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T_2-OBJECTS IN CATEGORY OF STACK CONVERGENCE SPACES

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ABSTRACT. There are eight different ways to characterize T -spaces in the category of topological spaces. All eight methods are canonical, i.e they can be easily formulated in a general setting, where they, in general, do not coincide. In the following, the characterizations of each of these T_2- objects in the categories of stack and constant stack convergence spaces are given. Moreover, some invariance properties of each them and the other separation properties as well as interrelationship among their various forms are established.

YAKINSAK YIGIN UZAYLAR KATEGORISINDE T_2-OBJELERI

ÖZET. Topolojik uzaylarda T_2-uzaylarını karakterize etmek icin sekiz degisik yol vardır. Butun bu yollar kanoniktir, yani kolaylıkla genellestirilebilir fakat genelde bu genellestirmeler birbirinden farklıdır. Bu calısmada, yakınsak yıgın ve yakınsak sabit yıgın uzaylarda T_2-objelerin herbirinin karakterizasyonu verildi. Bundan baska, bunların herbirisinin ve diger ayrılma aksiyomlarının bazı invaryant ozellikleri incelendigi gibi bunların degisik formları arasındaki iliskiler arastırıldı.

1. INTRODUCTION.

Several known generalizations of the usual T_2-axiom of topology to topological categories are given in [1]. We want to compare them for topological categories of stack convergence spaces and constant stack convergence spaces. Furthermore, we investigate some invariance properties of them (e.g the cartesian product, subspaces, and the quotient space).

Let A be a set and K be a function on A whose value K(a) at each a in A is a set of nonempty stacks on A. A pair (A,K) is said to be a stack convergence space iff the following conditions hold:

1. For each $a \in A$, $[a] = \{ B \subset A \mid a \in B \} \in K(a)$.
2. If α and β are stacks on A and $\alpha \subset \beta$, then $\beta \in K(a)$ if $\alpha \in K(a)$.

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A morphism $f : (A,K) \to (B,L)$ between stack convergence spaces is a function $f : A \to B$ such that $f\alpha \in L(f(a))$ if $\alpha \in K(a)$, where $f\alpha$ denotes the stack $\{ U \mid U \subset B$ and $U \supset f(C)$ for some $C \in \alpha \}$. We denote by SCO, the category so formed. See [7] p. 354.

The category of constant stack convergence spaces, ConSCO is the full subcategory of SCO determined by those spaces (A,K) where K is a constant function [7] p.354.

2. INVARIANCE PROPERTIES.

We now, for convenience, give the characterizations of the separation properties for SCO and ConSCO for later use.

2.1 THEOREM. let X = (B,K) be in SCO (ConSCO).

1. If X is T_0, then for each distinct pair of points x and y in B, $[x] \notin K(y)$ or $[y] \notin K(x)$, (X is T_0 iff B is a point or the empty set) see [5].
2. If X is T_0' , then X is discrete i.e for each x in B, $K(x) = \{ \alpha \mid \alpha \supset [x] \}$ ($K = \{ \alpha \mid \alpha \supset [a]$ for some $a \in B \}$. See [5].
3. X is T_0 iff $\forall x \neq y$ in B, $[x] \cap [y] \notin K(x)$ or $[x] \cap [y] \notin K(y)$ ($[x] \cap [y] \notin K$). This follows from Lemma 2.1 of [8] p. 318.
4. X is T_1 , T_2', T_2, ST_2, or ΔT_2 iff B is a point or the empty set [3] and [5].
5. X is $PreT_2$ iff X is indiscrete i.e $\forall x \in B$, $K(x) = STK(B)$, the set of all stacks on B ($K = STK(B)$). See [5].
6. X is $PreT_2'$ iff B is a point or the empty set [5].

2.2 THEOREM. X = (B,K) in SCO or ConSCO is LT_2, MT_2, or NT_2 iff B is a point or the empty set.

PROOF. It follows from the parts (1), (2), (6), and (7) of 2.1 and definitions.

2.3 THEOREM. X = (B,K) in SCO or ConSCO is KT_2 iff X is a point or the empty set.

PROOF. X is KT_2 iff X is T_0' and $PreT_2$. Hence, the result follows from this and Theorem 2.1.

2.4 REMARK.

1. All of the T_2' , T_2, ST_2, $\triangle T_2$, KT_2, LT_2, MT_2, and NT_2 are equal.

2. Each of $T_3$and T_3' implies T_3 but the converse is not true, in general. For example, Let B = { x,y } be two point set and define K by K(x) = { [x], [y], [x] ∩ [y], [Φ], [x] ∪ [y] } and K(y) = { [y], [x] ∪ [y], [Φ]}. Clearly X = (B,K) is T_3 but not T_3' and T_3.

2.5 THEOREM. Let X_i= (B_i,K_i) be T_2-objects in SCO or ConSCO.

1. The cartesian product of each of T_2-objects, X_i is T_2-object.
2. A subspace of each of T_2-objects is T_2-object.
3. The quotient space of each of T_2-objects is T_2-object.
4. The coproduct of each of T_2-objects is not T_2-object.

PROOF. It follows easily from Theorems 2.1, 2.2, and 2.3.

2.6 REMARKS. 1.There are four various ways (see[1]) to define each of T_3 and T_4-objects in arbitrary topological categories [6]. It follows from these definitions and Theorem 2.1 that X = (B,K) in SCO or ConSCO is T_i-object i = 3,4 iff B is a point or the empty set.

2. Theorem 2.5 holds for each T_i-objects i = 3,4.However,in TOP it is well-known,that the Cartesian Product of T_4-spaces, the subspace of T_4-space is not necessarily T_4-space.

Let U : E → Sets be topological, X an object in E, and p a point in UX. Recall,[1],that p-axial subspace is the initial lift of the principal p-axis map A_p : $BV_pB \to UX^2 = B^2$, and p-wedge is the final lift of the canonical injections i_1, i_2 : $UX \to BV_pB$.

We now give conditions on an object X = (B,K) and a point p in B so that its p-axial subspace is the same as the its p-wedge.

2.7 THEOREM. Let X = (B,K) be in SCO or ConSCO. The p-axial subspace and the p-wedge are equivalent iff B = {p}.

PROOF. If B = {p}, then K(p) = { [p], [Φ] } = K =STK(B), and consequently the p-axial subspace and the p-wedge are equal.

Conversely, suppose they are equivalent and $B \neq \{p\}$. Then there exists x in B such that $x \neq p$. We first claim that $K(p) = STK(B)$ ($K = STK(B)$). To this end, let α = ["x-axis", "y-axis"] and note that $\pi_1 A_p \alpha = [p] = \pi_2 A_p \alpha \in K(p)$ (or in K), where π_i are the projections $B^2 \to B$ for i = 1,2. Hence, by assumption we have $\alpha \supset i_1\alpha_1$ or $i_2\alpha_1$ for some $\alpha_1 \in K(p)$ ($\alpha_1 \in K$). We show that $\alpha_1 = [B]$. Clearly, $[B] \subset \alpha_1$ since B is in every stack. On the other hand, if $U \in \alpha_1$, then i_1U or i_2U is in $i_1\alpha_1 \subset \alpha$ or $i_2\alpha_1 \subset \alpha$, and consequently i_1U or i_2U is in α. It follows easily that U = B i.e $\alpha_1 \subset [B]$. Hence, $\alpha_1 = [B]$ and then $K(p) = STK(B)$ ($K = STK(B)$). Next, let $\beta = [(x,p)] \cap [(p,x)]$ and note that by Lemma 2.2 of [2] $\pi_1 A_p \beta = [x] \cap [p] = \pi_2 A_p \beta$, and both are in K(p) (in K). But clearly $\beta \not\supset i_1\beta_1$ and $\beta \not\supset i_2\beta_1$ for all $\beta_1 \in K(p)$ ($\beta_1 \in K$), a contradiction. Hence, $B = \{p\}$.

2.8 THEOREM. Let X = (B,K) be in SCO (ConSCO). Axial subspace and wedge space are equal iff B is a point or the empty set.

PROOF. If B is a point or the empty set, then clearly the axial subspace and the wedge space are equal.

Conversely, suppose the axial subspace and the wedge space are equal and $\Phi \neq B \neq \{b\}$, a point. Then there exists x in B such that $x \neq b$. Let $\alpha = (\pi_1 A)^{-1}[b] \cup (\pi_2 A)^{-1} [b] \cup (\pi_3 A)^{-1}[b]$, where π_i's are the projection maps $B^3 \to B$, i = 1,2,3 and A is the principal axis map defined in [1] or [2]. Note that α is proper and by Lemma 2.2 of [2], $\pi_1 A \alpha = \pi_2 A \alpha = [b] = \pi_3 A \alpha \in K(b)$ (all are in K). Since the axial subspace and the wedge space are equal, it follows that $\alpha \supset i_k\alpha_1$ for some $\alpha_1 \in K^2(b,b)$ ($\alpha_1 \in K^2$) and k = 1,2 , where K^2 is the product structure on B^2. But this is a contradiction since no element in α is entirely contained in one compenent of the wedge. Hence, $B = \{\Phi\}$ or $\{b\}$, a point.

Let X = (B,K) be in SCO or ConSCO and $\Phi \neq F \subset B$. We now define the notion of (strongly) closure of F and give some algebraic properties of it. Recall, [3], that F is strongly closed iff F = B, and F is closed iff for any x in B, if x is not in F, then K(x) = { α ! $\alpha \supset$ [x] } (K = { α ! $\alpha \supset$ [b] for some b $\in$ B } in ConSCO). (Strongly) closure of F is the smallest closed set containing F i.e $\bar{F} = \cap$ { A ! A $\supset$ F and A is (strongly) closed }. It is easy to see

1. $\bar{\Phi} = \Phi$ since Φ is (str.) closed, $\bar{F}$ is (strongly) closed, and $\bar{F} \supset F$.

2. $\overline{\bigcup_{i\in I} \bar{F}_i} = \bigcup_{i\in I} \bar{F}_i$ and $\overline{\bigcap_{i\in I} F_i} = \bigcap_{i\in I} \bar{F}_i$ (if $\bigcap_{i\in I} F_i$ is not empty). Note that in the case of TOP, the category of topological spaces (2) does not hold, in general.

3. F is (strongly) closed if and only if $\bar{F}$ = F.

4. Arbitrary product of strongly closed subsets of B is strongly closed. Strongly closedness is cohereditary closed i.e if F is strongly closed and D $\supset$ F, then D is strongly closed.

5. Strongly closedness implies closedness but , in general, the converse is not true. For example, let B = {x,y} with x $\neq$ y, and K(x) = { [x], [x] $\cup$ [y], [Φ] } and K(y) = { [y], [x] $\cup$ [y], [Φ]}, and F = {x}. Clearly F is closed but not strongly closed.

6. Note that strongly closedness and closedness are equal iff X is T_1. Moreever, if X = (B,K) is " T_i-object" i = 1,2 3,4, then every subset of B is both closed and strongly closed. This not true in TOP, in general.

2.9 THEOREM. Let (B,K) and (A,L) be in SCO or ConSCO, and let f : (A,L) $\rightarrow$ (B,K) be an initial lift of f : A $\rightarrow$ B. If (B,K) is any of " T_i-object" for i = 1,2,3,4, then (A,L) is indiscrete i.e $\forall$ x $\in$ A L(x) = STK(A) (L = STK(A)).

PROOF: It is sufficient to show that [A] $\in$ L(x) for all x in A ([A] $\in$ L) since every stack α on A contains A. Since (B,K) is " T_i-object" for i = 1,2,3,4, then by 2.1, 2.2, 2.3, and 2.6 B is a point or the empty set. If B is the empty set, then clearly A is the empty set and the result follows. If B = {b}, a point, then K(b) = { [b], [Φ] }. Hence, f must be constant. Since f ([A]) = [f(A)] = [b] $\in$ L(b) ([b] $\in$ L) and f is an initial lift, then [A] $\in$ L(x) ([A] $\in$ L) for all x in A.

2.10 THEOREM. Let $f : (A,L) \to (B,K)$ be an initial lift and (B,K) be "T_i-object" $i = 1,2,3,4$. Then (A,L) is "T_i-object" $i = 1,2,3,4$ iff f is mono.

PROOF. If f is mono, then by 2.1, 2.2, 2.3, and 2.6 (A,L) is "T_i-object" for i = 1,2,3,4. If (A,L) is " T_i-object" for i = 1,2,3,4, then by 2.1, 2.2, 2.3, and 2.6 A is a point or the empty set, and consequently f is mono.

2.11 REMARK.1. Let X = (A,K) be "T_i-object" for i = 1,2,3,4 in SCO or ConSCO. Then X is always an abelian group.

2. By [4], Theorems 2.5, 2.9, 2.10, and remark 2.6 do hold for separation properties at a point p. Furthermore, points in X are closed and strongly closed iff X is T_0 and T_1, respectively (see Theorem 2.1 and [3]).

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