Erc. Ünv. Fen Bil. Derg. 13,1-2 (1997)1-10

ON THE SHEAF OF RELATIVE HOMOTOPY GROUPS

Ayhan ŞERBETÇİ and Sabahattin BALCI
University of Ankara, Faculty of Sciences, Department of Mathematics,
06100 Tandoğan Ankara, TURKEY[1]

ABSTRACT.

Let X be a topological space and A be a connected, locally path connected, and semilocally simple connected subspace of X. Let $S_n(X,A)$ be the disjoint union of the n-th homotopy groups of the pair (X,A) obtained for each $x \in A$, i.e., $S_n(X,A) = \bigvee_{x \in A} \pi_n(X,A,x)$. We show that $S_n(X,A)$ is an algebraic sheaf over the pair (X,A) by giving a topological structure to the set $S_n(X,A)$. We show that if $f:(X,A) \to (Y,B)$ is a continuous mapping, then there exists a sheaf homomorphism between the sheaves of relative homotopy groups of the pairs (X,A) and (Y,B). Furthermore, if f is a homeomorphism, then there exists a group isomorphism between the groups of sections.

RÖLATİF HOMOTOPİ GRUPLARININ DEMETİ

ÖZET.

X bir topolojik uzay ve A, X' in irtibatlı, lokal eğrisel irtibatlı ve yarı lokal basit irtibatlı bir altuzayı olsun. $S_n(X,A)$, (X,A) çiftinin herbir $x \in A$,için bulunan *n.* homotopi gruplarının ayrık birleşimi olsun, yani, $S_n(X,A) = \bigvee_{x \in A} \pi_n(X,A,x)$ dır. Bu çalışmada $S_n(X,A)$ üzerine bir topolojik yapı vererek (X,A) çifti üzerinde bir cebirsel demet olduğunu gösterdik. Eğer $f:(X,A) \to (Y,B)$ bir sürekli dönüşüm ise, bu durumda (X,A) ve (Y,B) çiftleri üzerindeki rölatif homotopi gruplarının demetleri arasında bir demet homomorfizminin varlığını gösterdik. Ayrıca eğer f bir homeomorfizm ise, bu durumda kesitlerin grupları arasında bir grup izomorfizminin varlığını gösterdik.

1.INTRODUCTION

Let X be a topological space with base point x_0 .For $n \geq 1$, the n-th homotopy group $\pi_n(X,x_0)$ is defined as the totality of homotopy classes of the mappings $\alpha:(I^n, \dot{I}^n) \to (X,x_0)$ such that $\alpha(\dot{I}^n) = x_0$, where I^n is

---

1 AMS Subject Classification (1991): 55N30
Key Words : Homotopy Groups, Sheaf, Section

Euclidean n-cube, consisting of points $(t_1,t_2,\ldots,t_n)$ with $0\le t_i\le 1$ and $\dot{I}^n$ is its boundary. Equivalently, $\pi_n(X,x_0)$ may be regarded as the homotopy classes of the base point preserving mappings $\alpha:(S^n,P_0)\to(X,x_0)$ such that $\alpha(P_0)=x_0$, where S^n is the unit n-sphere given by $t_1^{\,2}+t_2^{\,2}+\ldots+t_n^{\,2}=1$ and $P_0=(1,0,\ldots,0)$. This group is abelian for $n\ge 2$. Hence we write $[\alpha]+[\beta]=[\alpha+\beta]$ instead of $[\alpha][\beta]=[\alpha\beta]$ for $[\alpha],[\beta]\in\pi_n(X,x_0)$. Let X be a connected, locally path connected, and semilocally simple connected topological space and $S_n(X)$ be the disjoint union of the n-th homotopy groups obtained for each $x\in X$, i.e., $S_n(X)=\underset{x\in X}{V}\pi_n(X,x)$. Then $S_n(X)$ is an algebraic sheaf $[2]$. It is a sheaf of abelian groups for $n\ge 2$. $S_n(X)$ is called the sheaf of n-th homotopy groups of X. For $n=1$, $S_1(X)$ is called the sheaf of fundamental groups of X [1] . Steenrod showed that $S_n(X)$ is a bundle of coefficients [5].

Now let X be a topological space and A be a subspace of X and let $x_0\in A$. I^{n-1} is the face of I^n given by $t_n=0$. Let J^{n-1} be the union of remaining (n-1) faces of I^n, i.e., $J^{n-1}=I^n-\operatorname{int}I^{n-1}$. For $n\ge 2$, n-th relative homotopy groups $\pi_n(X,A,x_0)$ of the pair (X,A), with base point in A is defined as the set of all homotopy classes of the mappings $\alpha:(I^n,I^{n-1},J^{n-1})\to(X,A,x_0)$, where $\alpha(I^n)\subset X$, $\alpha(I^{n-1})\subset A$, and $\alpha(J^{n-1})=x_0$ (or equivalently $\alpha:(E^n,S^{n-1},P_0)\to(X,A,x_0)$, where E^n is Euclidean n-cell, consisting of points $(t_1,t_2,\ldots,t_n)$ with $0\le t_i\le 1$ and $t_1^{\,2}+t_2^{\,2}+\ldots+t_n^{\,2}\le 1$, S^{n-1} is its boundary). $\pi_n(X,A,x_0)$ is an abelian group for $n\ge 3$, while $\pi_1(X,A,x_0)$ is only a set. In this context, if $\alpha,\beta:(I^n,I^{n-1},J^{n-1})\to(X,A,x_0)$ are two continuous mappings, then the group structure is induced by

$$\delta(t_1,t_2,\ldots,t_n)=(\alpha+\beta)(t_1,t_2,\ldots,t_n)=$$
$$\begin{cases}\alpha(t_1,t_2,\ldots,t_{k-1},2t_k,t_{k+1},\ldots,t_n), & 0\le t_k\le\frac{1}{2}\\ \beta(t_1,t_2,\ldots,t_{k-1},2t_k-1,t_{k+1},\ldots,t_n), & \frac{1}{2}\le t_k\le 1\end{cases}$$

for $1 < k \leq n$. It should be noted that the relative homotopy group $\pi_n(X, A, x_0)$, reduces to the homotopy group $\pi_n(X, x_0)$, if $A = \{x_0\}$.

2. THE SHEAF OF RELATIVE HOMOTOPY GROUPS

Let X be a topological space and A be a connected, locally path connected, and semilocally simple connected subspace of X and let $x \in A$. Let $S_n(X, A)$ be the disjoint union of the n-th relative homotopy groups obtained for each $x \in A$, i.e., $S_n(X, A) = \underset{x \in A}{V} \pi_n(X, A, x)$. Thus, $S_n(X, A)$ is a set over A. Let us define a mapping $\varphi: S_n(X, A) \to A$ as, for each $\sigma \in S_n(X, A)$, $\varphi(\sigma) = \varphi([\alpha]_x) = x$, where $x \in A$ and $\sigma = [\alpha]_x \in \pi_n(X, A, x)$. Now, if $x_0 \in A$ is an arbitrarily fixed point, then there exists an open path connected neighborhood $W = W(x_0)$ *of* x_0 in A such that for any two points x_1 and x_2 in W every pair of paths in W joining x_1 to x_2 are homotopic in A with endpoints held fixed since A is locally path connected and semilocally simple connected. If γ is a path from x_0 *to* x for any $x \in W$, then γ induces an isomorphism

$$(\gamma^*)_n : \pi_n(X, A, x_0) \to \pi_n(X, A, x)$$

for all n, given by $(\gamma^*)_n([\alpha_0]_{x_0}) = [\alpha]_x$ for arbitrarily fixed $[\alpha_0]_{x_0} \in \pi_n(X, A, x_0)$.

To topologize $S_n(X, A)$; first, define a mapping $s: W(x_0) \to S_n(X, A)$ by $s(x) = (\gamma^*)_n([\alpha_0]_{x_0}) = [\alpha]_x \in \pi_n(X, A, x)$. Clearly, s is a well-defined mapping. Indeed, if δ is a different path than γ in W from x_0 *to* x, then they are homotopic in A with endpoints held fixed, hence $(\gamma^*)_n = (\delta^*)_n$. Moreover, $s(x_0) = 1^*([\alpha_0]_{x_0}) = [\alpha_0]_{x_0} \in \pi_n(X, A, x_0)$ and $\varphi \circ s = 1_W$.

Next, we consider $s(W) = \{s(x) = [\alpha]_x \in S_n(X, A) : x \in W \subset A\}$ as a neighborhood of $[\alpha_0]_{x_0}$ in $S_n(X, A)$ and we prescribe that all these sets be open sets. Then $\{s(W) : W \subset A\}$ forms a basis for the topology on $S_n(X, A)$: Indeed, let W_1, W_2 be any two path connected subsets of A, and $\sigma \in s_1(W_1) \cap s_2(W_2)$. Then s_1, s_2 agree at $\varphi(\sigma) = \varphi([\alpha]_x) = x$ $(x \in W_1 \cap W_2)$ and by the definition of the mappings

s_1, s_2, $s_1(W_1 \cap W_2) = s_2(W_1 \cap W_2)$. Therefore σ has a basic neighborhood $s_1(W_1 \cap W_2) = s_2(W_1 \cap W_2)$ inside $s_1(W_1) \cap s_2(W_2)$. It follows that s is a continuous mapping with respect to this topology. Moreover, since $(\gamma^*)_n$ is an isomorphism, there exists a unique $[\alpha]_x \in (S_n(X,A))_x$ corresponding to $[\alpha_0]_{x_0}$, i.e., $(S_n(X,A))_x \cap s(W) = [\alpha]_x$. Therefore the topology on $S_n(X,A)_x$ is discrete.

Furthermore, $\varphi^{-1}(W) = \underset{i \in I}{\cup} s_i(W)$, where the index set I corresponds to the totality of all equivalence class in $\pi_n(X,A,x_0)$. The sets $s_i(W)$ are pair wise disjoint: if $\sigma \in s_i(W) \cap s_j(W)$, then for any $x \in W$, $\sigma = s_i(x) = s_j(x) \Rightarrow (\gamma^*)_n([\alpha_i]_{x_0}) =$ $(\gamma^*)_n([\alpha_j]_{x_0}) \Rightarrow [\alpha_i]_{x_0} = [\alpha_j]_{x_0}$, where γ is a path from x_0 *to* x and $(\gamma^*)_n$ is the isomorphism induced by γ. Hence for every $x \in W$, $s_i(x) = s_j(x)$, i.e., $s_i(W) = s_j(W)$. Then the projection φ is well-defined.

Now if the set $W \subset A$ is any open set in the relative topology on A, then $W = \underset{i \in I}{\cup} W_i$, where for each $i \in I$, W_i is a path connected open set in A. Hence, we can define a mapping $s: W \to S_n(X,A)$ as follows: If $x \in W$, then $x \in W_i$ for any $i \in I$ and there exists a mapping $s_i: W_i \to S_n(X,A)$. We define the mapping s by $s(x) = s_i(x)$. Clearly, s is continuous and $\varphi \circ s = 1_W$. Therefore φ is continuous with respect to the topology on $S_n(X,A)$, since for any open W in A

$$\varphi^{-1}(W) = s(W) = \underset{i \in I}{\cup} \{ s_i(W) : W_i \subset W \},$$

where W_i is path connected and open subset of W. In addition φ is a locally homeomorphism since on $s_i(W_i)$ it has the continuous inverse s_i.

Therefore we can state the following theorem.

Theorem 1. Let X be a topological space and A be a connected, locally path connected, and semilocally simple connected subspace of X. Let $\pi_n(X,A,x)$ be corresponding n-th relative homotopy group for each $x \in A$, and $S_n(X,A) = \underset{x \in A}{V} \pi_n(X,A,x)$. If the mapping $\varphi: S_n(X,A) \to A$ is defined as above, then there exists a natural topology on $S_n(X,A)$ such that

φ is locally topological with respect to this topology. Thus, $(S_n(X,A),\varphi)$, is a sheaf over (X,A).

Definition 2. The sheaf $(S_n(X,A),\varphi)$ given by the Theorem 1 is called the sheaf of the n-th relative homotopy groups of the pair (X,A). $\varphi^{-1}(x)=\pi_n(X,A,x)$ is called the stalk of the sheaf and denoted by $(S_n(X,A))_x$ for every $x\in A$. A continuous mapping $s:W\to S_n(X,A)$ such that $\varphi\circ s=1_W$ is called a section of $S_n(X,A)$ over open set $W\subset A$.

Then the collection of all sections of $S_n(X,A)$ over a fixed open subset W of A is denoted by $\Gamma(W,S_n(X,A))$. The set $\Gamma(W,S_n(X,A))$ is a group with the point wise addition. If $s_1,s_2\in\Gamma(W,S_n(X,A))$ are obtained by means of the elements $[\alpha_1]_x,[\alpha_2]_x\in\pi_n(X,A,x)$, respectively, then

$$(s_1+s_2)(x)=s_1(x)+s_2(x)=[\alpha_1]_x+[\alpha_2]_x=[\alpha_1+\alpha_2]_x.$$

It is easy to see that $\Gamma(W,S_n(X,A))$ is a group. Thus, the operation (+): $S_n(X,A)\oplus S_n(X,A)\to S_n(X,A)$ defined by $(\sigma_1,\sigma_2)\to\sigma_1+\sigma_2$ for every $\sigma_1,\sigma_2\in S_n(X,A)$ is continuous. Hence $(S_n(X,A),\varphi)$ is an algebraic sheaf. It is a sheaf of abelian groups for $n\geq 3$ since $\varphi^{-1}(x)=\pi_n(X,A,x)$ is an abelian group for each $x\in A$.

It is easily seen that if W is any open path connected subset of A, then it has all the properties required of an elementary neighborhood in the definition of covering space. Therefore $S_n(X,A)$ is a covering space of A.

3.CHARACTERIZATIONS

Definition 3. Let X,Y be arbitrary topological spaces and A,B be connected, locally path connected, and semilocally simple connected subspaces of X,Y, respectively. Let the sheaves $(S_n(X,A),\varphi)$ and $(S_n(Y,B),\psi)$ be the corresponding sheaves to the pairs (X,A), (Y,B), respectively. It is said that there is a homomorphism between these sheaves and it is written $F=(f,f^*)$: $[(X,A),S_n(X,A)]\to[(Y,B),S_n(Y,B)]$, if there exists a pair $F=(f,f^*)$ such that

1. $f:(X,A)\to(Y,B)$ is a continuous mapping such that $f(A)\subset B$,
2. $f^*:S_n(X,A)\to S_n(Y,B)$ is a continuous mapping,
3. f^* preserves the stalks with respect to f, i.e., $f\,o\varphi=\psi\, o f^*$

4. for every $x \in A$, the restricted mapping

$$f^*|((S_n(X,A))_x : ((S_n(X,A))_x \to ((S_n(X,A))_{f(x)}$$

is a homomorphism.

$F=(f,f^*)$ is called an isomorphism, if the mappings f^* and f are topological, then the sheaves $S_n(X,A)$ and $S_n(Y,B)$ are said to be isomorphic.

Theorem .2 Let the sheaves $S_n(X,A)$ and $S_n(Y,B)$ be given. A continuous mapping $f:(X,A)\to(Y,B)$ induces a homomorphism of sheaves $f^*:S_n(X,A)\to S_n(Y,B)$.

Proof. Let $x \in A$ be an arbitrary fixed point. Then $f(x)\in B$ and $\pi_n(X,A,x)=(S_n(X,A))_x$ and $\pi_n(Y,B,f(x))=(S_n(Y,B))_{f(x)}$ are the corresponding stalks. If $\alpha_1,\alpha_2:(I^n,I^{n-1},J^{n-1})\to(X,A,x)$ are two continuous mappings, then we can define the continuous mappings $\beta_1,\beta_2:(I^n,I^{n-1},J^{n-1})\to(Y,B,f(x))$ as $\beta_1=fo\alpha_1, \beta_2=fo\alpha_2$. If $\alpha_1\sim\alpha_2$ then $\beta_1\sim\beta_2$. Thus, the correspondence $[\alpha]_x\to[fo\alpha]_{f(x)}$ is well-defined. Since the point $x\in A$ is arbitrary fixed, the above correspondence gives us a map $f^*:S_n(X,A)\to S_n(Y,B)$ such that $f^*([\alpha]_x)=[fo\alpha]_{f(x)}$ for every $[\alpha]_x\in S_n(X,A)$. Now, we will show that $F=(f,f^*)$ is a homomorphism.

1. f^* is a continuous mapping: If $V\subset f^*(S_n(X,A))\subset S_n(Y,B)$ is any open set, then $(f^*)^{-1}(V)=U\subset S_n(X,A)$ is an open set. Indeed, since V is open, $V=\bigcup_{i\in I}t_i(W_i)$ and $\varphi_2(V)=\bigcup_{i\in I}W_i\subset f(A)\subset B$ is open, where W_i is a path connected open neighborhood in B for every $i\in I$ and $t_i\in\Gamma(W_i,S_n(Y,B))$. Since f is continuous, for every $i\in I$. $A\cap f^{-1}(W_i)=G_i$ is open path connected subset of A. Hence there exists a section $s_i:G_i\to S_n(X,A)$ and $\bigcup_{i\in I}s_i(G_i)\subset S_n(X,A)$ is an open set. We will show that $U=\bigcup_{i\in I}s_i(G_i)$. If $\sigma_1=[\alpha]_x\in U$, $x\in A$ then there exists a $\sigma_2=[\beta]_y\in V$ such that $(f^*)^{-1}(\sigma_2)=\sigma_1$ and $\varphi_2(\sigma_2)=\varphi_2\left([\beta]_y\right)=y$ where y = f(x) and $\beta=fo\alpha$. Thus, if $y\in W_i$ for some $i\in I$, then $x\in G_i$ and $\sigma_1=[\alpha]_x\in\bigcup_{i\in I}s_i(G_i)$. Hence, $U\subset\bigcup_{i\in I}s_i(G_i)$. On the other hand, if

$\sigma_1 \in \bigcup_{i\in I} s_i(G_i)$ then $\sigma_1 \in s_i(G_i)$ for some $i \in I$ and $\varphi_1(\sigma_1) = \varphi_1([\alpha]_x) = x \in G_i$. Hence $f(x) \in W_i$ and $[fo\alpha]_{f(x)} = \sigma_2 \in V$ and then, $(f^*)^{-1}(\sigma_2) = \sigma_1 \in U$. Therefore $\bigcup_{i\in I} s_i(G_i) \subset U$ and $U = \bigcup_{i\in I} s_i(G_i)$.

2. f^* preserves the stalks with respect to f. Indeed, for any $[\alpha]_x \in S_n(X,A)$

$$(fo\varphi_1)([\alpha]_x = f(\varphi_1([\alpha]_x)) = f(x)$$

$$(\varphi_2 of^*)([\alpha]_x) = \varphi_2(f^*([\alpha]_x) = \varphi_2([fo\alpha]_{f(x)}) = f(x)$$

that is, $fo\varphi_1 = \varphi_2 of^*$.

3. For every $x \in A$ the mapping

$f^*|(S_n(X,A))_x : (S_n(X,A))_x \to (S_n(X,A))_{f(x)}$ is a homomorphism. Indeed,

If $\alpha_1, \alpha_2 : (I^n, I^{n-1}, J^{n-1}) \to (X, A, x)$, and

$fo\alpha_1, fo\alpha_2 : (I^n, I^{n-1}, J^{n-1}) \to (Y, B, f(x))$

are continuous mappings, then

$$f^*([\alpha_1 + \alpha_2]_x) = [fo(\alpha_1 + \alpha_2)]_{f(x)} = [fo\alpha_1 + fo\alpha_2]_{f(x)}$$
$$= [fo\alpha_1]_{f(x)} + [fo\alpha_2]_{f(x)}$$
$$= f^*([\alpha_1]_x) + f^*([\alpha_2]_x)$$

Therefore $F = (f, f^*)$ is a homomorphism.

Theorem 3. Let the sheaves $S_n(X,A), S_n(Y,B), S_n(Z,C)$, be given. The existence of continuous mappings $f:(X,A) \to (Y,B)$ and $g:(Y,B) \to (Z,C)$, leads to the existence of a homomorphism $h^*: S_n(X,A) \to S_n(Z,C)$ such that $h = gof$ and $h^* = (gof)^* = g^* o f^*$

Proof. Since $h = gof$ is continuous, by Theorem 2, there exists a homomorphism $h^*: S_n(X,A) \to S_n(Z,C)$. It is sufficient to show that $h^* = g^* o f^*$. For any $[\alpha] \in S_n(X,A)$, $h^*([\alpha]) = [(g \circ f) \circ \alpha]$ and

$$(g^* \circ f^*)([\alpha]) = g^*(f^*([\alpha])) = g^*([f \circ \alpha]) = [g \circ (f \circ \alpha)].$$

Thus we must show that $(g \circ f) \circ \alpha \sim g \circ (f \circ \alpha)$. If $\alpha:(I^n, I^{n-1}, J^{n-1}) \to (X, A, x)$ is a continuous mapping, then $(g \circ f) \circ \alpha :(I^n, I^{n-1}, J^{n-1}) \to (Z, C, g(f(x)))$ and $g \circ (f \circ \alpha) :(I^n, I^{n-1}, J^{n-1}) \to (Z, C, g(f(x)))$. Let us define a mapping $F(t_1,\ldots,t_n,k):(I^n \times I,\, I^{n-1} \times I, J^{n-1} \times I) \to (Z, C, g(f(x)))$ as follows

$$F(t_1,t_2,\ldots,t_n,k) = \begin{cases} (g \circ f)(\alpha(t_1,t_2,\ldots,t_n)), & 0 \le t_1 \le 1-k \\ g \circ (f \circ \alpha)(t_1,t_2,\ldots,t_n)), & 1-k \le t_1 \le 1 \end{cases}.$$

It is clear that F is continuous. Furthermore, $F(t_1,\ldots,t_n,0) = (g \circ f)(\alpha)$, $F(t_1,\ldots,t_n,1) = g \circ (f \circ \alpha)$, and $F(t_1,\ldots,t_n,k) = g(f(x))$, for $(t_1,\ldots,t_n) \in J^{n-1}$. Therefore $(g \circ f) \circ \alpha \sim g \circ (f \circ \alpha)$ and $h^* = g^* o f^*$

Now we can state the following theorem.

Theorem 4. There is a covariant functor from the category of the pairs of topological spaces together with their connected, locally path connected, and semilocally simple connected subspaces and continuous mappings to the category of sheaves and continuous homomorphisms.

Proof. Let **D** be the category of the pairs of topological spaces together with their connected, locally path connected, and semilocally simple connected subspaces and **E** be the category of sheaves and continuous homomorphisms. Define a mapping F:**D**→**E** as follows: F((X,A))=$S_n(X,A)$ and if $f:(X,A) \to (Y,B)$ is any continuous mapping, then F(f)= $f^*:S_n(X,A) \to S_n(Y,B)$.

(1) If $f = 1_{(X,A)}$, then F($1_{(X,A)}$)= $1_{F(X,A)}$, since $(1_{(X,A)})^* = \left[1_{(X,A)} \circ \alpha\right] = [\alpha]$ for any $[\alpha]_x \in (S_n(X,A))_x$.

(2) If $f:(X,A) \to (Y,B)$ and $g:(Y,B) \to (Z,C)$ any two continuous mappings, then by the Theorem 3, $(gof)^* = g^* o f^*$ i.e., F($g \circ f$)=F(g)∘F(f). Therefore F is a covariant functor.

Theorem 5. Let the sheaves $S_n(X,A)$ and $S_n(Y,B)$ be given. A homeomorphism $f:(X,A) \to (Y,B)$ induces an isomorphism of sheaves $f^*:S_n(X,A) \to S_n(Y,B)$.

Proof. Since f is a homeomorphism, there exists the continuous inverse mapping f^{-1} such that $f \circ f^{-1} = 1_{(Y,B)}$, $f^{-1} \circ f = 1_{(X,A)}$. By theorem 2, there exists the sheaf homomorphism $(f^{-1})^*:S_n(Y,B) \to S_n(X,A)$. By

Theorem 4, $(f\circ f^{-1})^{*}=f^{*}\circ(f^{-1})^{*}=1_{F(Y,B)}$ and $(f^{-1}\circ f)^{*}=(f^{-1})^{*}\circ f^{*}=1_{F(X,A)}$. Hence $(f^{-1})^{*}=(f^{*})^{-1}$ and therefore f^{*} is a sheaf isomorphism.

Theorem 6. Let the sheaves $S_n(X,A)$ and $S_n(Y,B)$ be given and $f,g:(X,A)\to(Y,B)$ be two continuous mappings. If f and g are homotopic $rel\,A$, then their induced sheaf homomorphisms $f^{*},g^{*}:S_n(X,A)\to S_n(Y,B)$ are equal for every $n\geq 2$.

Proof. Let $[\alpha]\in S_n(X,A)$. Since $f\sim g\ rel\ A$, there exists a continuous mapping $F:X\times I\to Y$ such that

$$F(x,0)=f(x)\ and\ F(x,1)=g(x)\ for\ all\ x\in X,$$
$$F(a,t)=f(a)=g(a)\quad for\ all\ a\in A\ and\ t\in I,$$
$$F(A\times I)\subset B.$$

Then the composition $F\circ\alpha$ gives a homotopy connecting $f\circ\alpha$ and $g\circ\alpha$ i.e., $[f\circ\alpha]=[g\circ\alpha]$. Hence $f^{*}([\alpha])=g^{*}([\alpha])$.

Theorem 7. Let the sheaves $(S_n(X,A),\varphi)$, $(S_n(Y,B),\psi)$ be given and $\Gamma(W,S_n(X,A))$, $\Gamma(f(W),S_n(Y,B))$ be the groups of all sections of $S_n(X,A)$, $S_n(Y,B)$ over W and $f(W)$ respectively, where W is an open subset of A. Then a homeomorphism $f:(X,A)\to(Y,B)$ induces an isomorphism $f_{*}:\Gamma(W,S_n(X,A))\to\Gamma(f(W),S_n(Y,B))$.

Proof. By Theorem 5, the homeomorphism $f:(X,A)\to(Y,B)$ induces a sheaf isomorphism $f^{*}:S_n(X,A)\to S_n(Y,B)$. Let $s\in\Gamma(W,S_n(X,A))$ be a section over W, then $f^{*}\circ s\circ f^{-1}:f(W)\to S_n(Y,B)$ is a continuous mapping and $\psi\circ(f^{*}\circ s\circ f^{-1})=1_{f(W)}$. Hence $f^{*}\circ s\circ f^{-1}$ is a section over $f(W)$, i.e., $f^{*}\circ s\circ f^{-1}\in\Gamma(f(W),S_n(Y,B))$. Thus we can define a mapping $f_{*}:\Gamma(W,S_n(X,A))\to\Gamma(f(W),S_n(Y,B))$. with $f_{*}(s)=f^{*}\circ s\circ f^{-1}$, for all $s\in\Gamma(W,S_n(X,A))$. It is easy to see that f_{*} is a group isomorphism.

Now we give the functorial statement of this theorem:

Theorem 8. There is a covariant functor from the category of the pairs of topological spaces together with their connected, locally path connected, and semilocally simple connected subspaces and topological mappings to the category of groups and isomorphisms.

Proof. Let **D** be the category of the pairs of topological spaces together with their connected, locally path connected, and semilocally simple connected subspaces and **E** be the category of groups and isomorphisms. Define a mapping F:**D**→**E** as follows: $F((X,A)) = \Gamma(W, S_n(X,A))$. and if $f:(X,A) \to (Y,B)$ is any continuous mapping, then $F(f) = f_*: \Gamma(W, S_n(X,A)) \to \Gamma(f(W), S_n(Y,B))$. Then F is a covariant functor:

(1) If $f = 1_{(X,A)}$, then $F(1_{(X,A)})_*$. Thus $F(1_{(X,A)})_* = 1_{(X,A)}$.

(2) If $f:(X,A) \to (Y,B)$ and $g:(Y,B) \to (Z,C)$ any two continuous mappings, then

$F(g \circ f) = (g \circ f)_* = g_* \circ f_*$:

$\Gamma(W, S_n(X,A)) \to \Gamma(g(f(W)), S_n(Z,C))$, *i.e.*, $F(g \circ f) = F(g) \circ F(f)$.

REFERENCES

[1] Balcı, S., The Sheaf of the Fundamental Groups. Comm Fac. Sci.Univ. Ankara Ser.A_1,: Math., pp. 59-65 (1982).

[2] Canbolat, N., Yüksek Homotopi Gruplarının Demeti ve İlgili Karakterizasyonlar, Ph.D. Thesis, Ankara Univ., (1982).

[3] Hilton, P.J., An Introduction to Homotopy Theory, Cambridge University Press, (1961).

[4] Hu, S.T., Homotopy Theory, Academic Press, N.Y., (1959).

[5] Steenrod, N., The Topology of Fibre Bundles, Princeton University Press, N.J.,(1951).

[6] Tennison, B.R., Sheaf Theory, Cambridge University Press, (1975).

[7] Whitehead, G.W., Elements of Homotopy Theory, Springer Verlag, New York, (1978).