ respectively.

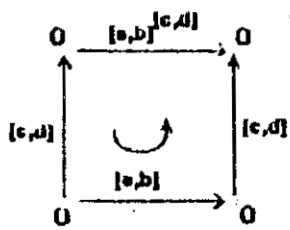




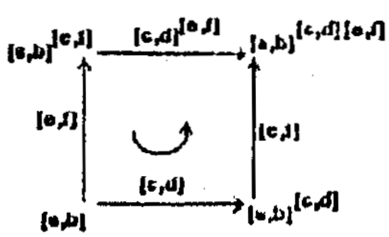
Note that $\partial(a^{\omega_1}.b^{\omega_2}) = \partial(a^{\omega_1}).\partial(b^{\omega_2})$ /1/.

3. THE COMMUTATOR RACK SPECIES

1. A commutator rack [X,X] defines a species S([X,X]) with a single vertex and [X,X] the set of edges. The preferred squares are of the following type.

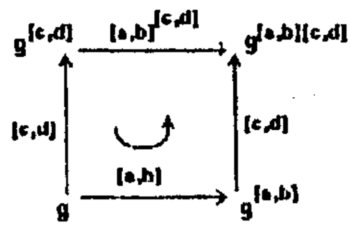


2. A commutator rack [X,X] determines a second species $S_{[X,X]}([X,X])$ by taking as vertices the set [X,X], and edges $[a,b] \xrightarrow{[c,d]} [a,b]^{[c,d]}$, and preferred squares of the following type

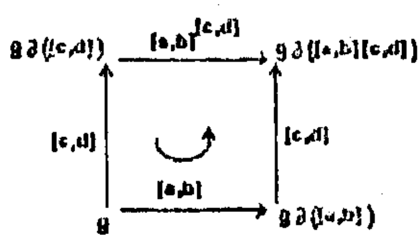


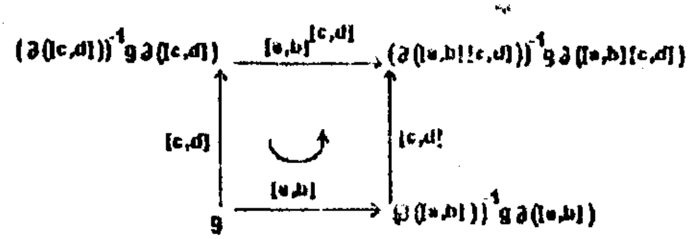
The species $S_{[X,X]}([X,X])$ covers the species S([X,X]).

3.A G-rack ∂:[X,X]$\rightarrow$ G determines a species $S_G([X,X])$, by taking as vertices the set G, and edges $g \xrightarrow{[a,b]} g^{[a,b]}$ for $[a,b] \in [X,X]$, Where $g^{[a,b]}$ is defined to be $(\partial([a,b]))^{-1}.g$. The preferred squares are of the following type.



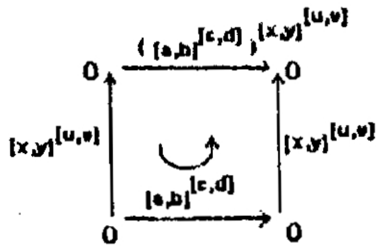
4. There are two similar species associated to a G-rack defined by redefining $g^{[a,b]}$ to be $g^{\partial([a,b])}$ and $(\partial([a,b]))^{-1}\, g\, \partial([a,b])$ respectively



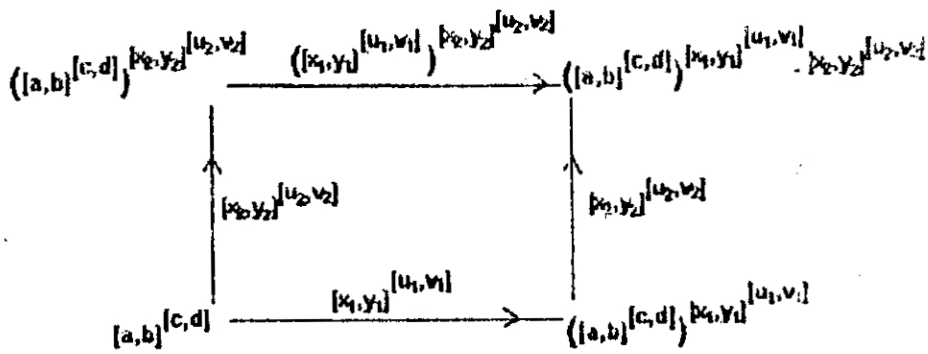


4.FREE COMMUTATOR RACK SPECIES

1. A free commutator rack FR([X,X]) defines a species SFR([X,X]) with a single vertex and with FR([X,X]) the set of edges. The preferred squares are of the following type.

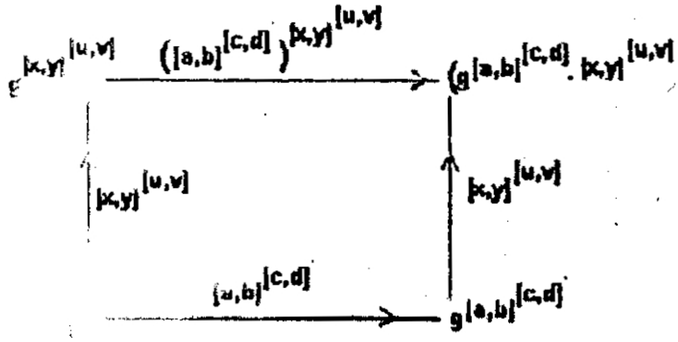


2. A free commutator rack determines a second pricies $S_{FR[X,X]}$ (FR[X,X]) by taking as vertices the set FR[X,X], and edges $[a,b]^{[c,d]} \xrightarrow{[x,y]^{[u,v]}} ([a,b]^{[c,d]})^{[x,y]^{[u,v]}}$ and preferred squares of the following type.

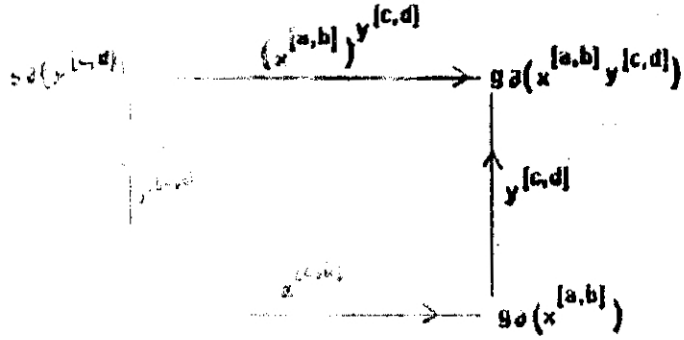


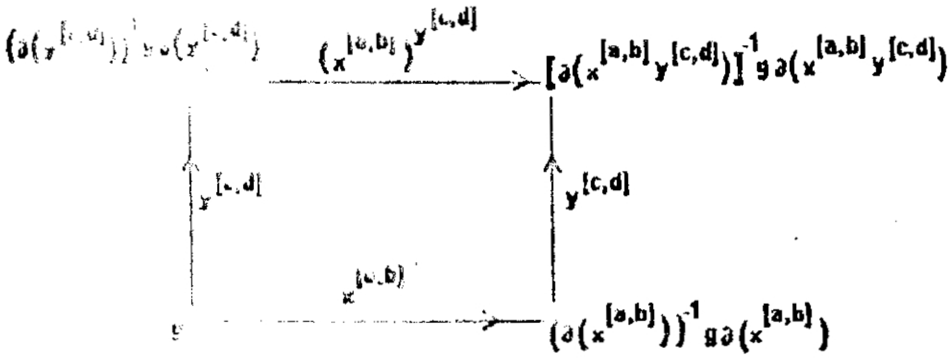
The species $S_{FR[X,X]}$ (FR[X,X]) covers the species S(FR[X,X]).

3. A G-rack ∂:FR[X,X] → G determines a species S_G (FR[X,X]), by taking as vertices the set G, and edges $g \xrightarrow{[a,b]^{[c,d]}} g^{[a,b]^{[c,d]}}$, for $[a,b]^{[c,d]} \in$ FR[X,X], where $g^{[a,b]^{[c,d]}}$ is defined to be $(\partial[a,b]^{[c,d]})^{-1}g$. The preferred squares are of the following type.



There are two similar species associated to a G-rack defined by redefining $g^{x^{[a,b]}}$ to be $g\partial(x^{[a,b]})$ and $(\partial(x^{[a,b]}))^{-1}g\partial(x^{[a,b]})$ respectively.

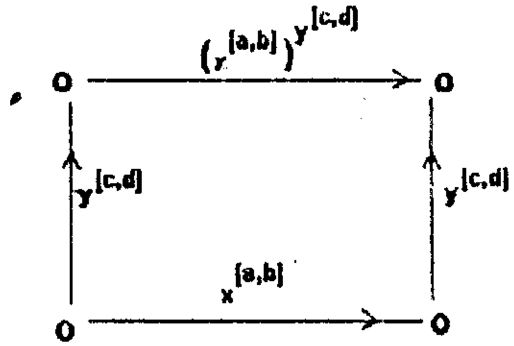




Note that [illegible]

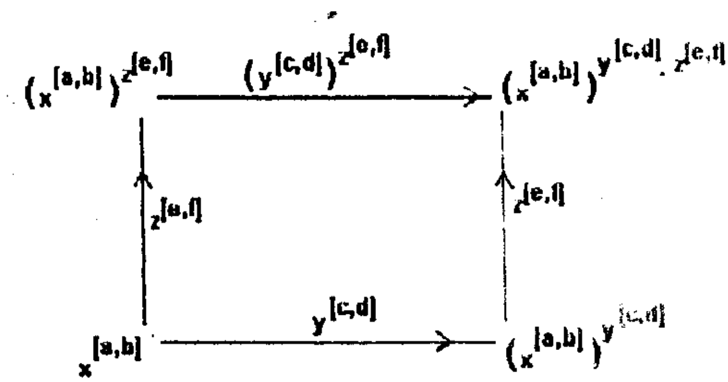
5.SEMI-FREE COMMUTATOR RACK SPECIES

1) A semi-free commutator rack FR[X,[X,X]] defines a species SFR[X,[X,X]] with a single vertex and with FR[X,[X,X]] the set of edges. The preferred squares are of the following type.



2) . A Semi - free commutator rack FR[X,[X,X]] determines a second species $S_{FR[X,[X,X]]}$ (FR[X[X,X]]) by taking as vertices the set FR[X,[X,X]], and edges

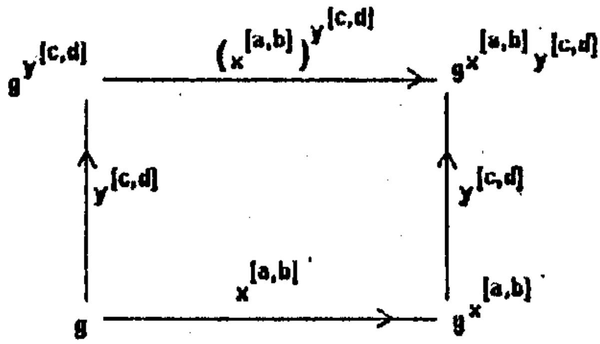
$x^{[a,b]} \xrightarrow{y^{[c,d]}} (x^{[a,b]})^{y^{[c,d]}}$ and preferred squares of the following type



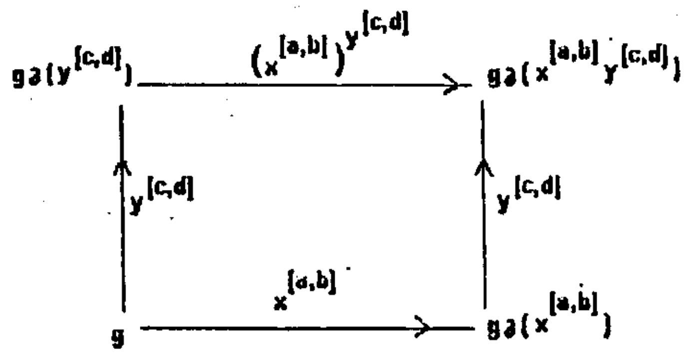
The species $S_{FR[X,[X,X]]}$ (FR([X,[X,X]]) covers the species SFR[X,[X,X]]

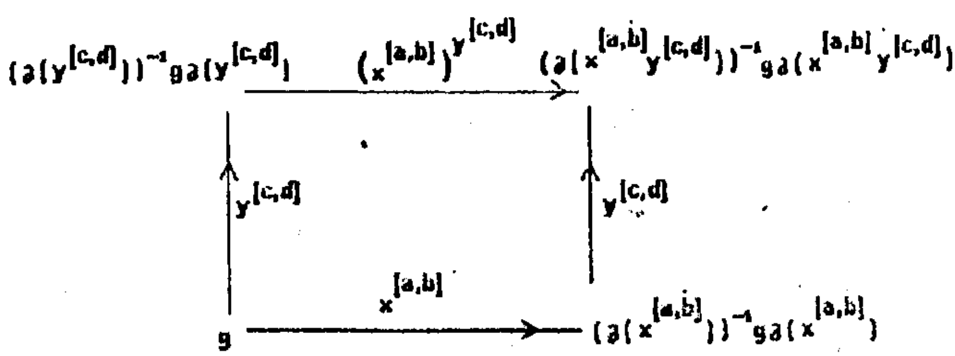
3) A G - rack ∂ :[X,[X,X]]→ G.determines a species S_G[X,[X,X]], by taking as vertices the set G, and edges

$g \xrightarrow{x^{[a,b]}} g^{x^{[a,b]}}$, for $x^{[a,b]}$ ε[X,[X,X]], where $g^{x^{[a,b]}}$ is defined to be $\partial(x^{[a,b]})^{-1}.g$. The preferred squares are of the followingtype.



4). There are two similer species associated to a G - rack defined by redefinig $g^{x^{[a,b]}}$, to be $g.\partial(x^{[a,b]})$ and $(\partial(x^{[a,b]})^{-1}.g.\partial(x^{[a,b]})$ respectively.





6. KNOT RACK SPECIES

Define 6.1 (Knot) : A homomorphic image (an embedding) K of the unit circle $S^1 = \{(x,y)|x^2+y^2=1\}$ into S^3(or R^3) is called a knot. Namely a knot is a simple closed curve in S^3.

Define 6.2 (Knot Group): If K is a Knot in R^3, the fundamental group $\pi_1(S^3\text{-}K)$ of the complement is called, simply the group of K and it is shown by $G=\pi_1(S^3\text{-}K) = \{x_1,x_2,\ldots,x_n;r_1,r_2,\ldots,r_n\}$. Where respectively $x_1,x_2,\ldots,x_n$ and $r_1,r_2,\ldots,r_n$ are the genarators and relations of the knot group.

Now, we consider the normal diagram of a knot in R^2. According to the Wirtinger presentation, the relation that were attached a Ci cross point is $xy\,x^{-1}z^{-1}=1$ as seen the following figure 1, where x,y, and z are generators of the relation.

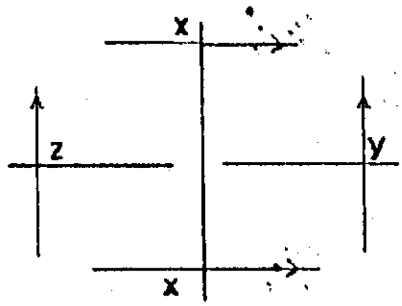


Figure 1.

We state this relation with the rack concept as the following. There are two underpasses (y,z) and a overpass (x) in the normal diagram of R2 as seen the figure 1. These underpasses are named respectively as the first and the second underpasses. If we move on the overpasses in a C_i cross point, we will call the underpass of the right side of C_i is the first underpasses and the underpass. We obtain the relation that is defined above as

(II.underpass)$^{\text{overpass}}$= I.underpass (i.a; $y=z^x$).

Example 6.1: The normal diagram of trefoil knot is given in the following figure

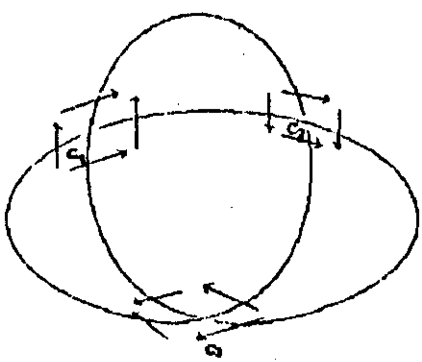
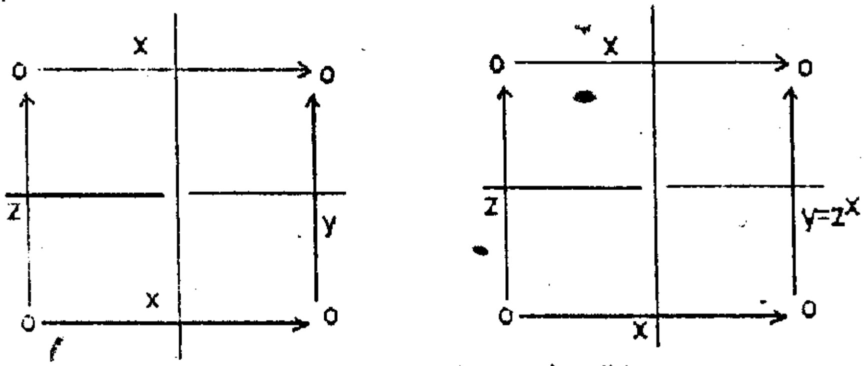


Figure 2.

According to the Wirtinger presentation the knot group of trefoil knot $G=\pi_1(S^3-K)=\{x,y,z: xzx^{-1}y^{-1}=1, zyz^{-1}x^{-1}, yxy^{-1}z^{-1}=1\}$. Now, if we write according to the above definition, we obtain $G=\{x,y,z: y^x=z, x^z=y, z^y=x\}$.

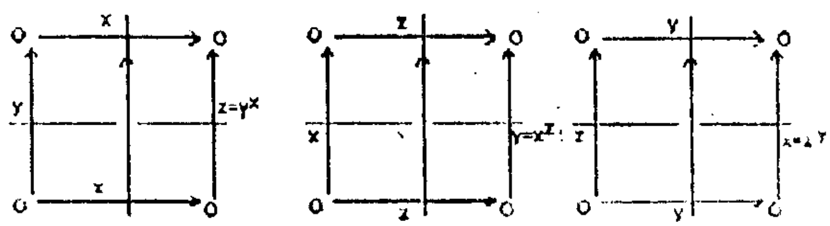
Definition 6.3 (The Knot Rack Species) : Lets G be the fundamental group of K knot and lets X be a set of the generators that is attached to the cross points of K. A X knot rack defines a species S(X) with a single vertex and with X the set of edges. The preferred squares are of the following type.



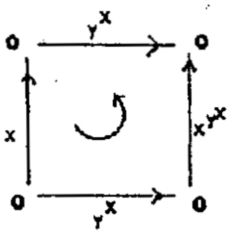
(a) $xy\overline{xz}=1$

(b) $xz^{x}x^{1}.z^{1}=1$

Example 6.2: Now, we introduce a species for each cross point of the trefoil knot. The specieses in the cross point C_1, C_2 and C_3 respectively are shown the following type.

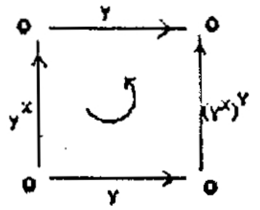


(C_1) (C_2) (C_3)



$(C_1 \circ C_2)$

$y^x \cdot x^{y^x} \cdot (y^x)^{-1} \cdot x^{-1} = 1$

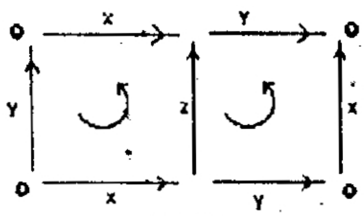


$(C_1 \circ C_3)$

$y \cdot (y^x)^y \cdot y^{-1} \cdot (y^x)^{-1} = 1$

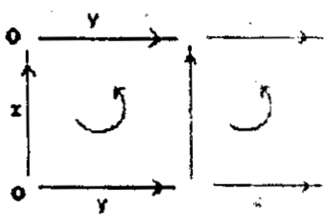
$(C_2 \circ C_3)$

$y^z \cdot z^{x^z} \cdot (x^z)$



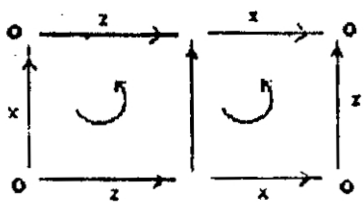
$(C_1 \cup C_3)$

$G = \{x, y, z : xyxy^{-1}x^{-1}y^{-1} = 1\}$



$(C_3 \cup C_2)$

$G = \{y, z : yzyz^{-1}y^{-1}z^{-1} = 1\}$



$(C_2 \cup C_1)$

$G = \{x, z : xzxz^{-1}x^{-1}z^{-1} = 1\}$

Lemma 6.1: $C_1 \circ C_1 = C_1$, $C_2 \circ C_2 = C_2$, $C_3 \circ C_3 = C_3$

Proof: It is clear.

7. MATRIX RACK

Definition 7.1 ***(The Matrix Rack)*** : Let X be a set of all square matrixes which are the same dimension (n.n) . X is a non-empty set. For $A \in X$, using the functions f_A given by $f_A : X \to X$, $f_A(B)=B^A$, and $f_A^{-1}(B)= B^{A^{-1}}$, we have a rack structure on X which is called the matrix rack. Infact this definition satisfies two conditions of the definition rack.

1) For each A_{nxn} , B_{nxn} $\in X$, there is a unique $C_{nxn} \in X$ such that $A=C^B$ we arrive $C=A^{B^{-1}} \in X$.

2) Foe each A_{nxn}, B_{nxn} , $C_{nxn} \in X$, we have $A^{BC} = A^{CB^C}$

Lemma 7.1 : Let $G=\{f_A : X\to X, f_A(B)=B^A , f_A^{-1}(B)= B^{A^{-1}} \}$ be a set of functions. Then (G,o) is a group where o is the composition operator of functions.

Proof :1) $\forall f_A, f_B \in G$, $(f_A \circ f_B)(C)= f_A(f_B(C))$

$$= f_A(C^B)= C^{BA} = f_{BA}(C) \in X, BA \in X$$

2) $\forall f_A, f_B, f_C \in G$, $(f_A \circ (f_B \circ f_C))(D)=(f_A \circ (f_B) f_C(D))=(f_A \circ (f_B))(D^C)$

$$= f_A(D^{CB})= D^{CBA} =((f_A \circ f_B)\circ f_C)(D)$$

3) $f_{I_{nxn}} \in G$ where I is unit matrix.

$$(f_I \circ f_A)(B)= f_I f_A(B)= f_I(B^A)= B^{AI} = B^A = f_A(B)$$
$$(f_A \circ f_I)(B)= f_A(f_I(B))= f_A(B^I)= B^{IA} = B^A = f_A(B)$$

4). $(f_A \circ f_A^{-1})(B)= f_A(f_A^{-1}(B))= f_A(B^{A^{-1}})= B^{A^{-1}A} = B^I = f_I(B)$

$$(f_A^{-1} \circ f_A)(B)= f_A^{-1}(f_A(B))= f_A^{-1}(B^A)= B^{AA^{-1}} = B^I = f_I(B)$$

***Lemma** 7.2:* If for each $A \in X$ the function f_A given by $f_A : X\to X$, $B \to B^A$ is an otomorphism of X.

Proof: Let $B.C \in X$ for all $B,C \in X$, $f_A(B.C) =(B.C)^A =B^A.C^A =f_A(B)f_A(C)$ (Where $B^A =A^{-1}.B.A$)

***Lemma** 7.3:* For each $A \in X$ the function f_A given by $f_A(B)=B^A$ is a bijection of X to itself.

Proof: It is clear.

Lemma 7.4: The matrix rack identity is a right self-distributive law as can be seen if we use the notation A.B for A^B; (A.B).C=(A.C).(B.C).

Proof: $(A.B).C=(A^B).C=(A^B)^C=A^{BC}=A^{CB^C}=(A^C)^{B^C}=(A.C).(B.C)$.

Lemma 7.5: Let Λ be the ring of Laurent polynomials $Z[t,t^{-1}]$ in the variable t. Any Λ-module M has the structure of a matrix rack with the rule $A^B:=tA+(1-t)B$.

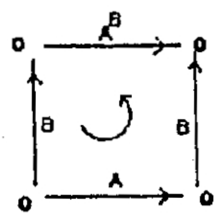
Proof: 1) For all $A,B \in X$ if $A=C^B=tC+(1-t)B \Rightarrow C=t^{-1}A+(1-t^{-1})B \in Z[t,t^{-1}]$
2) $A^{BC}:=tA+(1-t)BC$
$A^{CB^C}:=tA+(1-t)[C(tB+(1-t)C]$
$A^{BC}=A^{CB^C} \Leftrightarrow \alpha=\beta$
$BC=C(tB+(1-t)C)$
$C^{-1}BC=tB+(1-t)C$
$B^C:=tB+(1-t)C$

Lemma 7.6: $G\cong X$, here X is a set of matrix and $G=\{f_A:f_A:X\to X, f_A(B)=B^A\}$

Proof: Let $h:G\to X$ be a function given by $h(f_A)=A$. Since,
$(f_A\circ f_B)(C)=f_{BA}(C)$, $h(f_A\circ f_B)=h(f_{BA})=B.A=h(f_B).h(f_A)$.

8. THE MATRIX RACK SPECIES

1) A matrix rack X defines a species S[X] with a single vertex and with X the set of edges. The preferred squares are of the following type.



2) A matrix rack X determines a second species $S_{[X]}[X]$ by taking as vertices the set X, and edges $A \xrightarrow{B} A^B$, and preferred squares of the following type.