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ABOUT APPLICATION TO THE CRITICAL POINTS OF A POLYNOMIAL WITH ROUCHE THEOREM, RELATED TO THE CONJECTURE OF SENDOV

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ABSTRACT

Let $P = c(z-a)^k \prod_{v=1}^{4} (z-z_v)$ be a polynomial of degree n having its zeros in the closed

$$\left\{ z \in C : |z-a| \le \frac{2}{3}\left(|a| + \sqrt{2-|a|^2}\right)\right\}$$

If k=2 and in $\left\{ z \in C : |z-a| \le \dfrac{\left(\sqrt{(k+1)^2 - |a|^2 (2k+1)} + |a|^k\right)}{k+1} \right\}$

If k 3 where $a \in \overline{D}(0,1)$ let that a is zero of P multiplycity k. We seek out to determine in that discs the critical points of P with respect to Rouche theorem with help Conjecture of Sendov.

SENDOV'UN KONJEKTÖRÜ İLE İLİŞKİLİ OLARAK, ROUCHE TEOREMİ YARDIMI İLE BİR POLİNOMUN KRİTİK NOKTALARINA UYGULAMA HAKKINDA

ÖZET

$P = c(z-a)^{k}\prod_{v=1}^{4}(z-z_v)$ n inci dereceden

$$\left\{ z \in C : |z-a| \le \frac{2}{3}\left(|a| + \sqrt{2-|a|^{2}}\right)\right\}$$

kapalı diskinde k=2 ve k 3 ise

$\left\{ z \in C : |z-a| \le \left(\sqrt{(k+1)^{2}-|a|^{2}\,(2k+1)}+|a|\right)/(k+1)\right\}$ kapalı diskinde bir polinom (burada $a \in \overline{D}(0,1)$ ve a P nin k katlı bir sıfırı olsun. Sendov'un konjektörü yardımıyla P nin kritik noktalarını Rouche Teoremine göre araştırdık.

Auxilary Theorem 1 (Rouche Theorem) : If D is the domain inside the trajectory of a Jordan contour in the Complex Plane, if f and g are functions that are analytic in some open set which contains D, and if the inequality

$$| f(z)-g(z)| < |f(z)| + |g(z)|$$

holds at every point z of ∂D, then f and g have the same number of zeros in D, provided that zero-counts are made with due regard for multiplicity.

Conjecture of Sendov

Theorem A:

Let |a|=1 If

$$P(z) = c(z-a)^{k}\prod_{j=1}^{n-k}(z-z_j)$$

is a polynomial of degree n(>k) such that $|z_j|$ 1 for (j=1,2,...,n-k), then P' has that least k zeros in

$$\overline{D}\left(\frac{a}{k+1},\frac{k}{k+1}\right) \subset \overline{D}\left(a,\frac{2k}{k+1}\right)$$

four different proofs of Theorem A are known in the case k=1. It was shown that ρ(n,k,a) $\frac{2k}{k+1}$ for all $a \in \overline{D}(0,1)$ and all $k \in N$ if k+1 n $(k+1)^2$

The results says in particular that if k=1 then $\overline{D}(a,1)$ contains at least one zero of P' for n 4. However, more is known in this case k=1. It was shown by Meir and Sharma that if k=1 and n=5 then

$$\overline{D}\left(a, |a| + \frac{\sqrt{2-|a|^2}}{2}\right)$$

contains at least one zero of P', i.e.

$$\rho(5,1,a) \leq \left(|a| + \frac{\sqrt{2-|a|^2}}{2}\right) \leq 1 \qquad (1)$$

The following extension of (1) was shown by Rahman and Tariq

***Theorem** : Let $a \in \overline{D}(0,1)$ and k an integer 2.

If

$$P = c(z-a)^k \prod_{v=1}^{4} (z - z_v)$$

is a polynomial of degree k+4 such that $|z_v|$ 1 for v=1,...,4 then P' has at least k zeros in

$$\overline{D}\left(a, \frac{2}{3}\left(|a| + \sqrt{2-|a|^2}\right)\right)$$

if k=2 and in

$$\overline{D}\left(a, \frac{\sqrt{(k+1)^2 - |a|^2(2k+1) + |a|^k}}{k+1}\right)$$

if k 3. The purpose of this paper is to determine in that disks the critical points of P by using the Rouche Theorem.

CONJECTURE : Without loss of generality we may assume 0<a<1. Let $P(z)=(z-a)^k q(z)$. Then

$$P'(z) = (z-a)^{k-1}\{kq(z) + (z-a)q'(z)\},\ q(z) = \prod_{v=1}^{4} (z - z_v)\ ),\ |z_v| \leq 1, v = 1, ..., 4)$$

Let $g(z)=k(z-a)^{k-1}q(z)$ and $f(z)=(z-a)_k q'(z)$.

$$|(z-a)^k q'(z)| - |k(z-a)^{k-1}q(z)| \quad |(z-a)^k q'(z) - k(z-a)^{k-1}q(z)|$$
$$|(z-a)^k q'(z)| + |k(z-a)^{k-1}q(z)|$$
$$|f(z)| + |g(z)|$$

According to *Theorem since $f(z)=(z-a)^k q'(z)$, ($a\in \overline{D}(0,1)$) is a polinomial of degree k+3 such that $|z_v|$ 1 for v=1,2,3 its has at least k-1 zeros in $\overline{D}\left(a, \frac{2}{3}(|a|+\sqrt{2-|a|^2})\right)$ if k=2.

According to Rouche Theorem the polinomials f(z) and f(z)+g(z)=P'(z) have the some number of zeros in $\overline{D}\left(a, \frac{2}{3}(|a|+\sqrt{2-|a|^2})\right)$. Similary,

$$f(z) = (z-a)^k q'(z)$$

is a polinomial of degree k+3 such that $|z_v|$ 1 for v=1,2,3, then f(z) has at least k-1 zeros in if k 3

$$\overline{D}\left(a, \frac{\sqrt{(k+1)^2-|a|^2(2k+1)}+|a|^k}{k+1}\right)$$

By using Rouche Theorem again, the polinomials f(z) and g(z)+f(z)=P'(z) have the same number of zeros in

$$\overline{D}\left(a, \frac{\sqrt{(k+1)^2-|a|^2(2k+1)}+|a|^k}{k+1}\right)$$

If k 3.

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