Erc.Ünv.Fen Bil.Derg.12,1-2 (1996) 37-49

THE EIGENVALUES OF SYMMETRIC TOEPLITZ MATRICES

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ABSTRACT

In this paper the eigenvalues of symmetric Toeplitz matrices as in the form are $T_n = T_n(a, b, c, b, \ldots, c, b)$ searched, where n is positive even integer.We have given a formulae for the eigenvalues of these matrices. We have shown that there was equality between the number of different elements of such matrices and the number of eigenvalues. Moreover, we have shown that the eigenvectors of such matrices unchanged, even if its elements change.

SİMETRİK TOEPLİTZ MATRİSLERİN ÖZDEĞERLERİ

ÖZET

Bu çalışmada, n pozitif çift sayı olmak üzere $T_n = T_n(a, b, c, b, \ldots, c, b)$ şeklindeki simetrik Toeplitz matrislerin özdeğerlerini araştırdık. Bu matrislerin özdeğerleri için bir formül verip, bu matrislerin farklı eleman sayıları ile farklı özdeğerleri arasında bir eşitlik olduğunu gösterdik. Ayrıca matrisin elemanları değişse de özvektörlerin değişmediğini gösterdik.

1. INTRODUCTION

There is a lot of paper about the eigenvalues of Toeplitz matrices in the literature. In [1] Weaver, determined the number of eigenvalues real and complex of nonnegative symmetric matrix. In [2] Trench, gave the formulas for the characteristic polynomial and eigenvectors of Toeplitz and band matrices. In [3] Gover, has shown that when the order of the symmetric tridiagonal 2-Toeplitz matrix is odd, the eigenvalues can be explicitly determined in terms of the zeros of the Chebyshev polynomials.

In this paper we give a formulae for the eigenvalue of symmetric Toeplitz matrix $T_n = T_n(a, b, c, b, \ldots, c, b)$, where the order of this matrix is even. We have shown that there were numerical equality between the number of different elements and the number of eigenvalues of such matrices. Moreover, we prove the eigenvectors of such matrix are independent its elements.

2. THE COMPUTATION OF EIGENVALUES

Definition 2.1. A matrix $T_n = T_n(a_0, a_1, \ldots, a_{n-1})$ nxn is called (real) symmetric Toeplitz matrix if its elements a_{ij} obey the rule $a_{ij} = a_{|i-j|}$ for all i,j = 1,...,n. The matrix T_n is a function of n parameters, i.e., $T_n = T_n(a_0, a_1, \ldots, a_{n-1})$. A symmetric Toeplitz matrix can be written in explicit form as

$$T_n = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ a_1 & a_0 & a_1 & \cdots & a_{n-2} \\ a_2 & a_1 & a_0 & \cdots & a_{n-3} \\ . & . & . & \cdots & . \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_0 \end{bmatrix}$$

In its most general form, the nxn Toeplitz matrix has the following structure:

$$T_n = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ a_{-1} & a_0 & a_1 & \cdots & a_{n-2} \\ a_{-2} & a_{-1} & a_0 & \cdots & a_{n-3} \\ . & . & . & \cdots & . \\ a_{-n+1} & a_{-n+2} & a_{-n+3} & \cdots & a_0 \end{bmatrix}$$

where the element of T_n are such that $a_{ij} = a_{i-j}$.

Lemma 2.1. Consider the matrix

$$T_n = \begin{bmatrix} a & -b & -c & \ldots & -b & -c & -b \\ -b & a & -b & \ldots & -c & -b & -c \\ -c & -b & a & \ldots & -b & -c & -b \\ . & . & . & \ldots & . & . & . \\ -b & -c & -b & \ldots & a & -b & -c \\ -c & -b & -c & \ldots & -b & a & -b \\ -b & -c & -b & \ldots & -c & -b & a \end{bmatrix}. \tag{2.1}$$

The matrix T_n in (2.1) has eigenvalues

$$\lambda_1 = a + \frac{n}{2}b + \frac{n-2}{2}c, \quad \lambda_2 = a - \frac{n}{2}b + \frac{n-2}{2}c, \quad \lambda_3 = a - c$$

where, a,b,cŒR, b c, n 4 and n is positive even integer.

Proof. Let the characteristic equation of the matrix in (2.1) be

$$|\lambda_k I - T_n| = 0 \tag{2.2}$$

where k =1,2,...,n. Hence we have

$$|\lambda_1 I - T_n| = \begin{bmatrix} t & -b & -c & \dots & -b & -c & -c \\ -b & t & -b & \dots & -c & -b & -c \\ -c & -b & t & \dots & -b & -c & -b \\ . & . & . & \dots & . & . & . \\ -b & -c & -b & \dots & t & -b & -c \\ -c & -b & -c & \dots & -b & t & -b \\ -b & -c & -b & \dots & -c & -b & t \end{bmatrix} \quad (2.3)$$

where

$$t = \frac{n}{2}b + \frac{n-2}{2}c, \quad s = -\frac{n}{2}b + \frac{n-2}{2}c, \quad (2.4)$$

If the first (n-1) rows of the determinant (2.3) are added to the last row, all the elements of the last row are zero. Hence $|\lambda_1 I - T_n| = 0$ and therefore λ_1 is eigenvalue of the matrix (2.1).

Now, let us show that λ_2 is also an eigenvalue of the matrix (2.1) i.e. $|\lambda_2 I - T_n| = 0$. Hence we have

$$|\lambda_2 I - T_n| = \begin{bmatrix} s & -b & -c & \dots & -b & -c & -b \\ -b & s & -b & \dots & -c & -b & -c \\ -c & -b & s & \dots & -b & -c & -b \\ . & . & . & \dots & . & . & . \\ -b & -c & -b & \dots & s & -b & -c \\ -c & -b & -c & \dots & -b & s & -b \\ -b & -c & -b & \dots & -c & -b & s \end{bmatrix} \quad (2.5)$$

If we product 1,3,...,(n-1)th the odd index rows in the determinant (2.5) with -1 and the first (n-1) rows of this determinant and also add to the last row, then all the elements of the last row are zero. Hence $|\lambda_2 I - T_n| = 0$. Thus λ_2 is an eigenvalue of the matrix (2.1).

Finally, let us show that λ_3 is the eigenvalue of the matrix (2.1). We consider the determinant $|\lambda_3 I - T_n| = 0$, i.e.

$$|\lambda_3 I - T_n| = \begin{bmatrix} -c & -b & -c & \dots & -b & -c & -b \\ -b & -c & -b & \dots & -c & -b & -c \\ -c & -b & -c & \dots & -b & -c & -b \\ . & . & . & \dots & . & . & . \\ -b & -c & -b & \dots & -c & -b & -c \\ -c & -b & -c & \dots & -b & -c & -b \\ -b & -c & -b & \dots & -c & -b & -c \end{bmatrix} \tag{2.6}$$

The rows 1,3,...,(n-1)th of the determinant in (2.6) are linear combination of each other and also the rows 2,4,...,nth. are. Then $|\lambda_3 I - T_n| = 0$. Hence λ_3 is an eigenvalue of the matrix (2.1). This is completed the proof of Lemma 2.1.

Theorem 2.1. There is not the eigenvalue of the matrix (2.1) different from the following eigenvalue

$$\lambda_1 = a + \frac{n}{2}b + \frac{n-2}{2}c, \quad \lambda_2 = a - \frac{n}{2}b + \frac{n-2}{2}c, \quad \lambda_3 = a - c \tag{2.7}$$

i.e. there are eigenvalues each other of the matrix (2.1). Where *a*,b,cŒR, b c, n 4

and n is positive even integer.

Proof. We have shown that λ_1, λ_2 and $\lambda_3$are eigenvalue of the matrix (2.1) in the proof of Lemma 2.1. Now, let us show that $\lambda_3 = \lambda_4 = \dots = \lambda_n = a - c$. We had optained the determinant (2.5) when we subsituted $\lambda_3 = a - c$ into (2.2). Since $c^2 - b^2 \neq 0$, the rank of the matrix corresponding to the determinant (2.6) is two. Thus the linear independent vectors are

[-c, -b, -c, ..., -b, -c] and [-b, -c, -b, ..., -c, -b]. (2.8)

Let us compute the eigenvectors corresponding to eigenvalue $\lambda_3 = a$ - c. We optain the eigenvectors from the solution of the linear homogen system

$$\begin{aligned} cx_1 + bx_2 + cx_3 + \ldots + bx_{n-2} + cx_{n-1} + bx_n &= 0 \\ bx_1 + cx_2 + bx_3 + \ldots + cx_{n-2} + bx_{n-1} + cx_n &= 0. \end{aligned}$$

If we solve this system, then we find

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} -(a_3 + a_5 + \ldots + a_{n-1}) \\ -(a_4 + a_6 + \ldots + a_n) \\ a_3 \\ \vdots \\ a_n \end{bmatrix} = a_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} + a_4 \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{bmatrix} + \ldots + a_{n-1} \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix} + a_n \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

where $x_3 = a_3,\ x_4 = a_4,\ \ldots,\ x_{n-1} = a_{n-1},\ x_n = a_n$ are arbitrary numbers.

If we represent the eigenvectors $v_3, v_4, \ldots, v_{n-1}, v_n$ then we have

$$v_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix},\ v_4 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{bmatrix},\ \ldots,\ v_{n-1} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix},\ v_n \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}. \qquad (2.9)$$

Since the number of the eigenvectors corresponding to eigenvalue $\lambda_3 = a$ - c are n-2, the order of the eigenvalue λ_3 is n-2 [4, p.351]. Thus

$$\lambda_3 = \lambda_4 =\ldots= \lambda_n = a - c.$$

We have computed eigenvectors n-2 of the matrix (2.1). Since at least eigenvector corresponding to every eigenvalue, two eigenvectors correspond to eigenvalues λ_1 and λ_2. Thus, since n eigenvectors correspond to eigenvalues λ_1, λ_2 and λ_3, there are eigenvalues of the matrix (2.1) as λ_1, λ_2 and λ_3.

Corollary 2.1. The eigenvectors of the matrix (2.1) are independent from its elements.

Proof. We have shown that the eigenvectors correspond to eigenvalues λ_k for k=3,4,...,n, are independent from the elements of the matrix. Now, we will show that the eigenvectors corresponding to eigenvalues λ_1 and λ_2 are independent from its elements.

Consider matrix corresponding the determinant (2.3). If the first (n-1) rows of matrix corresponding the determinant (2.3) are added to the last row, all the elements of the last row are zero. Hence we have,

$$\begin{bmatrix} t & -b & -c & \cdots & -b & -c & -b \\ -b & t & -b & \cdots & -c & -b & -c \\ -c & -b & t & \cdots & -b & -c & -b \\ . & . & . & \cdots & . & . & . \\ -b & -c & -b & \cdots & t & -b & -c \\ -c & -b & -c & \cdots & -b & t & -b \\ -b & -c & -b & \cdots & -c & -b & t \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = [0] \qquad (2.10)$$

Let us product the row (n-1)th with -1 and add this row to the rows 1,3,5,...,(n-3) th respectively and let us product the row (n-2)th with -1 and add this row to rows 2, 4, 6,...,(n-4)th respectively in the system of equations (2.10). Again let us the row (n-1) th add to the (n-2)th and let us product the row (n-2)th with n/2 and add to the row (n-2) all the rows except for the row (n-1)th, then we have

$$\begin{bmatrix} t+c & 0 & 0 & 0 & \dots & 0 & 0 & -t-c & 0 \\ 0 & t+c & 0 & 0 & \dots & 0 & -t-c & 0 & 0 \\ 0 & 0 & t+c & 0 & \dots & -t-c & 0 & -t-c & 0 \\ 0 & 0 & 0 & t+c & \dots & 0 & -t-c & 0 & 0 \\ . & . & . & . & \dots & . & . & . & . \\ 0 & 0 & 0 & 0 & \dots & t+c & 0 & -t-c & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & t+c & 0 & -t-c \\ -c & -b & -c & -b & \dots & -c & -b & t & -b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = [0] \tag{2.11}$$

Hence, we have

$$\begin{aligned} (t+c)\,x_1 - (t+c)\,x_{n-1} &= 0 \\ (t+c)\,x_2 - (t+c)\,x_{n-2} &= 0 \\ (t+c)\,x_3 - (t+c)\,x_{n-1} &= 0 \\ (t+c)\,x_4 - (t+c)\,x_{n-2} &= 0 \\ &\dots\dots\dots\dots \\ (t+c)\,x_{n-2} - (t+c)\,x_n &= 0 \\ -c\,x_1 - b\,x_2 - c\,x_3 - \dots - b\,x_{n-2} + t\,x_{n-1} - b\,x_n &= 0 \end{aligned} \tag{2.12}$$

If it is solved the equations system (2.12), we obtain

$$\begin{aligned} x_1 &= x_{n-1} \\ x_2 &= x_{n-2} \\ x_3 &= x_{n-1} \\ \\ x_4 &= x_{n-2} \\ \\ &\cdots \\ x_{n-2} &= x_n \end{aligned} \qquad (2.13)$$

If we substitute the values in (2.13) into the last equation in (2.12), we have

$$-c\,x_1 - b\,x_2 - c\,x_1 - \ldots - b\,x_{n-2} + t\,x_1 - b\,x_2 = 0$$
$$-(c + c + \ldots + c)x_1 - (b + b + \ldots + b)x_2 + t\,x_1 = 0$$
$$-\frac{n-2}{2}cx_1 - \frac{n}{2}bx_2 + tx_1 = 0$$
$$\frac{n}{2}bx_1 = \frac{n}{2}bx_2 \Rightarrow x_1 = x_2$$

Hence, we obtain $x_1 = x_2 = x_3 = \ \ldots \ = x_{n-1} = x_n$.

Since the rank of the coefficient matrix in (2.11) is n-1 and the number of the unknowns of the linear homogen equations system in (2.12) is n we find

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} k \\ k \\ \vdots \\ k \end{bmatrix} = k\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

where k is arbitrary constant and chose and $x_n = k$. Hence the eigenvector corresponding to eigenvalue λ_1 is

$$V_1 = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}.$$

Finally eigenvalue λ_1 corresponds only one eigenvector and this eigenvector is independent from the elements of the matrix (2.1).

Now, let us we compute the eigenvector corresponding to eigenvalue λ_2. Consider matrix corresponding the determinant (2.5). If we product 1,3,...,(n-1)th the odd index rows of matrix corresponding the determinant (2.5) with -1 and the first (n-1) rows the matrix and also add to the last row, then all the elements of the last row are zero. Hence we have following equation system

$$\begin{bmatrix} s & -b & -c & \dots & -b & -c & -b \\ -b & s & -b & \dots & -c & -b & -c \\ -c & -b & s & \dots & -b & -c & -b \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -b & -c & -b & \dots & s & -b & -c \\ -c & -b & -c & -b & -c & s & -b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = [0] \qquad (2.14)$$

Let us product the row (n-1)th with -1 and add this row to the rows 1,3,5,...,(n-3) th respectively and let us product the (n-2)th with -1 and add to the rows 2, 4, 6,..., (n-4)th respectively. Again If we product the rows (n-2)th and (n-1)th with n/2 and -n/2 respectively and add to the row (n-2)th, we have

$$\begin{bmatrix} s+c & 0 & 0 & 0 & \dots & 0 & 0 & -s-c & 0 \\ 0 & s+c & 0 & 0 & \dots & 0 & -s-c & 0 & 0 \\ 0 & 0 & s+c & 0 & \dots & -s-c & 0 & -s-c & 0 \\ 0 & 0 & 0 & s+c & \dots & 0 & -s-c & 0 & 0 \\ . & . & . & . & \dots & . & . & . & . \\ 0 & 0 & 0 & 0 & \dots & s+c & 0 & -s-c & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & s+c & 0 & -s-c \\ -c & -b & -c & -b & \dots & -c & -b & s & -b \end{bmatrix}_{(n-1)xn} \tag{2.15}$$

Let us write the linear homogen equations system corresponding to the matrix (2.15). Hence we have

$$\begin{aligned} &(s+c)x_1-(s+c)x_{n-1}=0 \\ &(s+c)x_2-(s+c)x_{n-2}=0 \\ &(s+c)x_3-(s+c)x_{n-1}=0 \\ &(s+c)x_4-(s+c)x_{n-2}=0 \\ &\dots\dots\dots\dots\dots\dots\dots\dots \\ &(s+c)x_{n-2}-(s+c)x_n=0 \\ &-cx_1-bx_2-cx_3-\dots-bx_{n-2}+sx_{n-1}-bx_n=0 \end{aligned} \tag{2.16}$$

From (2.16), we obtain

$$\begin{aligned} &x_1=x_{n-1} \\ &x_2=x_{n-2} \\ &x_3=x_{n-1} \\ &x_4=x_{n-2} \\ &\dots\dots\dots \\ &x_{n-2}=x_n \end{aligned} \tag{2.17}$$

Thus, we have

$x_1=x_3=\dots=x_{n-1}$ and $x_2=x_4=\dots=x_n$

If we substitute this values in (2.17) into the last equation in (2.16), we have

$$-cx_1 - bx_2 - cx_1 - \ \dots\ - bx_2 + sx_1 - bx_2 = 0$$
$$-(c + c+\dots+c)x_1 - (b + b+\dots+b)x_2 + tx_1 = 0$$
$$-\frac{n-2}{2}cx_1 - \frac{n}{2}bx_2 + sx_1 = 0$$
$$-\frac{n}{2}bx_1 = \frac{n}{2}bx_2 \Rightarrow x_1 = -x_2$$

Since the rank of the matrix (2.15) is n-1, choose $x_2 = r$ we find $x_1 = -r$. Hence we have

$$x_1 = x_3 = \ \dots\ = x_{n-1} = -r \quad \text{and} \quad x_2 = x_4 = \ \dots\ = x_n = r$$

where r is an arbitrary constant. The eigenvector corresponding to eigenvalue λ_2 is

$$V_2 = \begin{bmatrix} -1 \\ 1 \\ : \\ -1 \\ 1 \end{bmatrix}$$

Hence eigenvector V_2 is independent from the elements of the matrix (2.1).

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