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ON $|\bar{N},p_n|_k$ SUMMABILITY FACTORS OF INFINITE SERIES

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SUMMARY

In this paper, a theorem on $|\bar{N},p_n|_k$ summability factors, which generalizes the theorems of Sinha [2] and Bor [4] has been proved.

SONSUZ SERİLERİN $|\bar{N},p_n|_k$ TOPLANABİLME ÇARPANLARI HAKKINDA

ÖZET

Bu çalışmada, Sinha [2] ve Bor [4] 'un teoremlerini genelleştiren $|\bar{N},p_n|_k$ toplanabilme çarpanları ile ilgili bir teorem ispat edilmiştir.

1. INTRODUCTION

Let $\sum a_n$ be a given infinite series with the sequence of partial sums (s_n) and let (p_n) be a sequence of positive real constants such that

$$P_n = P_0 + p_1 + \dots + p_n \to \infty \quad , \text{ as } \quad n \to \infty \quad , \; (P_{-1} = p_{-1} = 0),$$

A series, $\sum a_n$ is said to be summable $|\bar{N},p_n|_k$, $k \geqslant 1$, if (see [3])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} | t_n - t_{n-1} |^k < \infty \quad ,$$

where

$$t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \quad .$$

It should be noticed that $|\bar{N}, p_n|_k$ summability is identical with

$$|\bar{N},p_n| \quad ([1]) \quad \text{for} \quad k = 1 \; .$$

Recently Sinha [2] proved the following theorem.

Theorem A. Let (ε_n) be a sequence such that

$$\text{(i)} \sum_{n=2}^{m} \frac{p_n}{P_n} |\varepsilon_n| = O(1), \quad \text{(ii)} \frac{P_n}{p_n} |\Delta\varepsilon_n| = O(|\varepsilon_n|).$$

If

$$\sum_{v=1}^{n} |s_v| \, p_v = O(P_n \gamma_n)$$

where (γ_n) is a positive non-decreasing sequence such that

$$\frac{P_{n+1}}{p_{n+1}} \gamma_n \Delta \left(\frac{1}{\gamma_n} \right) = O(1), \text{ as } n \to \infty ,$$

then $\sum \frac{a_n \varepsilon_n}{\gamma_n}$ is summable $|\bar{N}, p_n|$.

Quite recently Bor [4] proved following more general theorem.

Theorem B. Let (ε_n) be a sequence such that

$$\text{(i)} \sum_{n=2}^{m} \frac{p_n}{P_n} |\varepsilon_n| = O(1), \quad \text{(ii)} \frac{P_n}{p_n} |\Delta\varepsilon_n| = O(|\varepsilon_n|).$$

If

$$\sum_{v=1}^{n} p_v |s_v|^k = O(P_n \gamma_n), \text{ as } n \to \infty,$$

where (γ_n) is a positive non-decerasing sequence such that

$$\frac{P_{n+1}}{p_{n+1}} \gamma_n \Delta \left(\frac{1}{\gamma_n} \right) = O(1), \text{ as } n \to \infty,$$

then $\sum \frac{a_n \varepsilon_n}{\gamma_n}$ is summable $|\bar{N}, p_n|_k$, $(k \geq 1)$,

2. MAIN RESULTS

The aim of this paper is to prove the following theorem which generalizes the above theorems.

Theorem. Let (ϵ_n) be a sequence such that

$$\text{(i)} \quad \sum_{n=2}^{m} \frac{p_n}{P_n} |\epsilon_n| = O(1), \quad \text{(ii)} \quad \frac{P_n}{p_n} |\Delta\epsilon_n| = O(|\epsilon_n|).$$

If

$$\sum_{v=1}^{n} p_v |s_v|^k = O(P_n|\gamma_n|), \; n \to \infty$$

where (γ_n) is a sequence such that $(|\gamma_n|)$ is non-decreasing sepuence and

$$\frac{P_{n+1}}{p_{n+1}} |\gamma_n| \left|\Delta\left(\frac{1}{\gamma_n}\right)\right| = O(1), \text{ as } n \to \infty ,$$

then

$$\sum \frac{a_n \epsilon_n}{\gamma_n} \text{ is summable } |\bar{N}, P_n|_k, \quad (k \geq 1).$$

Proof of the theorem. Let T_n denote $(\bar{N}, p_n)$ mean of the series $\sum \frac{a_n \epsilon_n}{\gamma_n}$. Then

$$T_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v \sum_{i=0}^{v} \frac{a_i \epsilon_i}{\gamma_i} = \frac{1}{P_n} \sum_{v=0}^{n} (P_n - P_{v-1}) \frac{a_v \epsilon_v}{\gamma_v} .$$

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v-1}^{n} P_{v-1} \cdot \frac{a_v \epsilon_v}{\gamma_v} .$$

Using Abel's transformation, we get

$$\begin{aligned} T_n - T_{n-1} = & - \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} \frac{p_v s_v \epsilon_v}{\gamma_v} + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} \frac{s_v}{\gamma_v} P_v |\Delta\epsilon_v| \\ & + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} s_v P_v \epsilon_{v+1} \Delta\left(\frac{1}{\gamma_v}\right) + \frac{s_n p_n \epsilon_n}{P_n \gamma_n} \\ & = T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}, \text{ say.} \end{aligned}$$

To Prove the theorem, by Minkowski's inequality, it is sufficient to show

$$\sum_{n=1} \left(\frac{P_n}{p_n}\right)^{k-1} |T_{n,i}|^k < \infty, \text{ for } i=1,2,3,4.$$

Now, applying Hölder's inequality, we have

$$\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{k-1} |T_{n,1}|^k \leqslant \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} \frac{p_v |s_v|^k |\varepsilon_v|}{|\gamma_v|} \times \left[\frac{1}{P_{n-1}} \sum_{v=1}^{n-1} \frac{p_v |\varepsilon_v|}{|\gamma_v|}\right]^{k-1}$$

$$= O(1) \sum_{v=1}^{m} \frac{p_v |s_v|^k |\varepsilon_v|}{P_v |\gamma_v|}$$

$$= O(1) \sum_{v=1}^{m-1} \Delta\left(\frac{|\varepsilon_v|}{P_v |\gamma_v|}\right) P_v |\gamma_v| + O(1) |\varepsilon_m|$$

$$= O(1) \sum_{v=1}^{m-1} |\Delta\varepsilon_v| + O(1) \sum_{v=1}^{m-1} |\varepsilon_{v+1}| |\gamma_v| \left|\Delta\left(\frac{1}{\gamma_v}\right)\right|$$

$$+ O(1) \sum_{v=1}^{m-1} \frac{|\varepsilon_{v+1}| p_{v+1} P_v}{P_{v+1} P_v |\gamma_{v+1}|} + O(1)$$

$$= O(1) \sum_{v=1}^{m-1} \frac{p_v}{P_v} |\varepsilon_v| + O(1) \sum_{v=1}^{m-1} \frac{p_{v+1}}{P_{v+1}} |\varepsilon_{v+1}| + O(1) \sum_{v=1}^{m-1} \frac{p_{v+1}}{P_{v+1}} |\varepsilon_{v+1}| + O(1)$$

$= O(1)$, as $m \to \infty$, by the hypotheses of the theorem.

Also we obtain

$$\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{k-1} |T_{n,2}|^k \leqslant \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}^k} \left[\sum_{v=1}^{n-1} \frac{|s_v| P_v |\Delta\varepsilon_v|}{|\gamma_v|}\right]^k$$

$$\leqslant K \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}^k} \left[\sum_{v=1}^{n-1} \frac{|s_v| p_v |\varepsilon_v|}{|\gamma_v|}\right]^k$$

$$= O(1) \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} \frac{|s_v|^k p_v}{|\gamma_v|} |\varepsilon_v|$$

$$= O(1) \sum_{v=1}^{m} \frac{|s_v|^k p_v \, |\varepsilon_v|}{|\gamma_v| \, P_v} = O(1) \text{, as } m \to \infty ,$$

where K is a positive constant.

Again,

$$\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{k-1} |T_{n,3}|^k = \sum_{m=2}^{m+1} \frac{p_n}{P_n P_{n-1}^k} \left| \sum_{v=1}^{n-1} s_v P_v \varepsilon_{v+1} \, \Delta \left(\frac{1}{\gamma_v}\right) \right|^k$$

$$= O(1) \sum_{v=1}^{m} \frac{|s_v|^k \; p_v \, |\varepsilon_v|}{P_v \, |\gamma_v|} = O(1) \text{, as } m \to \infty .$$

Finally, we have

$$\sum_{n=1}^{m+1} \left(\frac{P_n}{p_n}\right)^{k-1} |T_{n,4}|^k = \sum_{n=1}^{m+1} \frac{|s_n|^k p_n \, |\varepsilon_n|^k}{P_n |\gamma_n|^k}$$

$$= \sum_{n=1}^{m+1} \frac{|s_n|^k p_n |\varepsilon_n|}{P_n \, |\gamma_n|} \left(\frac{|\varepsilon_n|}{|\gamma_n|}\right)^{k-1}$$

$$= O(1) \sum_{n=1}^{m+1} \frac{|s_n|^k p_n \, |\varepsilon_n|}{P_n \, |\gamma_n|} = O(1), \ m \to \infty .$$

Therefore, we get

$$\sum_{n=1} \left(\frac{P_n}{p_n}\right)^{k-1} |T_{n,i}|^k < \infty \quad , \ i=1,2,3,4.$$

This completes the proof of the theorem.

Corollary. If we take $\gamma_n > o$; $\gamma_n > o$ and k=1 in our theorem then we get theorem B, and theorem A, respectively.

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