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ON INVARIANT SEQUENCE SPACES

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ABSTRACT

In this paper we define v-invariantness of a sequence space and examine the v-invariantness of the sequence spaces m. c c_o and ℓ_p. Furthermore, we give Köthe-Toeplitz duals of some certain sequence spaces.

İNVARİYANT DİZİ UZAYLARI ÜZERİNE

ÖZET

Bu çalışmada, bir dizi uzayının v-invariyantlığını tanımladık ve m, c, c_o ve ℓ_p dizi uzaylarının v-invariyantlığını inceledik. Ayrıca, bazı dizi uzaylarının Köthe-Toeplitz duallerini verdik.

1. INTRODUCTION

Let $v=(v_k)$ be any fixed sequence of nonzero complex numbers satisfying

$$\liminf |v_k|^{1/k} = r \quad (0 < r \leqslant \infty). \tag{1}$$

Then, we define

$$m_v = \{ x=(x_k) : \sup_k |v_k x_k| < \infty \}$$

$$(c_o)_v = \{ x=(x_k) : |v_k x_k| \to 0 \quad \text{as } k \to \infty \}$$

$$c_v = \{ x=(x_k) : v_k x_k \to \ell \quad \text{as } k \to \infty \text{ , for some } \ell \}$$

$$(\ell_p)_v = \{x=(x_k) : \sum_k |v_k x_k|^p < \infty, \quad 0 < p < \infty\}.$$

([1]). These sequence spaces also may be regarded as spaces of analytic functions ([1], [3], [4]).

In this paper, m, c, and c_o will denote the sequence spaces of bounded, convergent and null sequences, respectively, and ℓ_p $(0 < p < \infty)$ will denote the space of the sequences $x=(x_k)$ such that $\sum_k |x_k|^p < \infty$.

2. INVARIANT SEQUENCE SPACES

In this section we define v-invariantness of a sequence space X and give necessary and sufficient conditions for m, c, and ℓ_p to be v-invariant.

DEFINITION 1. We said that a sequence space X is v-invariant if $X_v = X$, where $X_v = \{x=(x_k) : (v_k x_k) \in X\}$.

It is known that if a sequence space X is a Banach space then X_v is also a Banach space, and if X is separable, X_v is also separable ([1]).

Now, if X is v-invariant then we have the following results.

THEOREM 1. Let X be a v-invariant sequence space. Then

(i) X_v is a Banach space if and only if X is a Banach space,
(ii) X_v is separable if and only if X is separable.

Let $u=(u_k)$ and $v=(v_k)$ be any fixed sequences of nonzero complex numbers such that

$$\liminf_{k\to\infty} |u_k|^{1/k} \text{ and } \liminf_{k\to\infty} |v_k|^{1/k}$$

are positive (may be infinite).

If $v_k = \lambda$ for every k, then obviously m, c, c_o and ℓ_p are v-invarient, where λ is a scalar.

THEOREM 2. Let $w_k = u_k v_k^{-1}$ for each $k \in N$, where $v_k^{-1} = 1/v_k$, and let X denote one of the sequence spaces m, c_o and ℓ_p. Then

(i) $X_v \subset X_u$ if and only if $\sup_k |w_k| < \infty$,

(ii) $X_v = X_u$ if and only if

$$0 < \inf_k |w_k| \leqslant |w_k| \leqslant \sup_k |w_k| < \infty ,$$

PROOF. We prove the theorem for X=m. The proof for $X=c_o$ and $X=\ell_p$ is similar.

(i) Sufficiency is trivial, since

$$|u_k x_k| = |w_k||v_k x_k| \qquad (2)$$

For the necessity suppose that $m_v \subset m_u$ but $\sup_k |w_k| = \infty$. Then there exists a strictly increasing sequence (k_i) of positive integers such that $|w_{k_i}| > i$. We put $x_k = 0$ $(k \neq k_i)$, $x_k = i/u_{k_i}$ $(k=k_i)$. Then we have $|v_k x_k| < 1$ and $|u_k x_k| = i$, where $k=k_i$. Whence $x \in m_v - m_u$ contrary to the assumption that $m_v \subset m_u$.

(ii) To prove this, it is enough to show that $m_u \subset m_v$ if and only if $\inf_k |w_k| > 0$. It is obvious that $\inf_k |w_k| > 0$ if and only if

$$\sup_k |w_k^{-1}| < \infty .$$

Hence the result follows from (i).

THEOREM 3. Let X denote one of the sequence spaces m, c_o and ℓ_p. Then

(i) $X \subset X_v$ if and only if $\sup_k |v_k| < \infty$,

(ii) $X_v \subset X$ if and only if $\inf_k |v_k| > 0$,

(iii) $X_v = X$ if and only if

$$0 < \inf_k |v_k| \leqslant |v_k| \leqslant \infty \sup_k |v_k| < \infty .$$

PROOF. Taking $v=(1,1,\ldots)$ and replacing u by v in Theorem 2(i), we obtain (i). It is trivial that $\inf_k |v_k| > 0$ if and only if

$$\sup_k |v_k^{-1}| < \infty$$

Hence taking $u=(1,1,\ldots)$ in Theorem 2(i), we get (ii). Finally, taking $u=(1,1,\ldots)$ in THeorem 2(ii), since clearly $\inf_k |v_k^{-1}| > 0$ if and only if $\sup_k |v_k| < \infty$, we get (iii).

COROLLARY 1. Let X denote one of the sequence spaces m, c_o and ℓ_p. Then X is v-invariant if and only if

$$0 < \inf_k |v_k| \leqslant |v_k| \leqslant \sup_k |v_k| < \infty . \tag{3}$$

Proof follows from Theorem 3 (iii).

THEOREM 4. (i) $c_v \subset c_u$ if and only if $w=(w_k) \in c$.

(ii) $c_v = c_u$ if and only if $w \in c - c_o$.

PROOF. (i) The sufficiency is trivial by (2). For the necessity suppose that $c_v \subset c_u$ but $w \notin c$. Then, either $w \in m$ or $w \notin m$. Now we put $x=(w_k u_k^{-1})=(v_k^{-1})$. Then $(v_k x_k) = (1,1,\ldots)$ and $(u_k x_k)=(w_k)$. Whence $x \in c_v - c_u$, contrary to the assumption that $c_v \subset c_u$. Hence we obtain the necessity.

(ii) Sufficiency. Let $w \in c-c_o$, then $c_v \subset c_u$ by (i). Let $x \in c_u$, so that $(u_k x_k) \in c$. Now, since $w \in c-c_o$, $\lim_k w_k^{-1}$ exists and is finite. Therefore, from the equality $v_k x_k = w_k^{-1} u_k x_k$, we have $(v_k x_k) \in c$ and hence $c_u \subset c_v$.

Necessity. Suppose that $c_v = c_u$, that is, $c_v \subset c_u$ and $c_u \subset c_v$. Then

$$\lim_k w_k = \lim_k u_k v_k^{-1} \text{ and } \lim_k w_k^{-1} = \lim_k u_k^{-1} v_k^{-1}$$

exist.

It is trivial that $\lim_k w_k^{-1}$ exists if and only if $\lim_k w_k \neq 0$. Hence $w \in c - c_o$. This completes the proof.

THEOREM 5. (i) $c \subset c_v$ if and only if $v \in c$,

(ii) $c_v = c$ if and only if v c and $\lim_k v_k \neq 0$.

PROOF. Taking $v = (1,1, \ldots)$ and replacing u by v in Theorem 4(i), we obtain (i). Theorem 4(ii) gives us (ii) for $u=(1.1,\ldots)$.

Remark. If $v \in c$ and $\lim_k v_k = 0$, that is $v \in c_o$, then $c \subset (c_o)_v$.

If we consider Definition 1 and Theorem 5(ii), then we can express the following result:

COROLLARY 2. c is v-invariant if and only if $v \in c - c_o$.

3. KÖTHE-TOEPLITZ DUAL SPACES

DEFINITION 2 ([2]). Let X be a sequence space and define

(i) $X^\alpha = \{ a=(a_k) : \sum_k |a_k x_k| < \infty \text{ for all } x \in X \}$,

(ii) $X^\beta = \{a=(a_k) : \sum_k a_k x_k \text{ converges for all } x \in X\}$,

(iii) $X^\gamma = \{a=(a_k) : \sup_n |\sum_{k=1}^n a_k x_k| < \infty \text{ for all } x \in X\}$.

Then X^α, X^β and X^γ are called the α-, β- and γ-dual spaces of X, respectively. X^α is also called Köthe-Toeplitz dual space and X^β is also caller generalized Köthe-Toeplitz dual space. It is easy to show that $\phi\subset X^{\alpha}\subset X^{\beta}\subset X^{\gamma}$. If $X\subset Y$ then $Y^{\eta}\subset X^{\eta}$ for $\eta=\alpha$, β or γ.

LEMMA 1 ([2]).

(i) $c_o^{\eta} = c^{\eta} = m^{\eta} = \ell_1$,

(ii) $\ell_p^{\eta} = \ell_q$ $(1 < p < \infty$, $p^{-1} = q^{-1} = 1)$

(iii) $\ell_p^{\eta} = m$ $(0 < p \leqslant 1)$

where η stand for α-, β- or γ .

The α-, β- and γ-duals of the sequence spaces $(c_o)_v$, c_v, m_v and $(\ell_p)_v$ are known, ([1]), where $v=(v_k)$ satisfies (1) without any other restriction.

Now, if we consider Lemma 1, Corollary 1 and Corollary 2. Then we have the following resu'ts:

THEOREM 6. a) Let (3) is satisfied. Then, for $\eta=\alpha$, β and γ,

(i) $((c_o)_v)^{\eta} = (m_v)^{\eta} = \ell_1$,

(ii) $((\ell_p)_v)^{\eta} = \ell_q$ $(1 < p < \infty,\ p^{-1} + q^{-1} = 1)$,

(iii) $((\ell_p)_v)^{\eta} = m$ $(0 < p \leqslant 1)$.

b) If $v \in c-c_o$ then, for $\eta=\alpha$, β and γ,

(iv) $(c_v)^{\eta} = \ell_1$.

THEOREM 7. a) Let (3) is satisfied. Then, for $\eta=\alpha$, β and γ,

(i) $((c_o)_v)^{\eta\eta} = (m_v)^{\eta\eta} = m$,

(ii) $((\ell_p)_v)^{\eta\eta} = \ell_p \qquad (1 < p < \infty),$

(iii) $((\ell_p)_v)^{\eta\eta} = \ell_1 \qquad (0 \leqslant p < 1).$

b) If $v \in c - c_o$ then, for $\eta = \alpha$, β and γ,

(iv) $(c_v)^{\eta\eta} = m.$

REFERENCES

[1] R. Çolak., P.D. Srivastava and S. Nanda, On certain sequence spaces and their Köthe-Toeplitz duals, (under communication).

[2] P.K. Kamtham, M. Gupta, Sequence spaces and series, Marcel Dekker Inc. New York (1981).

[3] S. Nanda, Two applications of Functional Analysis I : Matrix transformations and sequence spaces, Queen's Papers in Pure and Appl. Math. Queen's Univ. Press No. 74 (1986).

[4] P.D. Srivastava, A study of some aspect of certain spaces and algebras of analytic functions, Ph.D. Thesis IIT, Kanpur, India (1978).