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ON THE PROPERTIES OF CURVES UNDER THE PARALLEL MAP PRESERVİNG THE CONNECTION

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ABSTRACT

In this study, a relationship between the absolute, normal and geodesic curvatures of the $\overline{M}$-vector field defined on M^r parallel hypersurface has been investigated. It is shown that the Z-geodesic curve and Z-asymptotic curve given in [5] is not preserved under the parallel map preserving the connection. Also, a necessary and sufficient condition for the two parallel hypersurfaces to be tangent along σ^r differentiable curve has been given.

KONNEKSİYONU KORUYAN PARALEL DÖNÜŞÜMLER ALTINDA EĞRİLERİN BAZI ÖZELLİKLERİ ÜZERİNE

ÖZET

Bu çalışmada M^r paralel hiperyüzeyi üzerinde tanımlanan $\overline{M}$ - vektör alanının mutlak, normal ve geodezik eğrilikleri arasındaki ilişki incelendi. Konneksiyonu koruyan paralel dönüşümler altında [5] de verilen Z-geodezik eğri ve Z-asimptotik olma durumunun korunmadığı gösterildi. Ayrıca iki paralel hiperyüzeyin diferensiyellenebilirbir σ^r eğrisi boyunca teğet olmaları için gerek ve yeter şart verildi.

1.INTRODUCTION

Let M be a hypersurface of a C^∞ Riemannian n-manifold $\overline{M}$ with the metric tensor $\langle , \rangle$. Let $\overline{D}$ be the riemannian connection on $\overline{M}$, and N be aa unit normal vector field that is C^∞ on M. We have $\langle N_p, N_p \rangle = 1$ and $\langle N_p, X \rangle = 0$, where $P \in M$ and $X \in \chi(M)$. Let $S : T_M(P) \to T_M(P)$ be the weingarten map defined by

$$S(X) = \overline{D}_X N. \qquad (1.1)$$

for any C^∞ vector field Y defined abouut P in M and X in $\chi(M)$ we have the gauss equation given by

$$\overline{D}_X N = D_X N - \langle S(X), Y \rangle N \qquad (1.2)$$

Equation (1.2) induces a Riemannian connection D on M . Hence D assings to each pair of C^∞ vector fields X and Y on an open set U of M, a C^∞ vector field $D_X Y$ called the covariant derivative of Y in the direction X. If X,Y are C^∞ vector fields on U and f is real valued C^∞ function on U, then we have the following relation which will be used later [4]:

$$D_X(fY) = (Xf)Y + fD_X Y. \qquad (1.3)$$

Let M be a hypersurface of the manifold $\overline{M}$ and let $N = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}$, $a_i \in C^\infty(M, IR)$ a unit normal vector field of M. Let us define a set $M^r = \{P + N_p | P \in M\}$ in which r is any real constant. A map $f : M \to M^r$ is called a parellel map for each member P of M if it is defined to be

$$f(P) = P + rN_p$$

In this case , M^r is a called parallel hypersurface of M. For each point P in $M, N_p = N^r_{f(p)}$[1].

The following theorem can be give for parallel hypersurfaces.

Theorem. Let $\overline{M}$ be a C^{∞} Riemannian n-manifold. Let M and M^{r} be respectively a hypersurface of $\overline{M}$ and a hypersurface being parallel to M of $\overline{M}$. Then , for each X in $\chi(M)$

$$f_*(X) = X + rS(X) \qquad (1.4)$$

$$S^r(f_*(X)) = S(X) \qquad (1.5)$$

in which S and S^r denote the Weingarten maps of M and M^r respectively , and f_* is a map defined to be

$$f_* : T_M(P) \rightarrow T_{M^r}(f(P))$$

which is known to be a jacobian map of f

Let M and M' be C^{∞} manifolds with connection D and D' respectively. A C^{∞} map $f\colon M \rightarrow M'$ is connection preserving if

$$f_*\left(D_X Y\right) = D'_{f_*(X)} f_*(Y) \qquad (1.6)$$

for all vector fields X and Y in $\chi(M)$ [2].

Let Z be mapping that attaches a vector Z_p in $T_{\overline{M}}(P)$ to each point P in M. Z is called an $\overline{M}$-vector field dfined on M and said to be C^{∞} on M if about each point P in M there are local coordinate functions $x_1 x_2, \dots, x_n$ such that $Z = \sum_{i=1}^{n} \lambda_i \frac{\partial}{\partial x_i}$ on U, where $\lambda_i \in C^{\infty}(M, IR)$ $1 \le i \le n$. Z can be decomposed uniquely into its tangential and normal components as

$$Z = Z_T + Z_N \qquad (1.7)$$

in which Z_T is tangent vector field on M and Z_N is a vector field of $\overline{M}$ defined on M,which is normal to M at every point. The normal vector Z_N of Z can be written as $Z_N = \alpha N$ if $\alpha = \langle Z, N\rangle$, where α is a C^{∞} function on the neighbourhood U of M and N is the unit normal vector to M at P along Z_N [5].

2. SOME PROPERTİES OF CURVES UNDER THE PARALLEL MAP PRESERVING THE CONNECTION

Let M be a hypersurface of $\overline{M}$-manifold and let M^r be parallel hypersurface of M. If σ is curve passing through apoint P on M and T is the tangent vector field of σ on M , then $\sigma^r = f o \sigma$ is a curve passing through a point $f(P)$ on M^r and $f_*(T) \in T_{M^r}(f(P))$ is a tangent of σ^r at $f(P)$.

Z defined by $\overline{M}$ vector field on M^r can also be decomposed into its tangential and normal components as

$$Z = Z_{T^r} + Z_{N^r} \tag{2.1}$$

where Z_{T^r} is tangent vector to be $Z_{T^r} = f_*(Z_T) = Z_T + rS(Z_T)$ on M^r and Z_{N^r} vector field of $\overline{M}$ is the normal vector field to M^r at every point of M^r

The second term on the right hand side of equation (2.1) can bewritten as $Z_{N^r} = \beta N^r$ to be $\beta = \langle Z, N^r \rangle$ since $N^r_{f(p)} = N_p$ and $\alpha(P) = \beta(f(P)), \forall P \in M$ this term can also be written as $Z_{N^r} = \alpha N$. Hence equation (2.1) takes the form

$$Z = Z_{T^r} + \alpha N \tag{2.2}$$

the covariant derivative of Z in the direction T^r is

$$\overline{D}_{T^r} Z = \overline{D}_{T^r}\left(Z_{T^r} + \alpha N\right)$$

or

$$\overline{D}_{T^r} Z = \overline{D}_{T^r} Z_{T^r} + \overline{D}_{T^r}(\alpha N) \tag{2.3}$$

where $T^r = \dfrac{f_*(T)}{\|f_*(T)\|}$. Using (1.1)-(1.7), the terms $\overline{D}_{T^r} Z_{T^r}$ and $\overline{D}_{T^r}(\alpha N)$ in equation (2.3) can be expresed in terms of invariants of M as follows:

Let D^r be a connection of M^r parallel hypersurface.

$$\overline{D}_{f_*(T)} Z_{T^r} = D^r_{f_*(T)} Z_{T^r} - \langle S^r(f_*(T)), Z_{T^r} \rangle N^r$$

or

$$\overline{D}_{f_*(T)} Z_{T^r} = D^r_{f_*(T)} f_*\left(Z_T\right) - \langle S^r\left(f_*(T)\right), Z_{T^r}\rangle N^r$$
$$= D_T Z_T + rS\left(D_T Z_T\right) - \left(\langle S(T), Z_T\rangle + r\langle S(T), S\left(Z_T\right)\rangle\right) N \qquad (2.4)$$

Using equation (1.3), but only D, X and fY are replaced by $\overline{D}, f_*(T)$ and N, respectively, we have

$$\overline{D}_{f_*(T)} Z_{T^r} = \alpha \overline{D}_{f_*(T)} N^r + \left(f_*(T)[\alpha]\right) N$$
$$= \alpha S^r\left(f_*(T)\right) + \left(T + rS(T)\right)[\alpha] N \qquad (2.5)$$
$$= \alpha S(T) + \left(\frac{d\alpha}{dt} + rS(T)[\alpha]\right) N$$

substitution of equations (2.4) and (2.5) into equation (2.3) gives

$$\overline{D}_{T^r} Z = \frac{1}{\|f_*(T)\|}\Big[D_T Z_T + \alpha S(T) + r\left(D_T Z_T\right)$$
$$+ \left(\frac{d\alpha}{dt} - \langle S(T), Z_T\rangle + r\left(S(T)[\alpha] - \langle S(T), S\left(Z_T\right)\rangle\right)\right) N\Big] \qquad (2.6)$$

Let

$$\|f_*(T)\| = \mathsf{l}, \quad \tan \overline{D}_{T^r} Z = \frac{1}{\mathsf{l}}\left(D_T Z_T + \alpha S(T) + r\left(D_T Z_T\right)\right)$$
$$nor\overline{D}_{T^r} Z = \frac{1}{\mathsf{l}}\left(\left(\frac{d\alpha}{dt} - \langle S(T), Z_T\rangle + r\left(S(T)[\alpha] - \langle S(T), S\left(Z_T\right)\rangle\right)\right) N\right)$$

then equation (2.6) can be written as

$$\overline{D}_{T^r} Z = \tan \overline{D}_{T^r} Z + nor\overline{D}_{T^r} Z \qquad (2.7)$$

$\overline{D}_{T^r} Z$, $\tan \overline{D}_{T^r} Z$ and $nor\overline{D}_{T^r} Z$ are called respectively the absolute, geodesic and normal curvature vectoors of vector field Z with respect to σ^r. Equation (2.7) caan be written in the form

$$\overline{K}^r \overline{N}^r = K^r_n N + K^r_g X \qquad (2.8)$$

iin which $\overline{N}^r$, N and X are all uniit vectors in the directions of the absolute ,normal and geodesic curvature vectors of the vector field Z with respect to σ^r, respectively. They are defined as follows:

$$\left(\overline{K}^r\right)^2 = \langle \overline{D}_{T^r} Z, \overline{D}_{T^r} Z\rangle$$
$$\left(K_n^r\right)^2 = \frac{1}{\mathrm{l}^2}\left(\frac{d\alpha}{dt} - \langle S(T), Z_T\rangle\right)^2 + ra \tag{2.9}$$

and

$$\left(K_g^r\right)^2 = \frac{1}{\mathrm{l}^2}\Big(\langle D_T Z_T + \alpha S(T), D_T Z_T + \alpha S(T) + rb\rangle\Big) \tag{2.10}$$

where

$$a = 2\left(\frac{d\alpha}{dt} - \langle S(T), Z_T\rangle\right)\Big(S(T)[\alpha] - \langle S(T), S(Z_T)\rangle\Big) + r\Big(S(T)[\alpha] - \langle S(T), S(Z_T)\rangle\Big) \tag{2.11}$$

and

$$b = \Big(\langle D_T Z_T + \alpha S(T), S\big(D_T Z_T\big)\rangle + r\langle S\big(D_T Z_T\big), S\big(D_T Z_T\big)\rangle\Big). \tag{2.12}$$

from equations (2.9) and (2.10), $\left(\overline{K}^r\right)^2$ can be expressed in terms of $\left(K_n^r\right)^2$ and $\left(K_g^r\right)^2$ as

$$\left(\overline{K}^r\right)^2 = \left(K_g^r\right)^2 + \left(K_n^r\right)^2 \tag{2.13}$$

K.Nirmala shown that

$$\overline{K}^2 = K_n{}^2 + K_g{}^2$$

in which

$$\overline{K}^2 = \langle \overline{D}_T Z, \overline{D}_T Z\rangle, \quad K_n{}^2 = \left(\frac{d\alpha}{dt} - \langle S(T), Z_T\rangle\right)$$

and

$$K_g^2 = \langle D_T Z_T + \alpha S(T), D_T Z_T + \alpha S(T)\rangle.$$

Therefore

$$\left(\overline{K}^r\right)^2 = \frac{1}{\mathrm{l}^2}\Big(K_n^2 + K_g^2 + r(a+b)\Big)$$

or

$$\left(\overline{K}^r\right)^2 = \frac{1}{\mathsf{l}^2}\left(\overline{K}^2 + r(a+b)\right),$$

where a and b are giveen by equations (2.11) and (2.12), respectively.

Corollary 1. A relaitonship between the absolute , normal and geodesic curvatures of the $\overline{M}$-vector field Z with respect to σ^r can be by equation (2.13).

Corollary 2. In the case of r=0, we obtain the same results given by K. Nirmala [5].

Therefore, the results given in above can be considered as a generalised of the results proposed by K. Nirmala [5].

From equation (2.8) we have

$$\overline{K}^r \cos\theta = K_n^r, \tag{2.14}$$

which is clearly the analogue of meusnier's theorem for the $\overline{M}$-vector field Z with respect to σ^r, where θ is the angle between the direction of $\overline{D}_{T^r} Z$ and N.

From equations (2.8) we can given the following definitions.

Definition 1. σ^r is in absolute geodesic curve of the absolute curvature of the vector field Z vanishes along σ^r. Thus absolute geodesic curvatures of Z with respect to σ^r satisfy the equation

$$\langle \overline{D}_{T^r} Z, \overline{D}_{T^r} Z\rangle = 0$$

Definition 2. Let σ^r be a curve on M^r. σ^r is defined to be a Z-geodesic curve if the geodesic curvature of the $\overline{M}$-vector field Z with respect to σ^r vanishes at every point of σ^r. Hence , Z-geodesic satisfy the equation

$$\langle D_T Z_T + \alpha S(T) + rS\left(D_T Z_T\right), D_T Z_T + \alpha S(T) + rS\left(D_T Z_T\right)\rangle = 0$$

Definition 3. The curve σ^r is called an asymptotic curvature of Z if K_n^r=0.

From equation (2.12) and definitions given above , we can give the following theorems.

Theorem 1. Z- geodesic and asymptotic curvatures of Z under the parallel map preserving the connection are not preserved.

Theorem 2. The curve σ^r in M^r is an absolute geodesic curvature of the $\overline{M}$-vector field Z if and only if σ^r is an asymptotic geodesic of Z.

Particulary, if $\overline{M}$-vector field Z is defined to be a tangent vector on M^r along σ^r, then $\overline{K}^r, K_n^r$ and K_g^r become absolute, normal and geodesic curvatures of M^r, respectively.

Now, let M_1 and M_2 be two hypersurfaces of Riemannian manifold $\overline{M}$. Let $\sigma = \{x(t) | 0 \le t \le 1\}$ be diferentiable curve on $M_1 \cap M_2$. Let D^1 and D^2 be Riemannian connections of M_1 and M_2, respectively, and N_1 and N_2 be unit normal vector fields of M_1 and M_2, respectively. Then it is easily shown that $fo\sigma = \sigma^r = \{y(t) | 0 \le t \le 1\}$ is also a differentiable curve on $M_1^r \cap M_2^r$, where f is a parallel map preserving the connection.

If S_1^r and S_2^r are respectively , Weingarten mappings defined on $T_{M_1^r}(y(t))$ and $T_{M_2^r}(y(t))$, then

$$S_1^r(f_*(X)) = \overline{D}_{f_*(X)} N_1 \, , \; f_*(X) \in T_{M_1^r}(y(t)),$$

$$S_2^r(f_*(Y)) = \overline{D}_{f_*(Y)} N_2 \, , \; f_*(Y) \in T_{M_2^r}(y(t)),$$

Let Z be $\overline{M}$-vector field on $M_1^r \cap M_2^r$. Then

$$Z = Z_T + \alpha_1 N = Z_T + \alpha_2 N$$

such that $\alpha_1 \in C^\infty(M_1, IR)$ and $\alpha_2 \in C^\infty(M_2, IR)$. The covariant derivatives of (2.15) in the direction of T^r along σ^r is

$$\overline{D}_{T^r} Z = \frac{1}{l}\Big\{ D_T^1 Z_T + \alpha_1 S_1(T) + r S_1\big(D_T^1 Z_T\big) + \Big(\frac{d\alpha_1}{dt} - \langle S(T), Z_T\rangle + r\big(S(T)[\alpha] - \langle S(T), S(Z_T)\rangle\big)\Big) N_1 \Big\}$$

$$= \frac{1}{l}\Big\{ D_T^2 Z_T + \alpha_2 S_2(T) + r S_2\big(D_T^2 Z_T\big) + \Big(\frac{d\alpha_2}{dt} - \langle S(T), Z_T\rangle + r\big(S(T)[\alpha] - \langle S(T), S(Z_T)\rangle\big)\Big) N_2 \Big\} \quad (2.16)$$

If the curve σ^r is chosen as a Z-geodesic on M_1^r then, $tan\,\overline{D}_{T^r} Z = 0$ which implies that $\overline{D}_{T^r} Z \in \chi(M_1^r)^{\perp}$.

Assume that parallel hypersurfaces M_1^r and M_2^r are tangents along σ^r. In this case, $T_{M_1^r}(y(t)) = T_{M_2^r}(y(t))\ \forall\, y(t) \in \sigma^r$. Thus the vectors $\overline{D}_{T^r} Z$ is normal to M_2^r at $y(t)$ This is meant to be

$$D_T^2 Z_T + \alpha_2 S_2(T) + r S_2\big(D_T^2 Z_T\big) = 0$$

In other words, σ^r on M_2^r is a Z-geodesic. Therefore, properties given in above can be summarised with the following theorem.

Theorem. Let M_1^r and M_2^r be parallel hypersurfaces of Riemannian manifold $\overline{M}$. If the curve σ^r is a Z-geodesic on M_1^r then is a Z-geodesic on M_2^r, also . Here σ^r is a curve on $M_1^r \cap M_2^r$ and Z is an $\overline{M}$-vector field defined on $M_1^r \cap M_2^r$.

Conversely, let σ^r be a Z-geodesic curvature on both M_1^r and M_2^r. Thus,

$$D_T^1 Z_T + \alpha_1 S_1(T) + r S_2\big(D_T^1 Z_T\big) = 0$$

and

$$D_T^2 Z_T + \alpha_2 S_2(T) + r S_2\big(D_T^2 Z_T\big) = 0$$

from (2.16), normal curvature vectors of Z with respect to σ^r in both M_1^r and M_2^r are the same with $\overline{D}_{T^r} Z$. So, tangent spaces of M_1^r and M_2^r are on each other. In other words, it is $T_{M_1^r}(y(t)) = T_{M_2^r}(y(t))$ along σ^r.

Therefore , we can give this as a corollary.

Corollary. If Z is an $\overline{M}$-vector field of an $\overline{M}$-Riemannian manifold on $M_1^r \cap M_2^r$ and $\sigma^r = \{y(t) | 0 \le t \le 1\}$ is differentiable curve on $M_1^r \cap M_2^r$ to be a geodesiic on both M_1^r and M_2^r, then two parallel hypersurfaces M_1^r and M_2^r of $\overline{M}$ are tangent along σ^r.

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