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THEORETICAL EVALUATIONS OF APPARENT MASSES FOR CERTAIN CLASSES OF BODIES IN FRICTIONLESS FLUID

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ÖZET

Bu makalede, hız potansiyeli fonksiyonu kullanılarak, basit geometriye haiz bazı cisimlerin, elipsoid, küre ve eliptik silindir gibi, ilave akışkan kütlelerinin teorik hesabı verilmiştir. Bu çalışmadan, ilave akışkan kütlelerinin sadece cismin geometrisine ve hareket doğrultusuna bağlı olduğu elde edilmiştir. Ayrıca, cismin keyfi doğrultuda iki boyutlu daimi olmayan öteleme hareketinde, cismin ana eksenleri doğrultusundaki ilave akışkan kütle bileşenlerinin değişmediği görülmüştür.

THEORETICAL EVALUATIONS OF APPARENT MASSES FOR CERTAIN CLASSES OF BODIES IN FRICTIONLESS FLUID

SUMMARY

In this study, using the velocity potential function the theoretical evaluations of apparent masses (or virtual masses) for certain classes of bodies, such as elipsoid, sphere and elliptic cylinder, are given. It was found that the apparent mass of the body depends on the geometry of the body and its direction of motion. It was also found that when a body moves in any arbitrary two dimensional unsteady pure translational manner, the associated apparent mass components in the directions of the main axes do not change.

1. INTRODUCTION

The existing theory for the determination of apparent mass is based on the fact that, when a solid body is moving in the incompressible potential flow, the entire velocity field depends upon the instantaneous velocity of the submerged body and is quite independent of the past history of the motion. Consequently, any change of the motion of the body would be propogated instantaneously to all the particles of the fluid. Thus, the total kinetic energy imparted to the fluid during translation must vary directly with the square of the linear velocity, and during rotational motion, with the square of the angular velocity.

A number of hydrodynamic theorems have been published by Munk [1] , Lagally [2]and Taylor [3] concerning the apparent masses of bodies and the forces and moments which act upon them. These theorems enable the apparent masses and hence the forces and moments to be determined when the singularity distribution of sources, sinks and doublets within the body which may be considered to generate the potential flow about it, are known.

For unsteady flow conditions, an impulse associated with any change in the motion of the body is directly proportional to the change in the value of the velocity potential function, Φ , which specifies the flow. Consequently, the determination of the apparent masses of bodies moving through an ideal fluid must depend on the calculation of the velocity potential or of the corresponding stream function. The apparent masses are determined from this velocity potential functioning calculating the kinetic energy T, by means of the integral,

$$2T = -\rho \int_S \Phi \frac{\partial \Phi}{\partial n} ds \qquad (1)$$

where S denotes the surface of the body, Φ the velocity potential function, ρ the tensity of the fluid and n denotes an elemental length drawn in the fluid normal to the surface s.

The general motion of a solid body of arbitrary shape has six degrees of freedom represented by three components of linear velocity (U, V, W) and three components of angular velocity (P, Q, R). For each velocity component there is a corresponding fluid velocity potential. Consequently, there are a number of different apparent mass components. If the velocity potential is defined in terms of velocity components V_i of the body, as $\Phi = V_i \Phi_i$, where i ranges from 1 to 6, the apparent mass components are given from the kinetic energy equation (1) as

$$2T = - \rho V_i V_j \int_S \Phi_i \frac{\partial \Phi_i}{\partial n} ds \qquad (2)$$

thus,

$$\alpha_{ij} = \rho \int\int \Phi_j \frac{\partial \Phi_i}{\partial n} ds \qquad (3)$$

in which i and j range from 1 to 6. α_{ij} represent a symmetric apparent mass tensor, i.e.,

$$\alpha_{ij} = \alpha_{ji} \qquad (4)$$

In the quadratic expressions for the energy, there could be six squares and fifteen products of velocity components and therefore 21 distinct apparent mass components

$$\alpha_{ij} = \begin{vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} & \alpha_{15} & \alpha_{16} \\ & \alpha_{22} & \alpha_{23} & \alpha_{24} & \alpha_{25} & \alpha_{26} \\ & & \alpha_{33} & \alpha_{34} & \alpha_{35} & \alpha_{36} \\ & & & \alpha_{44} & \alpha_{45} & \alpha_{46} \\ & & & & \alpha_{55} & \alpha_{56} \\ & & & & & \alpha_{66} \end{vmatrix} \qquad (5)$$

However, as shown certain components disappear when the body possesses axes or planes of symmetry.

For an irrotational ideal fluid, the equation (3) shows that the apparent mass components depend upon the shape and the orientation of the body as well as on the mode of its motion. However, they do not depend upon its linear or angular velocities and its linear or angular accelerations.

The apparent mass components can also be expressed in terms of a different form of (3). By applying Green's Theorem, this equation can be written as,

$$\alpha_{ij} = -\rho \int\int_S \Phi_i \frac{\partial \Phi_j}{\partial n} ds + \rho \int\int_\infty \left(\Phi_i \frac{\partial \Phi_j}{\partial n} - \Phi_j \frac{\partial \Phi_i}{\partial n}\right) ds \qquad (6)$$

where the last integral is taken over a surface which approaches infinity. At infinity, Φ is equivalent to the strength of the dipole, hence equation (2.6) becomes [4]

$$\alpha = 4 \pi \rho \sigma_A - \rho \quad \text{Volume of the object} \qquad (7)$$

Where σ_A is the dipole strength at infinity of the potential function. This equation is valid for any symmetrical object moving in an infinite fluid [4] .It remains to evaluate the dipole strength, σ_A , for a particular flow around a particular body.

If velocity potential singularities are known, i.e., the flow problem about the body has been solved, the apparent mass components can be calculated directly from equation (3). Otheswise, it is more convenient to use equation (6) [5] .

2. APPARENT MASSES OF AN ELLIPSOID

Lamb [6] shows how the principle of hydrodynamic theory can be used to determine the values of the apparent mass components for all classes of ellipsoids, moving in an ideal fluid with axial, transverse and rotational motions.

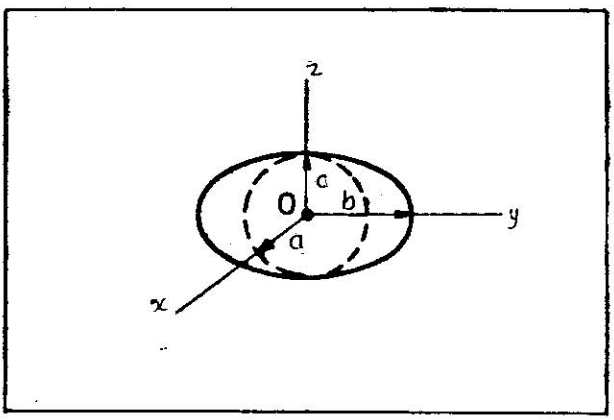

Fig.1 : Dimensions of an ellipsoid.

For an ellipsoid, for which

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \qquad (8)$$

where if $a = b > c$, an oblate ellipsoid is obtained which can be used as a model of parachute canopies. However, if $a = b < c$, a prolate spheroid is obtained, which could be used as a model for an airship hull.

By employing equation (3) and considering the ellipsoid motion apparent mass components can be determined directly from the defined velocity potential function on the ellipsoid surface.

For rectlinear motion parallel to the x-axis with a velocity U, as seen in Fig. 1, the velocity potential may be written [6] as

$$\Phi = C\,x \int_{\lambda}^{\infty} \frac{d\lambda}{(a + \lambda)\,\Delta} \qquad (9)$$

where

$$\Delta = \{ (a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda) \}^{1/2} \qquad (9a)$$

$$C = \frac{abc}{2 - \alpha_0}\,U \qquad (9b)$$

and

$$\alpha_0 = abc \int_0^{\infty} \frac{d\lambda}{(a^2 + \lambda)\,\Delta} \qquad (9c)$$

The surface condition is

$$\frac{\partial \Phi}{\partial \lambda} = - U \frac{\partial x}{\partial \lambda} \qquad \text{for} \quad \lambda = 0$$

For the motion in the direction of the other axes, (i.e. the y and z axes) the corresponding velocity potential and other variables (i.e. C and α_0)

can be written by substituting the appropriate variables into equations (9), (9a) and (9c). Hence, in the y and the z axes directions, the above equations become

$$\Phi = Cy \int_{\lambda}^{\infty} \frac{d\lambda}{(b^2 + \lambda)\,\Delta} \tag{10}$$

$$C = \frac{a\,b\,c\,V}{2 - \beta_0} \tag{10a}$$

$$\beta_0 = abc \int_0^{\infty} \frac{d\lambda}{(b^2 - \lambda)\,\Delta} \tag{10b}$$

and

$$\Phi = Cz \int_{\lambda}^{\infty} \frac{d\lambda}{(c^2 + \lambda)\,\Delta} \tag{11}$$

$$C = \frac{abc}{2 - \gamma_0}\,W \tag{11a}$$

$$\gamma_0 = abc \int_0^{\infty} \frac{d\lambda}{(c^2 + \lambda)\,\Delta} \tag{11b}$$

respectively.

When the ellipsoid rotates about the x axis, with the angular velocity ω, the velocity potential may be given as,

$$\Phi = Cyz \int_{\lambda}^{\infty} \frac{d\lambda}{(b^2 + \lambda)(c^2 + \lambda)\,\Delta} \tag{12}$$

where

$$C = \frac{c^2 - b^2}{2\,(c^2 - b^2) + (c^2 + b^2)(\beta_0 - \gamma_0)}\ abc\ \omega \tag{12a}$$

and the surface condition is

$$\frac{\partial\Phi}{\partial\lambda} = \omega\left(z\frac{\partial y}{\partial\lambda} - y\frac{\partial z}{\partial\lambda}\right)$$

It may be further seen that, for the motion about any other axis, the corresponding velocity potential can be written by substituting the appropriate variables into equations (12).

Thus, the kinetic energy T of the ellipsoid moving in the x-axis direction is evaluated from equation (1), as

$$2T = -\rho \iint_S \Phi \frac{\partial\Phi}{\partial n}\, ds = \frac{\alpha_0}{2 - \alpha_0}\, \rho\, U^2 \iint_S x\, \ell\, ds \tag{13}$$

where ℓ is the direction cosine in the x direction. Since the latter integral gives the volume of the ellipsoid, the energy equation becomes

$$2T = \frac{\alpha_0}{2 - \alpha_0}\, \frac{4}{3}\, \pi\, abc\, \rho\, U^2 \tag{14}$$

Consequently, the apparent mass in the x-axis direction may be written from equation (14),

$$\alpha_{xx} = \frac{\alpha_0}{2 - \alpha_0}\, \frac{4}{3}\, \pi\, abc\, \rho \tag{15}$$

The corresponding apparent mass coefficient which is defined as the ratio of the apparent mass to the mass of the fluid displaced by the ellipsoid is obtained from equation (15),

$$k_{xx} = \frac{\alpha_0}{2 - \alpha_0} \tag{16}$$

By the same approach, the apparent mass coefficients along the other two axes are determined as

$$k_{yy} = \frac{\beta_0}{2 - \beta_0} \tag{17}$$

$$k_{zz} = \frac{\gamma_0}{2 - \gamma_0} \tag{18}$$

Using the velocity potential function for the rotational motion of the ellipsoid, the apparent moment of inertia coefficient which is defined as the ratio of the apparent moment of inertia to the moment of inertia of the fluid displaced by the ellipsoid about the x-axis is found from equation (1),

$$k'_{xx} = \frac{e^4 (\beta_0 - \gamma_0)}{(2 - e^2)\,[(2e^2 - (2 - e^2)(\beta_0 - \gamma_0)]} \tag{19}$$

where e denotes the aspect ratio, as

$$e = 1 - \frac{c^2}{a^2} \quad \text{for an oblate ellipsoid}$$

$$e = 1 - \frac{a^2}{c^2} \quad \text{for a prolate ellipsoid.}$$

Then, from equation (9b) and (10b), the following equations result ; for oblate spheroid

$$\alpha_0 = \beta_0 = \frac{2}{e^2} \left[1 - \sqrt{1 - e^2}\, \frac{\sin^{-1} e}{e} \right] \tag{20a}$$

$$\gamma_o = \frac{\sqrt{1 - e^2}}{e^3} \sin^{-1} e - \frac{1 - e^2}{e^2} \qquad (20b)$$

and, for prolate spheroid

$$\alpha_o = \beta_o = 2 \left(\frac{1 - e^2}{e^3}\right) \left(1/2 \log \frac{1 + e}{1 - e} - e\right) \qquad (21a)$$

$$\gamma_o = \frac{1}{e^2} - \frac{1 - e^2}{2e^3} \log \frac{1 + e}{1 - e} \qquad (21b)$$

The result obtained for $k_{xx} = k_{yy}$, k_{zz} and k'_{xx} are given as a function of the aspect ratio in Fig. 2. For a = b = c, the shape of ellipsoid becomes a sphere and $k_{xx} = k_{yy} = k_{zz} = 0.5$, $k'_{xx} = 0.0$. When a = b = 2c, it becomes a hemisphere and the apparent mass components are obtained as,

$$k_{xx} = k_{yy} = 0.3 \quad , \quad k_{zz} = 1.1 \quad \text{and} \quad k'_{xx} = 0.4$$

Determining the strength of the dipole for a particular flow about spherical sheels, using equation (7), İbrahim (7) determined the apparent mass coefficient in the direction of the symmetry axis of the spherical cups. He assumed that the included mass which is the mass of the fluid inside the cup is a part of the apparent mass and consequently his results obtained for a sphere and for a hemisphere are higher than those previously calculated. If it is assumed that the included mass is carried by the spherical shell as if it were part of the solid, İbrahim's results become identical with those developed above, and as seen in Fig. 3 .

3. APPARENT MASSES IN ARBITRARY TWO-DIMENSIONAL MOTION

Elliptic Cylinder in Arbitrary Motion

Introducing the elliptic co-ordinates, ξ , η , the potential function for an elliptic cylinder moving parallel to its x-axis with a unit velocity, U, as shown in Fig. 4, is given [8] as,

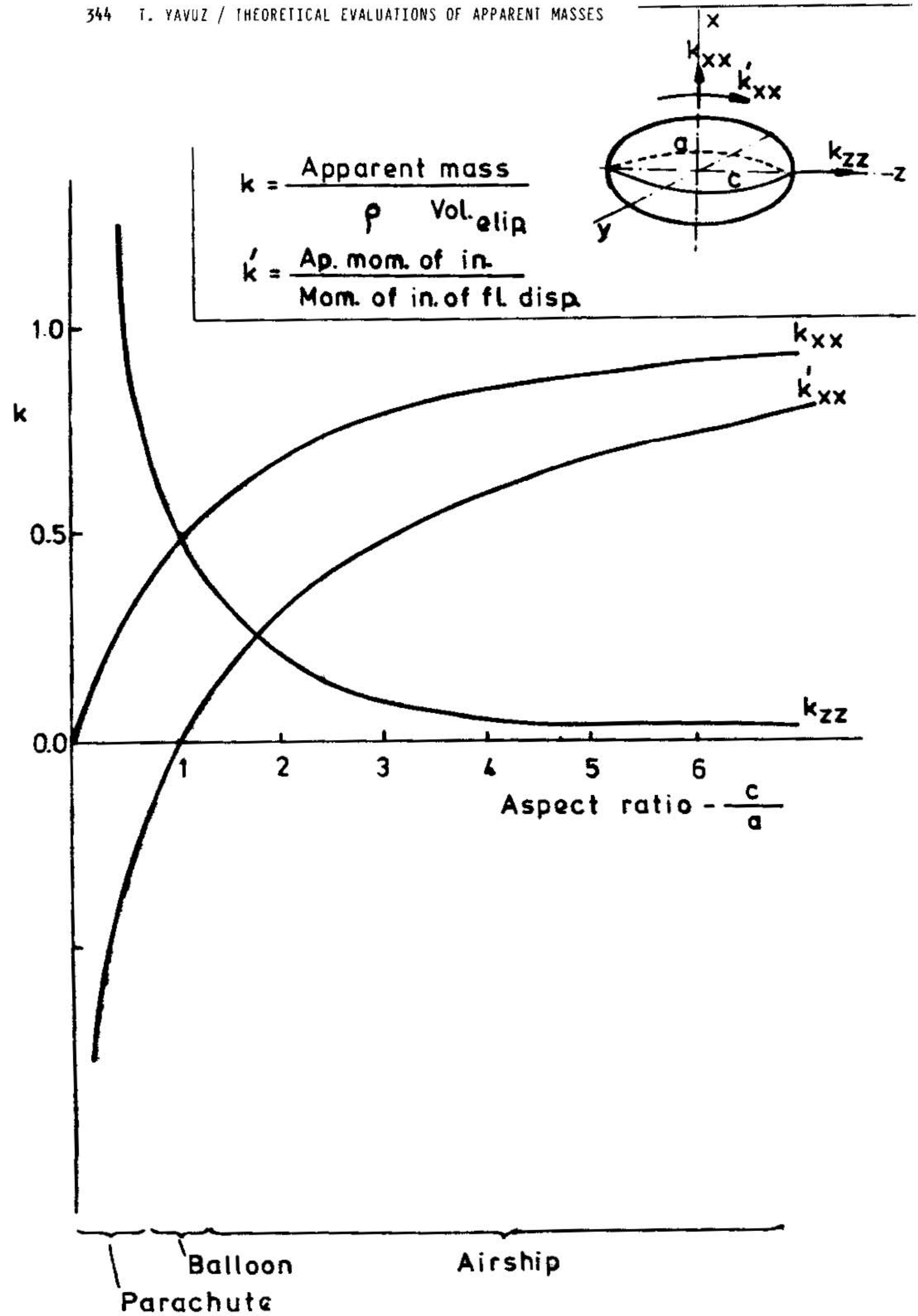

Fig.2 : Theoretical apparent mass coefficients for the parachute's aerodynamically related shapes.

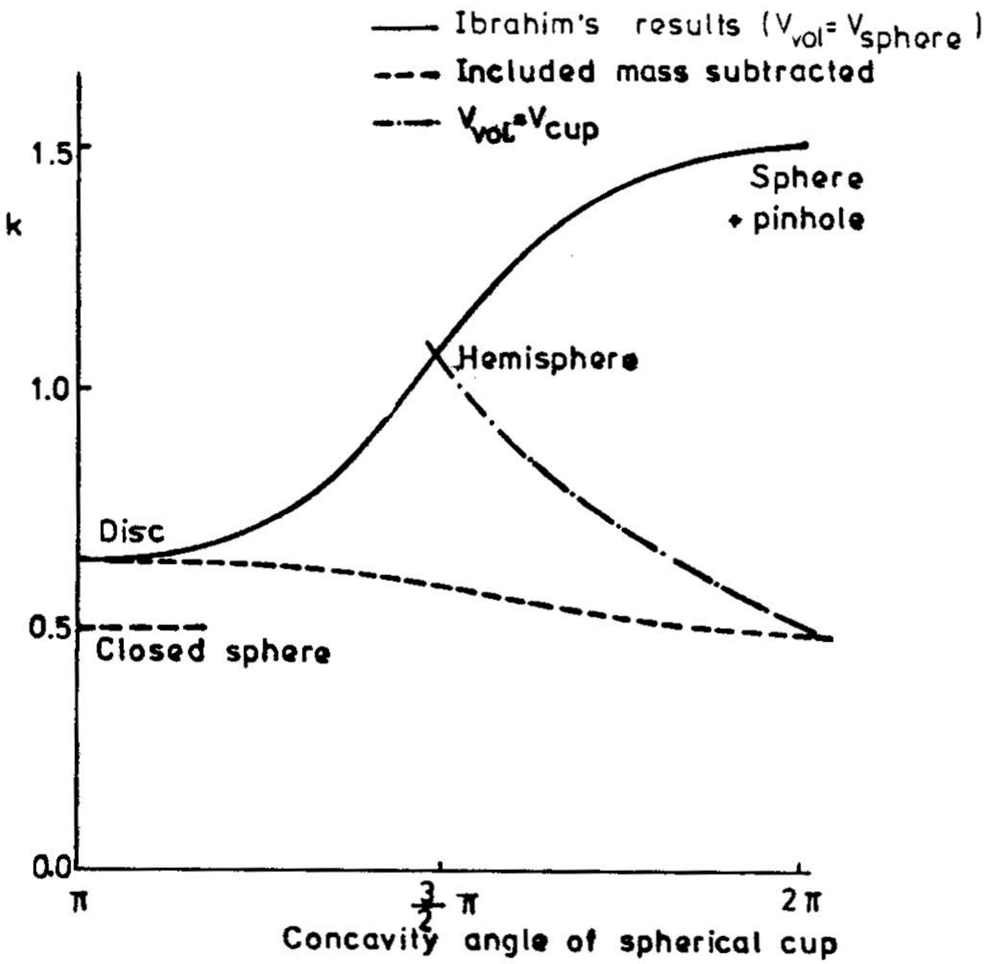

Fig. 3 : Tangential apparent mass coefficient, k_{33}, for spherical cup.

$$\Phi = - U.\ b \left(\frac{a + b}{a - b}\right)^{1/2} e^{-\xi} \sin \eta \qquad (22)$$

where a and b are the major and the minor axis if the motion of the elliptic cylinder is parallel to the y-axis, the equation will be,

$$\Phi = - V\ a \left(\frac{a + b}{a - b}\right) e^{-\xi} \sin \eta \qquad (23)$$

The resultant pressure, i.e. the force exerted by the surrounding fluid

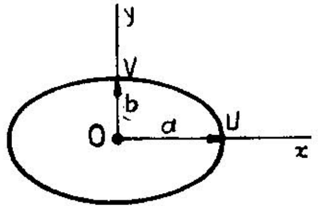

Fig. 4 : An elliptic cylinder in unsteady motion

on the unit length of the cylinder parallel to the x and the y axes respectively are determined by the equations,

$$F_x = - b \int_0^{2\pi} p \cos \eta \, d\eta \qquad (24)$$

$$F_y = - a \int_0^{2\pi} p \sin \eta \, d\eta \qquad (25)$$

where p is the pressure which is determined by the Bernoulli equation for unsteady flow conditions,

$$- \frac{p}{\rho} = \frac{\partial \Phi}{\partial t} + \frac{1}{2} q^2 \qquad (26)$$

where q is the velocity. Since the fluid is ideal and considering the fact that an ideal fluid exerts zero net force on a body of any shape wholly immersed in it (the d' Alembert paradox), the second term in the right hand side of the equation (26) does not make any contribution to force and only the first term will be considered. Hence, from equation (24) and (26), the apparent masses for an elliptic cylinder in the x-axis and the y-axis directions are found as

$$\alpha_{xx} = \pi \rho \, b^2$$

$$\alpha_{yy} = \pi \rho \, a^2$$

respectively.

Supposing the elliptic cylinder is moving in an arbitrary direction and U,V are the velocity components in the x-axis and the y-axis directions respectively. The potential function can then be expressed [9] as,

$$\Phi = -\left(\frac{a+b}{a-b}\right)^{1/2} e^{-\xi}\,(Ub\cos\eta + V a\sin\eta) \qquad (27)$$

Considering the resultant forces parallel to the x and the y axes separately, therefore,

$$\frac{\partial\Phi}{\partial t} = -\left(\frac{a+b}{a-b}\right)^{1/2} e^{-\xi}\left[\frac{\partial U}{\partial t} b\cos\eta + \frac{\partial V}{\partial t} a\sin\eta\right] \qquad (28)$$

can be obtained.

Substituting this into (26) and using equations (24) and (25) forces F_x and F_y are determined as,

$$F_x = -\pi\rho b^2 \frac{dU}{dt}$$

$$F_y = -\pi\rho a^2 \frac{dv}{dt}$$

and finally,

$$\alpha_{xx} = \pi\rho b^2$$

$$\alpha_{yy} = \pi\rho a^2$$

Their values remaining the same as if they were determined separately.

Ellipsoid in an Arbitrary Motion

If the ellipsoid moves in any arbitrary direction and U,V are the velocity components in the x and y directions respectively, as seen in Fig. 5, the velocity potential function is expressed as

$$\Phi = \frac{\alpha_0}{2 - \alpha_0} Ux + \frac{\beta_0}{2 - \beta_0} Vy \qquad (29)$$

where α_0 and β_0 are defined in section 2 .

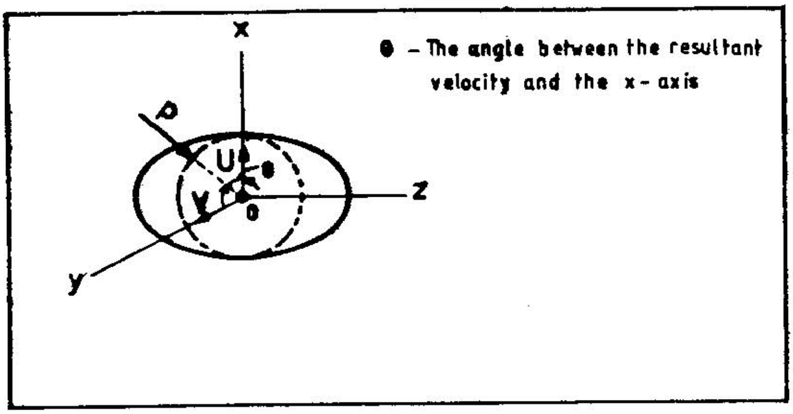

Fig. 5 : An ellipsoid in an unsteady two-dimensional motion.

The total forces acting on the ellipsoid in the x and y axes directions are evaluated by considering the pressure distribution over the surface. So,

$$F_x = - \int_S p \cos \theta \; d A \qquad (30)$$

$$F_y = - \int_S p \sin \theta \; d A \qquad (31)$$

where these associated integrals are taken over the surface and dA is small surface element. Since

$$-\frac{p}{\rho} = \frac{\partial\Phi}{\partial t} + 1/2\, q^2$$

then

$$\frac{\partial\Phi}{\partial t} = \frac{\alpha_o}{2-\alpha_o}\frac{dU}{dt}x + \frac{\beta_o}{2-\beta_o}\frac{dV}{dt}y \qquad (32)$$

From equation (2.30),

$$F_x = \rho \int_S \left(\frac{\alpha_o}{2-\alpha_o}\frac{dU}{dt}x + \frac{\beta_o}{2-\beta_o}\frac{dV}{dt}y\right)\cos\theta\, dA$$

(33)

$$F_x = \rho\frac{\alpha_o}{2-\alpha_o}\frac{dU}{dt}\int_S x\cos\theta\, dA + \rho\frac{\beta_o}{2-\beta_o}\frac{dV}{dt}\int_S y\cos\theta\, dA$$

The first integral gives volume of the ellipsoid. The becond is zero, thus

$$F_x = \frac{\alpha_o}{2-\alpha_o}\frac{4}{3}\pi abc\,\rho\frac{dU}{dt} \qquad (34)$$

and likewise,

$$F_y = \frac{\beta_o}{2-\beta_o}\pi\ abc\,\rho\frac{dv}{dt} \qquad (35)$$

where

$$k_{xx} = \frac{\alpha_o}{2-\alpha_o}$$

$$k_{yy} = \frac{\beta_o}{2-\beta_o}$$

which are the same expressions as those obtained when the ellipsoid moves parallel to its main axes separately.

4. CONCLUSIONS

It is concluded that, when the body moves in any arbitrary two-dimensional unsteady pure translational manner, the associated apparent mass components in the direction of the main axes do not change. As further evidence, İbrahim [7] determined the complex potential function for a circular arc moving in an arbitrary traslational direction making an angle γ with one of its main axis. He, thus, evaluated the apparent mass components ; namely, the apparent masses in the direction of the two main axes and also the coupling apparent mass component. He found that, the sum of the apparent masses in the directions of the two main axes is independent of the angle, γ , showing that the sum of the apparent masses in any two perpendicular directions of translational motion is invariant.

REFERENCES

1. Max, M. Munk, "Notes on Aerodynamic Forces", Tech. Notes 104 and 105, NACA, 1922.
2. Lagally, M. "Berechnung der Krafte und Momente, die Stromende Flussigkeiten auf ihre Begrenzug ausuben, 1922.
3. Taylor, G.I. "The Energy of a Body Moving in Infinite Fluid, with Application to Airship." Proc. Trans. Naval Arch. Mar. Eng., vol. 37, 1929.
4. Shifman, M. and Spenser, D.C., "The Flow in Ideal Incompressible Fluid about Lens." App. Mech., vol. 3, 1947.
5. Chia-Shun Yih, "Fluid Mechanics." Mc Graw - Hill, Inc, 1969.
6. Lamb, H., "Hydrodynamics", 6th edition, 1932, New York, Doherr Publ. 1945.
7. İbrahim, S.K., "Apparent Added Masses and Moment of Inertia of Cup-Shaped Bodies in Unsteady Incompressible Flow, Ph.D. Thesis, University of Minnesota, USA, 1965.
8. Milne Thomson, L.M., "Theoretical Hydrodynamics", Macmillen and Co. Ltd., London, 1938.
9. L. Milne - Thomson "Hydrodynamical Images", Proc. Camb. Phil. Soc., vol. 36, 1940.