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CONVERGENCE OF CERTAIN COSINE SUMS IN THE METRIC SPACE L^1

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SUMMARY

n this study a new and short proof of the theorem of Ahmad and Zaini [1] ıas been obtained, by considering the condition $S^2(r)$ instead of $S(r)$.

L^1 METRİK UZAYINDA BAZI KOSUNÜS TOPLAMLARININ YAKINSAKLIĞI

ZET

u çalışmada $S(r)$ şartı yerine $S^2(r)$ şartı alınarak Ahmad ve Zaini [1] nin eoreminin yeni ve kısa bir ispatı elde edilmiştir.

- INTRODUCTION

sequence (a_n) is said to be convex if $\Delta^2 a_n \geqslant 0$, and it is said to be uasi-convex if

$$\sum_{n=1}^{\infty} (n+1)|\Delta^2 a_n| < \infty \tag{1.1}$$

sequence (a_n) of positive numbers is said to be quasi-monotone if $\Delta a_n \geqslant -\gamma \frac{a_n}{n}$ or some positive γ . It is obvious that every null monotonic decreasing sequence is quasi-monotone. A sequence (a_n) is said to be r-quasi-monotone if $a_n \to 0$, $a_n > 0$ ultimately and $\Delta a_n \geqslant -r_n$, where (r_n) is a sequence of positive ımbers (see [2]). Clearly a null quasi-monotone sequence is r-quasi - onotone with $r_n = \gamma \frac{a_n}{n}$.

ıe concept of quasi-convex sequence was generalized by Sidon [5] and lyakovskii [7]. A sequence (a_n) is said to be belong to class S, or $a_n \in S$, $a_n \to 0$, as $n \to \infty$, and there exits a sequence of numbers (A_k) such that

(a) $A_k \downarrow 0$

(b) $\sum_{k=1}^{\infty} A_k < \infty$, and (1.2)

(c) $|\Delta a_k| \leq A_k$, for all k.

This class S of sequences has been further generalized to the class S' and $S(r)$ by Singh and Sharma [6] and, Zaini and Hasan [8], respectively:

$(a_n) \in S'$ if (1.2) holds with the condition (a) replaced by:

(a') (A_k) is quasi - monotone.

$(a_n) \in S(r)$ if (1.2) holds with the condition (a) replaced by:

(a") (A_k) is r-quasi - monotone and $\sum k\, r_k < \infty$.

For the class S the following equivalent definition was given by Garret, Rees and Stanojević [4] .

DEFINITION A. A null - sequence (a_k) belongs to class S^2 if there exits a null - sequence (A_k) of non - negative numbers such that

$$\sum_{k=1}^{\infty} k|\Delta A_k| < \infty \text{, and}$$

$$|\Delta a_k| \leq A_k \text{, for all } k \tag{1.3}$$

Now, we shall give the following definition.

DEFINITION : A sequence (a_k) belongs to class $S^2(r)$, if $a_k \to 0$, as $k \to \infty$, and there exits a sequence of numbers (A_k) such that it is r-quasi-monoton and $\sum k\, r_k < \infty$, and (1.3) holds.

2. Let

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

and

$$f_n(x) = \frac{1}{2} \sum_{k=0}^{n} \Delta a_k + \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta a_j \cos kx.$$

The following theorems are known:

THEOREM A ([6]). If $a_n \in S'$, then $f_n(x)$ converges to $f(x)$ in L-metric

THEOREM B ([1]). If $a_n \in S(r)$, then $f_n(x)$ converges to $f(x)$ in L-metric.

In this paper we shall prove the Theorem B by replacing the class of $S(r)$ by the class $S^2(r)$.

3. Now, we shall prove the following:

THEOREM. If $a_n \in S^2(r)$, then $f_n(x)$ converges to $f(x)$ in L-metric.

4. We need the following lemmas for the proof of our theorem.

LEMMA 1 ([3]). If the sequence of numbers (t_i) satisfies the condition $|t_i| \leq 1$, then

$$\int_0^{\pi} \left| \sum_{i=0}^{n} t_i D_i(x) \right| dx < C(n+1) \tag{4.1}$$

where $D_i(x) = \frac{1}{2} + \cos x + \cos 2x + \cos 3x + \dots + \cos ix$ and C is a positive absolute constant.

LEMMA 2. If (a_n) is r-quasi-monotone with $\sum n r_n < \infty$, then the convergence of $\sum a_n$ implies that $n a_n \to 0$ as $n \to \infty$. This lemma is a special case of Theorem 1 of Boas [2].

5. PROOF OF THE THEOREM. By summation by parts, we have

$$f(x) = \lim_{n\to\infty} \left\{ \frac{1}{2} a_0 + \sum_{k=1}^{n} a_k \cos kx \right\}$$

$$= \lim_{n\to\infty} \left\{ \frac{1}{2} a_0 + \sum_{k=1}^{n-1} D_k(x) \Delta a_k + a_n D_n(x) - \frac{1}{2} a_0 \right\}$$

$$= \lim_{n\to\infty} \left\{ \sum_{k=1}^{n-1} D_k(x) \Delta a_k + a_n D_n(x) \right\} = \sum_{k=0}^{\infty} D_k(x) \Delta a_k)$$

by the fact that $\lim_{n\to\infty} a_n D_n(x) = 0$ if $x \neq 0$, where

$$D_n(x) = \frac{1}{2} + \cos x + \cos 2x + \dots + \cos nx.$$

Similarly by summation by parts, we have

$$f_n(x) = \frac{1}{2} \sum_{k=0}^{n} \Delta a_k + \sum_{k=1}^{n} \left(\sum_{j=k}^{n} \Delta a_j \right) \cos kx = \sum_{k=0}^{n} D_k(x) \Delta a_k.$$

Again using summation by parts, we have

$$\sum_{k=n+1}^{N} D_k(x) \Delta a_k = \sum_{k=n+1}^{N} A_k D_k(x) \frac{\Delta a_k}{A_k} = \sum_{k=n+1}^{N-1} T_k(x) \Delta a_k + T_N(x) A_N - T_n(x) A_{n+1}$$

where $$T_n(x) = \sum_{k=0}^{n} D_k(x) \frac{\Delta a_k}{A_k}$$

Taking $t_k = \frac{\Delta a_k}{A_k}$, we observe by virture of the Lemma 1 that

$$\int_0^{\pi} \left| \sum_{k=n+1}^{N} D_k(x) \Delta a_k \right| dx \leq \int_0^{\pi} \left| \sum_{k=n+1}^{N-1} T_k(x) \Delta a_k(x) \right| dx$$

$$+ A_N \int_0^{\pi} | T_N(x) | dx + A_{n+1} \int_0^{\pi} | T_n(x) | dx$$

$$\leq \sum_{k=n+1}^{N-1} |\Delta A_k| \int_0^\pi |T_k(x)|\, dx + A_N \int_0^\pi |T_N(x)|\, dx + A_{n+1} \int_0^\pi |T_n(X)|\, dx$$

$$\leq C \sum_{k=n+1}^{N-1} (k+1)\, |\Delta A_k| + C(N+1)\, A_N + C\, (n+1)\, A_{n+1}$$

Making $N \to \infty$, we have

$$\int_0^\pi \left| \sum_{k=n+1}^{\infty} D_k(x)\, \Delta a_k \right| dx \leq C \sum_{k=n+1}^{\infty} (k+1) \left| \Delta A_k \right| + C\, (n+1)\, A_{n+1})$$

Since $(N+1)A_N = O(1)$ as $n \to \infty$, by Lemma 2.

Hence,

$$\int_0^\pi |f(x) - f_n(x)|\, dx = \int_0^\pi \left| \sum_{k=n+1}^{\infty} D_k(x)\, \Delta a_k \right| dx \leq C \sum_{k=n+1}^{\infty} (k+1) \left| \Delta A_k \right|$$

$$+ C(n+1)\, A_{n+1}$$

and therefore, by virtue of the hypothesis and Lemma 2, we obtain.

$$\lim_{n\to\infty} \int_0^\pi |f(x) - f_n(x)|\, dx = o(1).$$

This completes the proof of the theorem.

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