ON RELATION BETWEEN CONVERGENCE AND ABSOLUTE SUMMABILITY \(|N, p_n|\) OF INFINITE SERIES

Osman ÖZDEMIR, E.Ü. Fen-Edebiyat Fakültesi, Matematik Bölümü, KAYSERİ

SUMMARY

In this paper, we gave a relation between summability method \(|N, p_n|\) of that series with a given series \(\sum a_n\).

SÖNÜZ SERİLERİN YAKINSAKLIK VE MUTLAK TOPLANABİLİRLİĞİ ARASINDAKI İLİŞKİ ÜZERİNE

ÖZET

Bu çalışmada verilen bir \(\sum a_n\) serisi ile \(0\) serisinin \(|N, p_n|\) toplanabilirleme yöntemi arasındaki ilişkiyi verdik.

I. INTRODUCTION

Let \((p_n)\) be a sequence such that \(p_n > 0, P_n = \sum_{v=0}^{n} p_v\) as \(n \to \infty\). The transformation \((N, p_n)\) maps a sequence \((S_n)\) into the sequence \((U_n)\) by means of the equation:

\[
U_n = p_n^{-1} \sum_{v=0}^{n} p_v S_v
\]  \(1\)

where the sequence \((S_n)\) is the \(n\)th partial sum of a given series \(\sum a_n\) \([1]\).

A matrix method is said to be regular if it is preserving the limit for convergent sequences \([2]\).

The necessary and sufficient conditions for the regularity of \((1)\) is \(p_n \to \infty\) as \(n \to \infty\) \([2]\).
A series $\Sigma a_n$ with its partial sum $S_n$ is said to be summable $(\mathcal{N}, p_n)$ to sum $S$ if

$$U_n = \sum_{v=0}^{n-1} p_v S_v \rightarrow S \text{ as } n \rightarrow \infty \quad [1].$$

A series $\Sigma a_n$ with its partial sum $S_n$ is said to be absolutely summable $(\mathcal{N}, p_n)$ or summable $(\mathcal{N}, p_n)$ if

$$\sum_{n=0}^{\infty} |U_n - U_{n+1}| < \infty \quad [1].$$

Given two summability methods $A$ and $B$, we write $(A) \subseteq (B)$ if each series summable $A$ is summable $B$ [1].

2. WE SHALL NOW PROVE THE FOLLOWING THEOREMS

Theorem 1. Let $(a_n)$ be a sequence of positive numbers. Then the series $\Sigma a_n$ is convergent if and only if the series $\Sigma a_n$ is the summable $(\mathcal{N}, p_n)$.

Proof: (i) necessity: suppose that the series $\Sigma a_n$ is convergent and let $S_n$ be the $n$th partial sum of given series $\Sigma a_n$.

Then we must show that

$$\sum_{n=1}^{\infty} |U_n - U_{n-1}| < \infty$$

By applying Abel's transformation to the sums in the right side of equality:

$$U_n - U_{n-1} = \sum_{v=0}^{n-1} p_v S_v - \sum_{v=0}^{n-1} p_v S_v$$

we have
\[ \begin{align*} 
U_n - U_{n-1} &= p_n(p_{n-1})^{-1} \sum_{v=1}^{n} P_{v-1}a_v \\
(2) 
\end{align*} \]

Let
\[ \begin{align*} 
\sum_{n=1}^{m} |U_n - U_{n-1}| &= \sum_{n=1}^{m} n \sum_{v=1}^{n-1} \Sigma P_{v-1}a_v = W_m \\
(3) 
\end{align*} \]

To show that \( \sum_{n=1}^{\infty} |U_n - U_{n-1}| < \infty \), it is sufficient to show that the sequence \( (W_m) \) is convergent. Let us prove this now. Since

\[ \begin{align*} 
p_m(p_{m-1})^{-1} \sum_{v=1}^{m} P_{v-1}a_v > 0 
\end{align*} \]

for all \( m \in \mathbb{N} \),

\[ \begin{align*} 
W_m - W_{m-1} &= \sum_{n=1}^{m} n \sum_{v=1}^{n-1} \Sigma P_{v-1}a_v - \sum_{n=1}^{m-1} n \sum_{v=1}^{n-1} \Sigma P_{v-1}a_v \\
&= \sum_{v=1}^{m} \left( \sum_{n=v}^{m} P_{n-1}a_v \right) - \sum_{v=1}^{m-1} \left( \sum_{n=v}^{m-1} P_{n-1}a_v \right) \\
&= \sum_{v=1}^{m} \left( P_{n-1}a_v - P_{v-1}\right) \\
&= \sum_{v=1}^{m} a_v \\
&= W_m - W_{m-1} 
\end{align*} \]

is positive for all \( m \in \mathbb{N} \). That is, the sequence \( (W_m) \) is monotonically increasing.

Moreover,

\[ \begin{align*} 
W_m &= \sum_{n=1}^{m} n \sum_{v=1}^{n-1} \Sigma P_{v-1}a_v \\
&= \sum_{n=1}^{m} \left( \sum_{v=1}^{n-1} a_v \Sigma (p_{n-1} - p_v) \right) \\
&= \sum_{v=1}^{m} \left( \sum_{n=v}^{m-1} a_v \Sigma (p_{n-1} - p_v) \right) \\
&= \sum_{v=1}^{m} a_v \\
&= W_m - W_{m-1} 
\end{align*} \]

for all \( m \in \mathbb{N} \). Hence the sequence \( (W_m) \) is convergent.

(ii) sufficiency: suppose that the series \( \sum a_n \) is summable \( [N, p_n] \).
It means that $\sum_{n=1}^{n} |U_{n} - U_{n-1}| < \infty$. By (3), considering the fact that $P_{M} \rightarrow M^{-\infty}$, we have

$$\sum_{n=1}^{n} P_{n} (P_{n} P_{n-1})^{-1} \sum_{v=1}^{n} P_{v-1} a_{v}$$

$$= \sum_{v=1}^{n} P_{v-1} a_{v} \sum_{n=v}^{n} P_{n} (P_{n} P_{n-1})$$

$$= \sum_{v=1}^{n} P_{v-1} a_{v} \lim_{M^{-\infty}} \sum_{n=v}^{M} (P_{v-1} - P_{v})$$

$$= \sum_{v=1}^{n} P_{v-1} a_{v} \lim_{M^{-\infty}} (P_{v-1} - P_{v})$$

$$= \sum_{v=1}^{n} P_{v-1} a_{v} = \sum_{v=1}^{n} a_{v}.$$ Thus,

$$\sum_{n=1}^{n} |U_{n} - U_{n-1}| = \sum_{v=1}^{n} a_{v}$$

is obtained which means that the series $\Sigma a_{n}$ is convergent.

This completes the proof of the theorem.

Now, we want to give another theorem for the series with arbitrary terms:

Theorem 2. Suppose that the serie $\Sigma a_{n}$ is convergent. Then the series $\Sigma a_{n}$ is summable $|N, P_{n}|$.

Proof: The proof of this theorem can be made similarly as in theorem 1.

REFERENCES
