ABSOLUTE CESÁRO SUMMABILITY FACTORS OF INFINITE SERIES

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ABSTRACT

In this study a theorem on \( |C,1|_k \) summability factors, which generalizes a theorem of Mazhar [3], has been proved.

ÖZET

Bu çalışmada Mazhar [3] in bir teoremini genelleştiren \( |C,1|_k \) toplanabilme çarpanlarıyla ilgili bir teorem ispat edilmiştir.

1. INTRODUCTION

Let \( \Sigma a_n \) be a given infinite series with partial sums \( s_n \) and \( r_n = na_n \). By \( u^\alpha_n \) and \( t^\alpha_n \) we denote the n-th Cesàro means of order \( \alpha (\alpha > -1) \) of the sequences \( (s_n) \) and \( (r_n) \), respectively. The series \( \Sigma a_n \) is said to be summable \( |C, \alpha|_k \), \( k \geq 1 \), if (see [1])

\[
\sum_{n=1}^{\infty} \left| u^\alpha_n - u^\alpha_{n-1} \right|^k < \infty
\]

(1)

Since \( t^\alpha_n = n^\alpha (u^\alpha_n - u^\alpha_{n-1}) \) (see [2]), condition (1) can also be written as

\[
\sum_{n=1}^{\infty} \frac{1}{n} \left| t^\alpha_n \right|^k < \infty
\]

(2)


THEOREM A. If

\[
\sum_{v=1}^{n} \left| S_v \right| = O(n) \text{ as } n \to \infty
\]

(3)
and \((\lambda_n)\) is a bounded sequence such that

\[
\sum_{n=1}^{m} |\Delta \lambda_n| = O(1)
\]  

(4)

\[
\sum_{n=1}^{m} \frac{|\lambda_n|}{n} = O(1), \text{ and}
\]

(5)

\[
\sum_{n=1}^{m} n |\Delta^2 \lambda_n| = O(1) \text{ as } m \to \infty
\]  

(6)

then the series \(\sum a_n \lambda_n\) is summable \(|C,1|\).

3. Now we shall prove the following theorem.

THEOREM. Let \(k \geq 1\). If

\[
\sum_{v=1}^{n} |s_v|^k = O(n) \text{ as } n \to \infty
\]  

(7)

and \((\lambda_n)\) is a bounded sequence such that satisfies the conditions (4)-(6) of the Theorem A, then the series \(\sum a_n \lambda_n\) is summable \(|C,1|\)_k.

It should be noted that if we take \(k = 1\) in our theorem, then we get Theorem A.

4. PROOF OF THE THEOREM. Let \(T_n\) be the n-th Cesaro mean of order 1 of the sequence \((n a_n \lambda_n)\). That is

\[
T_n = \frac{1}{n+1} \sum_{v=1}^{n} v a_v \lambda_v
\]  

(8)

Now, applying Abel's transformation to the sum (8), we have that
\[ T_n = \frac{1}{n+1} \sum_{v=1}^{n-1} s_v \Delta \lambda_v - \frac{1}{n+1} \sum_{v=1}^{n-1} s_v \lambda_{v+1} + \frac{ns\lambda_n}{n+1} \]

\[ = T_{n,1} + T_{n,2} + T_{n,3}, \text{ say.} \]

To prove the theorem, by Minkowski's inequality, it is sufficient to show that

\[ \frac{1}{n} \sum_{n=1}^\infty |T_{n,r}|^k < \infty, \text{ for } r = 1, 2, 3, \text{ by (2)}. \]

Now, applying Hölder's inequality, we have that

\[ \frac{1}{n} \sum_{n=2}^{n+1} \frac{1}{n} |T_{n,1}|^k \leq \frac{1}{n} \sum_{n=2}^{n+1} \left\{ \frac{1}{n} \sum_{v=1}^{n-1} |s_v| |\Delta \lambda_v| \right\}^k \]

\[ \leq \frac{1}{n^2} \sum_{n=2}^{n+1} \frac{1}{n} \sum_{v=1}^{n-1} |s_v|^k |\Delta \lambda_v| \times \left\{ \frac{1}{n} \sum_{v=1}^{n-1} |\Delta \lambda_v| \right\}^{k-1} \]

Since

\[ \frac{1}{n} \sum_{v=1}^{n} |\Delta \lambda_v| \leq n \frac{1}{n} \sum_{v=1}^{n} |\Delta \lambda_v| \implies \frac{1}{n} \sum_{v=1}^{n} |\Delta \lambda_v| \leq \frac{1}{n} \sum_{v=1}^{n} |\Delta \lambda_v| = O(1) \]

as \( n \to \infty \), by (4), we have that

\[ \frac{1}{n} \sum_{v=1}^{n} |\Delta \lambda_v| = O(1). \]

Hence
\[ \frac{m-1}{\sum_{n=2}^{m} \frac{1}{n}} |T_{n,1}|^k = O(1) \frac{\sum_{v=1}^{m} |s_v|^k}{\sum_{v=1}^{m} |\Delta \lambda_v|^k} - \frac{1}{n^2} \]

\[ = O(1) \frac{\sum_{v=1}^{m} |s_v|^k}{\sum_{v=1}^{\lambda} |\Delta \lambda_v|^k} - O(1) \frac{\sum_{v=1}^{m-1} |\Delta \lambda_v|^k}{\sum_{v=1}^{\lambda} |s_v|^k} \]

\[ + O(1) |\Delta \lambda_m| \frac{\sum_{v=1}^{m} |s_v|^k}{\sum_{v=1}^{\lambda} |\Delta \lambda_v|^k} = O(1) \frac{\sum_{v=1}^{m-1} |\Delta \lambda_v|^k}{\sum_{v=1}^{\lambda} |s_v|^k} + O(1) m |\Delta \lambda_m| = O(1) \]

as \( m \to \infty \), by (6) and (7).

Again

\[ \frac{\sum_{n=2}^{m} \frac{1}{n}}{n} |T_{n,2}|^k \leq \frac{\sum_{n=2}^{m+1} \frac{1}{n}}{n} |s_v|^k \frac{|\lambda_{v+1}|^k}{|\lambda_v|^k} \left\{ \frac{\sum_{v=1}^{m-1} 1}{n} \right\}^{k-1} \]

\[ = O(1) \frac{\sum_{v=1}^{m} |\lambda_v|^k}{\sum_{v=1}^{\lambda} |s_v|^k} = O(1) \frac{\sum_{v=1}^{m-1} |\lambda_v|^k}{\sum_{v=1}^{\lambda} |s_v|^k} + O(1) \frac{\sum_{v=1}^{m-1} |\lambda_{v+1}|}{\sum_{v=1}^{\lambda} |s_v|^k} \]

\[ + O(1) |\lambda_m| = O(1) \text{ as } m \to \infty \], by the hypotheses.

Finally, we have

\[ \frac{\sum_{n=1}^{m} \frac{1}{n}}{n} |T_{n,3}|^k = O(1) \frac{\sum_{n=1}^{m} |\lambda_n|^k}{\sum_{n=1}^{\lambda} |s_n|^k} \cdot \frac{1}{n} \]

\[ = O(1) \frac{\sum_{n=1}^{m} |\lambda_n|}{n} |s_n|^k \]

Thus as in \( T_{n,2} \) we have that

\[ \frac{\sum_{n=1}^{m} \frac{1}{n}}{n} |T_{n,3}|^k = O(1) \frac{\sum_{n=1}^{m} |\lambda_n|^k}{\sum_{n=1}^{\lambda} |s_n|^k} = O(1) \text{ as } m \to \infty \].
Therefore, we get

\[
\sum_{n=1}^{\infty} \frac{1}{n} \left| T_{n,r} \right|^k < \infty, \text{ for } r = 1, 2, 3.
\]

This completes the proof of the theorem.

REFERENCES


3. S.M. Mazhar, \( |N.p| \) summability factors of infinite series, Kodai Math. Seminar Reports, 16 (1966), 96, 100.