ON INVARIANT SEQUENCE SPACES

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ABSTRACT

In this paper we define $v$-invariantness of a sequence space and examine the $v$-invariantness of the sequence spaces $m$, $c$, $c_0$ and $l_p$. Furthermore, we give Köthe-Toeplitz duals of some certain sequence spaces.

INVARIYANT DIZİ UZAYLARI ÜZERINE

ÖZET

Bu çalışmada, bir dizinin $v$-invaryantlığını tanımladık ve $m$, $c$, $c_0$ ve $l_p$ dizilerinin $v$-invaryantlığını inceledik. Ayrıca, bazı dizilerinin Köthe-Toeplitz duallerini verdik.

1. INTRODUCTION

Let $v=(v_k)$ be any fixed sequence of nonzero complex numbers satisfying

$$\liminf \frac{|v_k|^{1/k}}{v_k} = r \quad (0 < r < \infty). \quad (1)$$

Then, we define

$$m_v = \{ x=(x_k) : \sup_k |v_k x_k| < \infty \}$$

$$(c_0)_v = \{ x=(x_k) : |v_k x_k| \to 0 \text{ as } k \to \infty \}$$

$$c_v = \{ x=(x_k) : v_k x_k \to \varepsilon \quad \text{as } k \to \infty \text{, for some } \varepsilon \}$$

881
$$(\ell_p)_v = \{ x = (x_k) : \sum_k |v_k x_k|^p < \infty, \ 0 < p \leq \infty \}.$$  

([1]). These sequence spaces also may be regarded as spaces of analytic functions ([1], [3], [4]).

In this paper, $m$, $c$, and $c_0$ will denote the sequence spaces of bounded, convergent and null sequences, respectively, and $\ell_p$ \((0 < p \leq \infty)\) will denote the space of the sequences $x = (x_k)$ such that $\sum_k |x_k|^p < \infty$.

2. INvariant SEQUENCE SPACES

In this section we define $v$-invariantness of a sequence space $X$ and give necessary and sufficient conditions for $m$, $c$, and $\ell_p$ to be $v$-invariant.

**DEFINITION 1.** We said that a sequence space $X$ is $v$-invariant if $X_v = X$, where $X_v = \{ x = (x_k) : (v_k x_k) \subset X \}$.

It is known that if a sequence space $X$ is a Banach space then $X_v$ is also a Banach space, and if $X$ is separable, $X_v$ is also separable ([1]).

Now, if $X$ is $v$-invariant then we have the following results.

**THEOREM 1.** Let $X$ be a $v$-invariant sequence space. Then

(i) $X_v$ is a Banach space if and only if $X$ is a Banach space,

(ii) $X_v$ is separable if and only if $X$ is separable.

Let $u=(u_k)$ and $v=(v_k)$ be any fixed sequences of nonzero complex numbers such that

$$\liminf_{k \to \infty} |u_k|^{1/k} \quad \text{and} \quad \liminf_{k \to \infty} |v_k|^{1/k}$$

are positive (may be infinite).

If $v_k = \lambda$ for every $k$, then obviously $m$, $c$, $c_0$ and $\ell_p$ are $v$-invariant, where $\lambda$ is a scalar.
THEOREM 2. Let \( w_k = u_k v_k^{-1} \) for each \( k \in \mathbb{N} \), where \( v_k^{-1} = 1/v_k \), and let \( X \) denote one of the sequence spaces \( m, c_0 \) and \( l_p \). Then

(i) \( X_v \subseteq X_u \) if and only if \( \sup_k |w_k| < \infty \),

(ii) \( X_v = X_u \) if and only if

\[
0 < \inf_k |w_k| \leq |w_k| \leq \sup_k |w_k| < \infty ,
\]

PROOF. We prove the theorem for \( X = m \). The proof for \( X = c_0 \) and \( X = l_p \) is similar.

(i) Sufficiency is trivial, since

\[
|u_k x_k| = |w_k| |v_k x_k| \tag{2}
\]

For the necessity suppose that \( m_v \subseteq m_u \) but \( \sup_k |w_k| = \infty \). Then there exists a strictly increasing sequence \( (k_i) \) of positive integers such that \( |w_{k_i}| > i \). We put \( x_k = 0 \) \( (k \neq k_i) \), \( x_k = i/u_{k_i} \) \( (k = k_i) \). Then we have \( |v_k x_k| < 1 \) and \( |u_k x_k| = i \), where \( k = k_i \). Whence \( x \in m_v - m_u \) contrary to the assumption that \( m_v \subseteq m_u \).

(ii) To prove this, it is enough to show that \( m_u \subseteq m_v \) if and only if \( \inf_k |w_k| > 0 \). It is obvious that \( \inf_k |w_k| > 0 \) if and only if

\[
\sup_k |w_k| < \infty .
\]

Hence the result follows from (i).

THEOREM 3. Let \( X \) denote one of the sequence spaces \( m, c_0 \) and \( l_p \). Then

(i) \( X \subseteq X_v \) if and only if \( \sup_k |v_k| < \infty \),

(ii) \( X_v \subseteq X \) if and only if \( \inf_k |v_k| > 0 \),

883
R. ÇOLAK/ON INVARIANT SEQUENCE SPACES

(iii) \( X_v = X \) if and only if

\[ 0 < \inf_k |v_k| \leq |v_k| \leq \sup_k |v_k| < \infty. \]

PROOF. Taking \( v=(1,1,...) \) and replacing \( u \) by \( v \) in Theorem 2(i), we obtain (i). It is trivial that \( \inf_k |v_k| > 0 \) if and only if

\[ \sup_k |v_k^{-1}| < \infty. \]

Hence taking \( u=(1,1,...) \) in Theorem 2(i), we get (ii). Finally, taking \( u=(1,1,...) \) in Theorem 2(ii), since clearly \( \inf_k |v_k^{-1}| > 0 \) if and only if \( \sup_k |v_k| < \infty \), we get (iii).

COROLLARY 1. Let \( X \) denote one of the sequence spaces \( \ell^p \) and \( c_0 \). Then \( X \) is \( v \)-invariant if and only if

\[ 0 < \inf_k |v_k| \leq |v_k| \leq \sup_k |v_k| < \infty. \]

(3)

Proof follows from Theorem 3 (iii).

THEOREM 4. (i) \( c_v \subset c_u \) if and only if \( w=(w_k) \subset c \).

(ii) \( c_v = c_u \) if and only if \( w \subset c - c_0 \).

PROOF. (i) The sufficiency is trivial by (2). For the necessity suppose that \( c_v \subset c_u \) but \( w \not\subset c \). Then, either \( w \not\subset \ell^p \) or \( w \not\subset c \). Now we put \( x=(w_k^{-1})(v_k^{-1}) \). Then \( (v_k x_k) = (1,1,...) \) and \( (u_k x_k) = (w_k) \). Hence \( x \not\in c_v - c_u \), contrary to the assumption that \( c_v \subset c_u \). Hence we obtain the necessity.

(ii) Sufficiency. Let \( w \subset c - c_0 \), then \( c_v \subset c_u \) by (i). Let \( x \in c_u \), so that \( (u_k x_k) \subset c \). Now, since \( w \subset c - c_0 \), \( \lim_{k} w_k^{-1} \) exists and is finite.

Therefore, from the equality \( v_k x_k = w_k^{-1} u_k x_k \), we have \( (v_k x_k) \subset c \) and hence \( c_u \subset c_v \).

884
R. Çolak/ On Invariant Sequence Spaces

Necessity. Suppose that \( c_v = c_u \), that is, \( c_v \subseteq c_u \) and \( c_v \subseteq c_v \). Then

\[
\lim_{k \to \infty} w_k = \lim_{k \to \infty} u_k v_k^{-1} \quad \text{and} \quad \lim_{k \to \infty} w_k^{-1} = \lim_{k \to \infty} u_k^{-1} v_k^{-1}.
\]

exist.

It is trivial that \( \lim_{k \to \infty} w_k^{-1} \) exists if and only if \( \lim_{k \to \infty} w_k \neq 0 \). Hence \( w \in c \). This completes the proof.

**Theorem 5.** (i) \( c \subseteq c_v \) if and only if \( v \in c \),

(ii) \( c_v = c \) if and only if \( v \in c \) and \( \lim_{k \to \infty} v_k \neq 0 \).

**Proof.** Taking \( v = (1, 1, \ldots) \) and replacing \( u \) by \( v \) in Theorem 4(i), we obtain (i).

Theorem 4(ii) gives us (ii) for \( u = (1, 1, \ldots) \).

Remark. If \( v \in c \) and \( \lim_{k \to \infty} v_k = 0 \), that is \( v \in c_0 \), then \( c \subseteq (c_0)_v \).

If we consider Definition 1 and Theorem 5(ii), then we can express the following result:

**Corollary 2.** \( c \) is \( v \)-invariant if and only if \( v \in c_0 \).

3. Köthe-Teoplitz Dual Spaces

**Definition 2 ([2]).** Let \( X \) be a sequence space and define

(i) \( X^\alpha = \{ a = (a_k) : \sum_{k=1}^\infty |a_kx_k| < \infty \text{ for all } x \in X \} \)

(ii) \( X^\beta = \{ a = (a_k) : \sum_{k=1}^\infty a_kx_k \text{ converges for all } x \in X \} \)

(iii) \( X^\gamma = \{ a = (a_k) : \sup_{n} \sum_{k=1}^n |a_kx_k| < \infty \text{ for all } x \in X \} \).

Then \( X^\alpha, X^\beta \) and \( X^\gamma \) are called the \( \alpha \)-, \( \beta \)- and \( \gamma \)-dual spaces of \( X \), respectively. \( X^\alpha \) is also called Köthe-Teoplitz dual space and \( X^\beta \) is
also called generalized Köthe-Toeplitz dual space. It is easy to show that $\varphi X^\alpha \subset X^\beta \subset X^\gamma$. If $X \subset Y$ then $Y^n \subset X^n$ for $\eta = \alpha$, $\beta$ or $\gamma$.

**Lemma 1 ([2]).**

(i) $c^n_0 = c^n = m^n = \ell_1$,

(ii) $\ell^n_p = \ell_q$ \hspace{1cm} ($1 < p < \infty$, $p^{-1} + q^{-1} = 1$)

(iii) $\ell^n_p = m$ \hspace{1cm} ($0 < p < 1$)

where $\eta$ stand for $\alpha$, $\beta$ or $\gamma$.

The $\alpha$, $\beta$ and $\gamma$-duals of the sequence spaces $(c_0)_v$, $c_v$, $m_v$ and $(\ell_p)_v$ are known, ([1]), where $v = (v_k)$ satisfies (1) without any other restriction.

Now, if we consider Lemma 1, Corollary 1 and Corollary 2. Then we have the following results:

**Theorem 6.** a) Let (3) is satisfied. Then, for $\eta = \alpha$, $\beta$ and $\gamma$,

(i) $((c_0)_v)^n = (m_v)^n = \ell_1$,

(ii) $((\ell_p)_v)^n = \ell_q$ \hspace{1cm} ($1 < p < \infty$, $p^{-1} + q^{-1} = 1$),

(iii) $((\ell_p)_v)^n = m$ \hspace{1cm} ($0 < p < 1$).

b) If $v \in c_0$, then, for $\eta = \alpha$, $\beta$ and $\gamma$,

(iv) $(c_v)^n = \ell_1$.

**Theorem 7.** a) Let (3) is satisfied. Then, for $\eta = \alpha$, $\beta$ and $\gamma$,

(i) $((c_0)_v)^{n \eta} = (m_v)^{n \eta} = m$. 

886
(ii) \((\ell_p)_v)^{\infty} = \ell_p \quad (1 < p < \infty)\).

(iii) \((\ell_p)_v)^{\infty} = \ell_1 \quad (0 \leq p \leq 1)\).

b) If \(v \in c_0\) then, for \(n = a, b\) and \(c\),

(iv) \((c_v)^{\infty} = m\).

REFERENCES


