T₂-OBJECTS IN CATEGORY OF STACK CONVERGENCE SPACES

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ABSTRACT. There are eight different ways to characterize T₂-spaces in the category of topological spaces. All eight methods are canonical, i.e. they can be easily formulated in a general setting, where they, in general, do not coincide. In the following, the characteristics of each of these T₂-objects in the categories of stack and constant stack convergence spaces are given. Moreover, some invariance properties of each and the other separation properties as well as interrelationship among their various forms are established.

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1. INTRODUCTION.

Several known generalizations of the usual T₂-axiom of topology to topological categories are given in [1]. We want to compare them for topological categories of stack convergence spaces and constant stack convergence spaces. Furthermore, we investigate some invariance properties of them (e.g. the cartesian product, subspaces, and the quotient space).

Let A be a set and K be a function on A whose value K(a) at each a in A is a set of nonempty stacks on A. A pair (A, K) is said to be a stack convergence space iff the following conditions hold:

1. For each a ∈ A, \([a] = \{ B ⊆ A \mid a ∈ B \} \) ∈ K(a).
2. If x and y are stacks on A and x ⊆ y, then y ∈ K(a) if x ∈ K(a).

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A morphism \( f : (A, X) \rightarrow (B, L) \) between stack convergence spaces is a function \( f : A \rightarrow B \) such that \( f(x) \in L(f(a)) \) if \( x \in X(a) \), where \( f(x) \) denotes the stack \( \{ U \mid U \subseteq B \text{ and } U \supseteq f(C) \text{ for some } C \subseteq x \} \). We denote by SCO, the category so formed. See [7] p. 354.

The category of constant stack convergence spaces, ConSCO is the full subcategory of SCO determined by those spaces \((A, K)\) where \( K \) is a constant function [7] p. 354.

2. INVARIANCE PROPERTIES.

We now, for convenience, give the characterizations of the separation properties for SCO and ConSCO for later use.

2.1 THEOREM. Let \( X = (B, K) \) be in SCO (ConSCO).

1. If \( X \) is \( T_0 \), then for each distinct pair of points \( x \) and \( y \) in \( B \), \( [x] \not\subseteq K(y) \) or \( [y] \not\subseteq K(x) \), \( (X \) is \( T_0 \) iff \( B \) is a point or the empty set \( ) \) see [5].

2. If \( X \) is \( T_\Delta \), then \( X \) is discrete i.e. for each \( x \) in \( B \), \( K(x) = \{ \alpha \mid \alpha \supseteq [x] \} \) ( \( K = \{ \alpha \mid \alpha \supseteq [a] \text{ for some } a \in B \} \). See [5].

3. \( X \) is \( T_0 \) iff \( \forall x \neq y \in B \), \( [x] \cap [y] \not\subseteq K(x) \) or \( [x] \cap [y] \not\subseteq K(y) \) ( \( [x] \cap [y] \not\subseteq K \)). This follows from Lemma 2.1 of [8] p. 318.

4. \( X \) is \( T_1 \), \( T^{\prime}_2 \), \( T_\Delta \), \( ST_\Delta \), or \( \Delta T_\Delta \) iff \( B \) is a point or the empty set [3] and [5].

5. \( X \) is \( PreT_\Delta \) iff \( X \) is indiscrete i.e. \( \forall x \in B \), \( K(x) = STK(B) \), the set of all stacks on \( B \) ( \( K = STK(B) \) ). See [5].

6. \( X \) is \( PreT_\Delta \) iff \( B \) is a point or the empty set [3].

2.2 THEOREM. \( X = (B, K) \) in SCO or ConSCO is \( LT_\Delta \), \( MT_\Delta \), or \( NT_\Delta \) iff \( B \) is a point or the empty set.

PROOF. It follows from the parts (1), (2), (6), and (7) of 2.1 and definitions.

2.3 THEOREM. \( X = (B, K) \) in SCO or ConSCO is \( KT_\Delta \) iff \( X \) is a point or the empty set.

PROOF. \( X \) is \( KT_\Delta \) iff \( X \) is \( T_0 \) and \( PreT_\Delta \). Hence, the result follows from this and Theorem 2.1.

2.4 REMARK.
4. All of the $T_2^*, T_2$, $ST_2$, $\Delta T_2$, $KT_2$, $LT_2$, $MT_2$, and $NT_2$ are equal.

2. Each of $T_0$ and $T_0'$ implies $T_0$ but the converse is not true, in general. For example, let $B = \{ x, y \}$ be two point set and define $K$ by $K(x) = \{ [x], [y], [x] \cap [y], [\emptyset] \}$, $K(y) = \{ [y], [x] \cup [y], [\emptyset] \}$. Clearly $X = (B, K)$ is $T_0$ but not $T_0'$ and $T_0$.

2.5 THEOREM. Let $X_1 = (B_1, K_1)$ be $T_2$-objects in SCO or ConSCO.

1. The cartesian product of each of $T_2$-objects, $X_1$ is $T_2$-object.

2. A subspace of each of $T_2$-objects is $T_2$-object.

3. The quotient space of each of $T_2$-objects is $T_2$-object.

4. The coproduct of each of $T_2$-objects is not $T_2$-object.

PROOF. It follows easily from Theorems 2.1, 2.2, and 2.3.

2.6 REMARKS. 1. There are four various ways (see[1]) to define each of $T_3$ and $T_4$-objects in arbitrary topological categories [6]. It follows from these definitions and Theorem 2.1 that $X = (B, K)$ in SCO or ConSCO is $T_4$-object iff $B$ is a point or the empty set.

2. Theorem 2.5 holds for each $T_4$-objects iff $3, 4$. However, in TOP it is well-known, that the Cartesian Product of $T_4$-spaces, the subspace of $T_4$-space is not necessarily $T_4$-space.

Let $U : E \to \text{Sets}$ be topological, $X$ an object in $E$, and $p$ a point in $UX$. Recall[1], that $p$-axial subspace is the initial lift of the principal $p$-axis map $A_p : BV_p B \to UX^2 = B^2$, and $p$-wedge is the final lift of the canonical injections $i_1, i_2 : UX \to BV_p B$.

We now give conditions on an object $X = (B, K)$ and a point $p$ in $B$ so that its $p$-axial subspace is the same as the its $p$-wedge.

2.7 THEOREM. Let $X = (B, K)$ be in SCO or ConSCO. The $p$-axial subspace and the $p$-wedge are equivalent iff $B = \{ p \}$.

PROOF. If $B = \{ p \}$, then $K(p) = \{ [p], [\emptyset] \} = K = \text{STK}(B)$, and consequently the $p$-axial subspace and the $p$-wedge are equal.
Conversely, suppose they are equivalent and \( B \neq \{p\} \). Then there exists \( x \in B \) such that \( x \neq p \). We first claim that \( K(p) = STK(B) \) (\( K = STK(B) \)). To this end, let \( \alpha = \{ "x\text{-axis}" \, , \, "y\text{-axis}" \} \) and note that \( \pi_i A p \alpha = \{p\} = \pi_3 A p \alpha \in K(p) \) (or in \( K \)), where \( \pi_i \) are the projections \( B^3 \to B \) for \( i = 1, 2 \). Hence, by assumption we have \( \alpha \ni i_1 \alpha \) or \( i_2 \alpha \) for some \( \alpha, \varepsilon \in K(p) \) \( (\alpha \in K) \). We show that \( \alpha \ni [B] \). Clearly, \( [B] \subset \alpha \) since \( B \) is in every stack. On the other hand, if \( U \in \alpha \), then \( i_1 U \) or \( i_2 U \) is in \( i_1 \alpha \subset \alpha \) or \( i_2 \alpha \subset \alpha \), and consequently \( i_1 U \) or \( i_2 U \) is in \( \alpha \). It follows easily that \( U = B \) i.e. \( \alpha \subset [B] \). Hence, \( \alpha = [B] \) and then \( K(p) = STK(B) \) (\( K = STK(B) \)). Next, let \( \varepsilon = \{(x,p)\} \cap \{(p,x)\} \) and note that by Lemma 2.2 of [2] \( \pi_i A \varepsilon = \{x\} \cap \{p\} = \pi_3 A \varepsilon \), and both are in \( K(p) \) (in \( K \)). But clearly \( \varepsilon \ni i_1 \varepsilon \) and \( \varepsilon \ni i_2 \varepsilon \) for all \( \varepsilon \in K(p) \) \( (\varepsilon \in K) \), a contradiction. Hence, \( B = \{p\} \).

2.6 THEOREM. Let \( X = (B,K) \) be in \( SOC \) (ConSOC). Axial subspace and wedge space are equal iff \( B \) is a point or the empty set.

PROOF. If \( B \) is a point or the empty set, then clearly the axial subspace and the wedge space are equal.

Conversely, suppose the axial subspace and the wedge space are equal and \( \Phi \neq B \neq \{b\} \), a point. Then there exists \( x \in B \) such that \( x \neq b \). Let \( \alpha = \{\pi_i A\}^{-1}[b] \cup \{\pi_i A\}^{-1} \{b\} \cup \{\pi_i A\}^{-1}[b] \), where \( \pi_i \)'s are the projection maps \( B^3 \to B \), \( i = 1, 2, 3 \) and \( A \) is the principal axis map defined in [1] or [2]. Note that \( \alpha \) is proper and by Lemma 2.2 of [2], \( \pi_1 A \alpha = \pi_2 A \alpha = \{b\} = \pi_3 A \alpha \in K(b) \) (all are in \( K \)). Since the axial subspace and the wedge space are equal, it follows that \( \alpha \ni i_k \alpha \) for some \( \alpha, \varepsilon \in K^2(b,b) \) \( (\alpha \in K^2) \) and \( k = 1, 2 \), where \( K^2 \) is the product structure on \( B^3 \). But this is a contradiction since no element in \( \alpha \) is entirely contained in one component of the wedge. Hence, \( B = \{\Phi\} \) or \( \{b\} \), a point.

Let \( X = (B,K) \) be in \( SOC \) or ConSOC and \( \Phi \neq F \subset B \). We now define the notion of (strongly) closure of \( F \) and give some algebraic properties of it. Recall, [3], that \( F \) is strongly closed iff \( F = B \).
and \( F \) is closed iff for any \( x \) in \( B \), if \( x \) is not in \( F \), then \( K(x) = \{ \varnothing \mid \varnothing \supseteq [x] \} \) ( \( K = \{ \varnothing \mid \varnothing \supseteq [b] \) for some \( b \in B \) \) in \( \text{ConSCO} \). 

(Strongly) closure of \( F \) is the smallest closed set containing \( F \). \( F = \bigcap \{ A : A \supseteq F \) and \( A \) is (strongly) closed \). It is easy to see \( \emptyset \emptyset = \emptyset \) since \( \emptyset \emptyset \) is (str.) closed, \( F \) is (strongly) closed, and \( F \supseteq F \).

2. \( \bigcup_{i \in I} F_i = \bigcup_{i \in I} \bar{F}_i \) and \( \bigcap_{i \in I} \bar{F}_i = \bigcap_{i \in I} F_i \) (if \( \bigcap_{i \in I} F_i \) is not empty). Note that in the case of \( \text{TOP} \), the category of topological spaces (2) does not hold, in general.

3. \( F \) is (strongly) closed if and only if \( \bar{F} = F \).

4. Arbitrary product of strongly closed subsets of \( B \) is strongly closed. Strongly closedness is cohereditary closed i.e if \( F \) is strongly closed and \( D \supseteq F \), then \( D \) is strongly closed.

5. Strongly closedness implies closedness but, in general, the converse is not true. For example, let \( B = \{x, y\} \) with \( x \neq y \), and \( K(x) = \{ [x], [x] \cup [y], [\emptyset] \} \) and \( K(y) = \{ [y], [x] \cup [y], [\emptyset] \} \). and \( F = \{x\} \). Clearly \( F \) is closed but not strongly closed.

6. Note that strongly closedness and closedness are equal iff \( X \) is \( T_1 \). Moreover, if \( X = (B,K) \) is "\( T_1\)-object" \( i = 1, 2, 3, 4 \), then every subset of \( B \) is both closed and strongly closed. This not true in \( \text{TOP} \), in general.

2.9 Theorem. Let \( (B,K) \) and \( (A,L) \) be in \( \text{SCO} \) or \( \text{ConSCO} \), and let \( f : (A,L) \rightarrow (B,K) \) be an initial lift of \( f : A \rightarrow B \). If \( (B,K) \) is any of "\( T_1\)-object" for \( i = 1, 2, 3, 4 \), then \( (A,L) \) is indiscrete i.e \( \forall x \in A \) \( L(x) = \text{STK}(A) \) ( \( L = \text{STK}(A) \).

Proof. It is sufficient to show that \( [A] \subseteq L(x) \) for all \( x \) in \( A \) ( \( [A] \subseteq L \)) since every stack \( \varnothing \) on \( A \) contains \( A \). Since \( (B,K) \) is "\( T_1\)-object" for \( i = 1, 2, 3, 4 \), then by 2.1, 2.2, 2.3, and 2.6 \( B \) is a point or the empty set. If \( B \) is the empty set, then clearly \( A \) is the empty set and the result follows. If \( B = \{b\} \), a point, then \( K(b) = \{ [b], [\emptyset] \} \). Hence, \( f \) must be constant. Since \( f ([A]) = [f(A)] = [b] \subseteq L(b) \) ( \( [b] \subseteq L \)) and \( f \) is an initial lift, then \( [A] \subseteq L(x) \) ( \( [A] \subseteq L \)) for all \( x \) in \( A \).
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2.10 THEOREM. Let \( f : (A,L) \Rightarrow (B,K) \) be an initial lift and \( (B,K) \) be "\( T_i \)-object" \( i = 1,2,3,4 \). Then \( (A,L) \) is "\( T_i \)-object" \( i = 1,2,3,4 \) iff \( f \) is mono.

PROOF. If \( f \) is mono, then by 2.1, 2.2, 2.3, and 2.6 \( (A,L) \) is "\( T_i \)-object" for \( i = 1,2,3,4 \). If \( (A,L) \) is "\( T_i \)-object" for \( i = 1,2,3,4 \), then by 2.1, 2.2, 2.3, and 2.6 \( A \) is a point or the empty set, and consequently \( f \) is mono.

2.11 REMARK.1. Let \( X = (A,K) \) be "\( T_i \)-object" for \( i = 1,2,3,4 \) in SCO or ConSCO. Then \( X \) is always an abelian group.

2. By [4], Theorems 2.5, 2.9, 2.10, and remark 2.6 do hold for separation properties at a point \( p \). Furthermore, points in \( X \) are closed and strongly closed iff \( X \) is \( T_o \) and \( T_i \), respectively (see Theorem 2.1 and [3]).

REFERENCES


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