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SOME CLASSES OF RICCATI EQUATIONS INTEGRABLE IN QUADRATURES

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ABSTRACT. As it is known, the second-order ordinary linear differential equation with variable coefficients is solvable in case if related Riccati equation can be integrated by quadratures. This paper considers establishment of correspondence between such equations by the authors' method which means the second-order equation representation by a chain of the first-order equations. The algorithm of special Riccati equation solving is demonstrated (coefficients of these Riccati equations satisfy special conditions). One more peculiarity of this paper stands in consideration of exact applicational example – the Riccati equation which describes the magnetotellurics impedance behavior in geological media.

1. INTRODUCTION

Some mathematical problems have an amazing fate – for several centuries these have been attractive objects for the mathematics fans' attention. Of course, first of all, it is worth to mention the Fermat's Last Theorem. By the way, it is interesting to note that the theorem about the equation $x^3 + y^3 = z^3$ unsolvability in natural numbers appears in Abu-Mahmud Khojandi's (Xth century) investigations long before Pierre de Fermat.

In ordinary differential equations theory the same role belongs to the Riccati equations. Despite the fact that in 1841 Joseph Liouville proved that the general solutions of such equations are usually impossible to be express by quadratures in terms of elementary functions, a huge number of scientific papers are still devoted to the Riccati equations study till nowadays.

There are two main reasons for this popularity. Firstly, the Riccati equations are used in the mathematical description of a huge number of problems in the algebraic geometry and the theory of completely integrable Hamiltonian systems, in the calculus of variations and the conformal map theory, the quantum field theory, economics, biology, geophysics and etc. One of such application problems is presented

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in the current paper. Secondly, like in the case of the Fermat's Last Theorem, here is a simple formulation of the problem: it is needed to find solutions of the ordinary differential equation $y' = a(x) + b(x)y + c(x)y^2$. We hope the materials provided in this paper will help to make some contributions to these amazingly interesting and important equations study.

2. The linear differential equation of the second order with constant coefficients

The linear differential equation of the second order

$$y'' + ay' + by = f(x) \tag{2.1}$$

with constant coefficients a and b is representable like chain of differential equations of the first order $z' - pz = f(x)$ and $y' - qy = z$, where coefficients p and q are roots of the characteristic equation $k^2 + ak + b = 0$. To prove this statement it is needed to substitute the expression for z from the second equation to the first, equalize coefficients of this obtained equation and the basic one and find p and q values from outcome relations. It is important to note that this statement is true for linear equations of the higher order as well [1–3].

Example 1. Integrate the ordinary differential equation

$$y'' - 6y' + 9y = \frac{e^{3x}}{\sqrt[3]{x}}.$$

Using roots of a characteristic equation $k^2 – 6k + 9 = 0$ it is possible to expand the given equation in a view of the following chain of equations: $z' - 3z = \frac{e^{3x}}{\sqrt[3]{x}}$ and $y' - 3y = z$. Solving the first equation it can be found that $z = (1.5\sqrt[3]{x^2} + C)e^{3x}$, where $C \in \mathbb{R}$. Placing this result into the right side of the second equation, we can find that $y = (0.9\sqrt[3]{x^5} + Cx + C_1)e^{3x}$, where $C \in \mathbb{R}$ and $C_1 \in \mathbb{R}$.

3. The linear differential equation of the second order with variable coefficients

The chain expansion of the linear differential equation with constant coefficients shown above allows to suppose that the same approach is applicable to equations with variable coefficients, i.e. the linear differential equation of the second order with variable coefficients can be represented in the view of a chain of the liner differential equations of the first order with variable coefficients.

Let's suppose that the equation

$$y'' + a(x)y' + b(x)y = f(x), \tag{3.1}$$

where $a(x)$, $b(x)$ and $f(x)$ are some functions can be replaced by the chain of next equations:

$$z' - p(x)z = f(x),\ y' - q(x)y = z. \tag{3.2}$$

Substituting the expression for z from the second chain equation into the first, it can be found that the suggested chain leads to an equation

$$y'' + [-q(x) - p(x)]y' + [q(x)p(x) - q'(x)]y = f(x),$$

from which together with (2.1) it follows that

$$-p(x) - q(x) = a(x) \text{ and } q(x)p(x) - q'(x) = b(x).$$

Applying the expression for $p(x)$ function which follows from the first relation into the second one, the next equation of $q(x)$ can be obtained:

$$q'(x) + q(x)a(x) + q^2(x) = -b(x). \tag{3.3}$$

Solving the last equation we will have a chance to expand the basic equation (3.1) into the chain (3.2). However, (3.3) is Riccati equation which is unsolvable in general case in terms of elementary functions by quadratures [4]. Thus, from over here the next known result [4] follows once more: unfortunately, in general case, linear differential equations of the second order with variable coefficients cannot be integrated by quadratures. That's why we are suggesting to concentrate attention on some exact special cases.

4. The Euler differential equation

Chain approach usage allows to integrate not only equations with constant coefficients, but also some types of equations with variable coefficients as well. These equations include the Euler equations.

Theorem 4.1. *The Euler equation*

$$y'' + \frac{p}{x}y' + \frac{q}{x^2}y = f(x) \tag{4.1}$$

where p and q are constant coefficients, can be represented in a view of the chain of linear differential equations of the first order

$$z' - \frac{k}{x}z = f(x), \;\; y' - \frac{m}{x}y = z, \tag{4.2}$$

where coefficients k and m are solutions of the algebraic equations system

$$\begin{cases} k + m = -p; \\ (k+1)m = q. \end{cases} \tag{4.3}$$

Proof. It won't be complicated to proof this theorem's statement. Differentiating the second chain (4.2) equation it can be found that

$$z' = y'' + \frac{m}{x^2}y - \frac{m}{x}y'.$$

Using this result in the first equation of the chain (4.2) the next relation can be obtained:

$$y'' + \frac{-m-k}{x}y' + \frac{(k+1)m}{x^2}y = f(x).$$

Comparing coefficients of the last equation and the basic equation (4.1) it is easy to find that k and m coefficients are really solutions of the system (4.3).

□

Example 2. To integrate an equation

$$y'' - \frac{5}{x}y' + \frac{8}{x^2}y = x^3e^x \tag{4.4}$$

let's consider an algebraic system

$$\begin{cases} k + m = 5; \\ (k+1)m = 8. \end{cases}$$

Using one pair of this system roots (such as $k = 3$ and $m = 2$) it is possible to represent the equation (4.4) in a chain form

$$z' - \frac{3}{x}z = x^3 e^x, \ y' - \frac{2}{x}y = z.$$

From over here it follows that the first chain equation general solution takes the view of $z = x^3(e^x + C), \ C \in \mathbb{R}$, which means that solution of the equation (4.4) coincides with solution of an equation

$$y' - \frac{2}{x}y = x^3(e^x + C).$$

Thus, the general solution of the basic equation (4.4) is

$$y = x^2(xe^x - e^x + C_1x^2 + C_2), \ C_1 \in \mathbb{R}, \ C_2 \in \mathbb{R}.$$

5. The Euler-Riccati differential equation

In chapter 3 it was shown that integrability of linear differential equation of the second order by quadratures is defined by solvability of the related Riccati equation. We are going to call such equations by Euler-Riccati equations to consider these in the current section.

Theorem 5.1. *The Euler-Riccati equation*

$$y' = ay^2 + \frac{b}{x}y + \frac{c}{x^2} \tag{5.1}$$

where a, b and c are constant coefficients, is integrable by quadratures.

Proof. Following the traditional approach to find a general solution of the equation (5.1) it is possible to use its' particular solution $y = \frac{s}{x}$, where s is a root of equation $-s = as^2 + bs + c$. Substitution of $y = z + \frac{s}{x}$ allows to obtain the related Bernoulli Equation and, finally, deal with the linear differential equation of the first order. But we are going to discuss alternative approach to solve equation (5.1). Multiplying (5.1) by x^2, we have an equation $x^2y' = ax^2y^2 + bxy + c$, whose right side is represented by the square trinomial of unknown function xy. Adding xy to both sides of the last equation we will have a new one $x^2y' + xy = ax^2y^2 + (b+1)xy + c$, left side of it can be transformed by the next way: $x^2y' + xy = x(xy' + x'y) = x(xy)'$. Finally, usage of the substitution $u = xy$ leads to deal with a separable differential equation $xu' = au^2 + (b+1)u + c$.

□

Example 3. Solve Euler-Riccati equation

$$y' = y^2 - \frac{9}{x}y + \frac{17}{x^2}.$$

Let's follow the algorithm demonstrated above. Multiplying the given equation by x^2 and applying the substitution $u = xy$, we have an equation $xu' = u^2 - 8u + 17$ which can be solved by variables separation:

$$\int \frac{du}{u^2 - 8u + 17} = \int \frac{dx}{x}.$$

As
$$\int \frac{du}{u^2-8u+17} = \int \frac{du}{(u-4)^2+1} = arctg(u-4)+A,\ A \in \mathbb{R}.$$
so the general solution of the basic equation is $arctg(xy-4) = C|lnx|,\ C \in \mathbb{R}.$.

It should be noted that the reasoning used in the process of the Euler-Riccati equations solving can be applied in some other cases.

6. The Riccati differential equation

Theorem 6.1. *The Riccati equation which has a view of*

$$y' = ax^q y^2 + \frac{by}{x} + \frac{c}{x^{q+2}} \tag{6.1}$$

where a, b, c and q are constant numbers, is integrable by quadratures.

Proof. Let's rewrite (6.1) to the view of $x^{q+2}y' = ax^{2q+2}y^2 + bx^{q+1}y + c$. Adding the addendum $(q+1)x^{q+1}y$ and applying substitution $z = x^{q+1}y$ we will have an separable variables equation $xz' = az^2 + (b+q+1)z + c$.

□

Example 4. Solve the differential equation
$$y' = x^{0.5}y^2 - 2.5\frac{y}{x} - \frac{6}{x^{2.5}}.$$

Multiplying the given equation by $x^{2.5}$, adding expression $1.5x^{1.5}y$ to both sides of equation, and using the substitution $z = x^{1.5}y$, we will have a new equation $xz' = z^2 - z - 6$. From the last one it follows that
$$\int \frac{dz}{(z+2)(z-3)} = \int \frac{dx}{x},$$
which means
$$\frac{5}{z+2} - 1 = x^5C \text{ or } x^{1.5}y + 2 = \frac{5}{x^5C+1}.$$

So, the general solution of the given equation is
$$y = \frac{1}{x^{1.5}}\left(\frac{5}{x^5C+1} - 2\right),\ C \in \mathbb{R}.$$

7. The inverse Euler-Riccati differential equation

Theorem 7.1. *The inverse Euler-Riccati differential equation*

$$y' = \frac{a_1y^2}{(qx+r)^2} + \frac{b_1y}{qx+r} + c \tag{7.1}$$

where a_1, b_1 and c are constant numbers, is integrable by quadratures.

Proof. It is easy to find that the equation (7.1) can be written as
$$y' = \frac{ay^2}{(x+p)^2} + \frac{by}{x+p} + c,$$
where $a = a_1/q^2,\ b = b_1/q$. Dividing the last equation by $(x+p)$ and subtracting $y/(x+p)^2$ from both sides of the last equation, we will have a new one:
$$\frac{y'}{x+p} - \frac{y}{(x+p)^2} = \frac{a}{x+p}\frac{y^2}{(x+p)^2} + \frac{(b-1)y}{(x+p)^2} + \frac{c}{x+p}.$$

Let's collapse the left side of the previous equation by the rule:

$$\frac{y'}{x+p} - \frac{y}{(x+p)^2} = \left(\frac{y}{x+p}\right)'$$

and apply the substitution $u = y/(x+p)$. Finally these allow to obtain a separable variables equation

$$u' = \frac{au^2 + (b-1)u + c}{x+p}.$$

□

Example 5. Let's consider the Riccati equation which describes a magnetotelluric impedance in a one-dimensional geological media:

$$\frac{dZ(z)}{dz} - \sigma(z)Z^2(z) = i\omega\mu_0, \tag{7.2}$$

where $Z(z)$ is an unknown impedance function depends on spatial coordinate z (z-axis directed into the depths of the Earth); i is an imaginary 1, i.e. $i^2 = -1$; ω is an electromagnetic field frequency; μ_0 is a vacuum permeability constant; $\sigma(z)$ is an electrical conductance of the media [5].

It is easy to solve the equation (7.2) in case if $\sigma(z) = \sigma \equiv const$, because in this case (7.2) becomes an equation with separable variables:

$$\frac{dZ}{\sigma Z^2 + i\omega\mu_0} = dz.$$

In case if $\sigma(z) = \sigma_0(1+pz)^{-2}$, where σ_0 and p are positive real numbers, the equation (7.2) takes the view $Z' = \sigma_0(1+pz)^{-2}Z^2 + i\omega\mu_0$. Dividing the last equation by $(1+pz)$ and subtracting fraction $\frac{pZ}{(1+pz)^2}$ from both sides of this equation we will have:

$$\frac{Z'}{1+pz} - \frac{pZ}{(1+pz)^2} = \frac{\sigma_0}{1+pz}\left(\frac{Z}{1+pz}\right)^2 + \frac{i\omega\mu_0}{1+pz} - \frac{pZ}{(1+pz)^2}$$

or, transforming to view:

$$\left(\frac{Z}{1+pz}\right)' = \frac{\sigma_0}{1+pz}\left(\frac{Z}{1+pz}\right)^2 + \frac{i\omega\mu_0}{1+pz} - \frac{p}{1+pz}\frac{Z}{(1+pz)^2}.$$

Let's rewrite the previous equation view denoting $u = \frac{Z}{1+pz}$:

$$u' = \frac{\sigma_0}{1+pz}u^2 - \frac{p}{1+pz}u + \frac{i\omega\mu_0}{1+pz}$$

or

$$(1+pz)u' = \sigma_0 u^2 - pu + i\omega\mu_0.$$

Obtained equation is an equation with separable variables

$$\frac{du}{\sigma_0 u^2 - pu + i\omega\mu_0} = \frac{dz}{1+pz},$$

general solution of which takes a form:

$$u = \frac{1}{2\sigma_0}\left(p - \nu - \frac{2\nu}{C(1+pz)^\nu - 1}\right),$$

where $\nu = \sqrt{p^2 - 4i\omega\mu_0\sigma_0}$ and C is a real number represents a constant of integration.

To turn back to the initial function Z and find the general solution of equation (7.2) let's use the relation $u = \frac{Z}{1+pz}$:

$$Z = \frac{1+pz}{2\sigma_0}\left(p - \nu - \frac{2\nu}{C(1+pz)^{\nu} - 1}\right), \ \nu = \sqrt{p^2 - 4i\omega\mu_0\sigma_0}, \ C \in \mathbb{R}.$$

8. Conclusion

The process of linear ordinary differential equations solving in analytical view is an essential element in teaching mathematics for future engineers, economists, chemist, etc. Very often, the corresponding mathematical courses are overloaded with long preliminary discussions about the linear independence of particular solutions, the basis and other special terms. The approach represented in this paper allows to join the world of differential equations solutions of which can be "touched" without requiring deep prior knowledge. Demonstrated in the work direct connection between the second-order differential equations and the Riccati equations can be served as a basis for students and junior scientists beginning their research activity.

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References

[1] S.K. Kydyraliev, A.B. Urdaletova, Solving Linear Differential Equations by Operator Factorization. The College Mathematics Journal, USA, Vol.27, No.3, pp.199-204 (1996).

[2] S.K. Kydyraliev, A.B. Urdaletova, E.S. Burova, Advantages of the chain method for solving linear differential and difference equations, Bulletin of the Kyrgyz-Russian Slavic University, Vol.20, No.8, pp.16-20 (2020).

[3] S.K. Kydyraliev, A.B. Urdaletova, The linear ordinary differential equations' solution formula, Bulletin of the Kyrgyz-Russian Slavic University, Vol.20, No.8, pp.11-15 (2020).

[4] A.D. Polyanin, V.F. Zaitsev, Handbook of Ordinary Differential Equations: Exact Solutions, Methods, and Problems, Chapman Hall/CRC Press, Boca Raton, pp.1496 (2017).

[5] M.N. Berdichevskii, V.I. Dmitriev, Magnetotelluric Sounding of Horizontally Homogeneous Media. Moscow, Nedra, pp.250 (1992).

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