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ON THE HOMOGENIZATION OF SHEAVES

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ABSTRACT

In this study we show that homogenization and dehamogenization are true for the (pre)-sheaves of rings and homogenization functor commutes with sheafification whereas the same is true for dehomogenization when the topological space is (quasi) compact. Moreover we show that the campactness hypothesis cannot be dropped.

ÖZET

Bu çalışmada, halkaların demetleri (öndemetleri) için homogen izeyşin ve dehomogenizeyşinin doğru olduğu ve topolojik uzay (quası) kompakt olduğunda demetleştirme, homogenizeyşin funktoru ile dehomogenizeyşin funktorunu komutatif yaptığı gösterilmiştir. Ayrıca gösterilmiştir ki kompaklık hipotezden kaldırılamaz.

1. INTRODUCTION

In (commutative) algebraic geometry, homogenization and dehomogenization techniques are used to reduce questions about projective varieties to questions about associated affine varieties (at least if these questions can be answered by looking at the local rings, cfr.e.g. [2].

After the introduction of noncommutative affine and projective shemes (Cfr [1, 3, 4]), the question arose whether a similar technique could be developped in this noncommutative setting. The aim of this study is to prove that this is indeed possible. In the those sections we treat the most general case, i.e. homogenization and dehomogenization of (pre)-sheaves of rings. It is proved that the homogenization functor commutes with sheafification whereas the same is true for dehomogenization when the topological spaces is (quasi) compact. An example is given to show that this compactness hypothesis cannot be dropped.

Definitions 1.

(a) : A ring R is said to be graded (of type Z) if there is a family of additive subgroup $\{R_n : n \in Z\}$ of R such that $R = \oplus R_n$ and $\forall i, j \in Z : R_i R_j \subset R_{i+j}$.

(b) An $M \in R$-mod is said to be a graded left R-module if there is a family $\{M_n : n \in Z\}$ of additive subgroup of M with the properties: $M = \oplus M_n$ and $R_i M_j \subset M_{i+j}$ for all $i, j \in Z$. The elements of $h(R) = \bigcup R_n$ and $h(M) = \bigcup M_n$ are called the homogeneous element of R and M resp.

(c) If $m \neq 0$, $m \in M_i$; then m is called an homogeneous element of degree i and we write deg m=i. Any non-zero $m \in M$ may be written, in a unique way, as a finite sum $m_1 + \cdots + m_k$ with $\deg m_1 < \deg m_2 < \cdots < \deg m_k$; the elements $m_1, \dots, m_k$ (all non-zero) are called the homogeneous components of m.

R-gr is the Grothendieck-Category with objects the graded left R-modules and morphisms the gradation preserving R-module homomorphisms.

Let X be a fixed topological space. All sheaves and presheaves in this paper will be defined over Open (X), the category obtained from X in the usual way. Let R be a Gr-Ring, i.e. a sheaf of graded rings over Open (X) such that for any $U, V \in$ Open (X) with $V \subset U$, the restriction morphisms $\rho_V^U : R(U) \to R(V)$ preserve the gradation.

We will consider the following Grothendieck-Categories:

$\pi(R)$ (resp.σ (R)) will be category of presheaves (resp. sheaves) of modules over the sheaf R.

$g\pi(R)$ (resp.$g\sigma(R)$) will be the category of presheaves (resp. sheaves) of graded modules over the sheaf R, i.e., $\mathcal{M} \in g\pi(R)$ if and only if $\underline{\mathcal{M}} \in \pi(R)$ where $(\underline{\ }) : g\pi(R) \to \pi(R)$ is the functor defined by forgetting the gradaiton and for all $U \in \text{Open}(X); \mathcal{M}(U) \in R-gr$; $\forall U, V \in \text{Open}(X)$, $V \subset U$: $\mathcal{M}_V^U : \mathcal{M}(U) \to \mathcal{M}(V)$ is gradation preserving.

2. HOMOGENIZATION AND DEHOMOGENIZATION

From [3] we recollect the following definitions: Let R be a graded ring. The ring of polynomials R[T] may be made into a graded ring by putting:

$$\deg T = 1;\ R[T]_n = \left\{ \sum_{i+j=n} r_i T^j ; r_i \in R_i \right\}$$

In the same way, we construct the graded module of polynomials M[T] starting from an $M \in R-gr$. If we decompose $x \in M$ into homogeneous elements; $x = x_{-m} + \cdots + x_o + \cdots + x_n$ $(x_i \in M_i)$, then we may associate to it an homogeneous element x* in M[T] which is given by :

$x^* = x_{-m}T^{m+n} + \cdots + x_o T^n + \cdots + x_n$. We say that x_* is the homogenized of x.

Conversely, if u is homogeneous element of M[T], say $u = u_{-m}T^{m+n+p} + \cdots + u_o T^{n+p} + \cdots + u_n T^p$ with $u_i \in M_i$, then $u_* = u_{-m} + \cdots + u_o + \cdots + u_n$ is said to be the dehomogenned of u.

If $M \in R-gr$ and N is a (not, necessarely graded) R-submodule of M, then by N^* we mean the R[T] submodule of M[T] generated by the n^*, $n \in N$.

Of course N^* is a graded submodule of M[T], it is called the homogenized of N. Any $n \in N^*$ is of the form $T^r n_1^*$, $n_1 \in N$ and $r \geq 0$

Conversely, to a graded R[T]-submodule L of M[T] we may associate $L_* = \{u_* : u \in h(L)\}$. It is clear that L_* is an R-submodule of M.

Remark 2.1. Now, Let R be a Gr-Ring. We will form the Ring of polynomials R[T] as follows: if $U \in \text{Open}(X)$, we put R[T] (U)=R(U) [T] with gradation as in above if U, $V \in \text{Open}(X)$ with $V \subset U$, the restriction morphism $(\rho^*)_V^U$ is given by.

$$(\rho^*)_V^U\left(T^{m+n+p}x_{-m} + \cdots + T^{n+p}x_0 + \cdots + T^p x_n\right) = T^{m+n+p}\rho_V^U(x_{-m}) + \cdots + T^{n+p}\rho_V^U(x_o) + \cdots + T^p \rho_V^U(x_n).$$

Because ρ_V^U preserves the gradation, the same is true for $(\rho^*)_V^U$, hence R[T] is a Gr-Ring (the verification that R[T] is a sheaf is proved along the lines of lemma 2.1. below).

If $\mathcal{M} \in g\pi(R)$, we can define the module of polynomials $\mathcal{M}[T]$ in a similar way.

Now, let $N \in \pi(R)$ be a subpresheaf of $\mathcal{M} \in g\pi(R)$. For all $U \in \text{Open}(X)$, define $\mathsf{N}^*(U) = \mathsf{N}(U)^* \subset \mathcal{M}[T]$. The restriction morphisms $(\mathsf{N}^*)_V^U$ for $V \subset U \in \text{Open}(X)$ are given by.

$$(\mathsf{N}^*)_V^U\left(T^{m+n+p}x_{-m} + \cdots + T^{n+p}x_0 + \cdots + T^p x_n\right) = T^{m+n+p}\mathcal{M}_V^U(x_{-m}) + \cdots + T^{n+p}\mathcal{M}_V^U(x_o) + \cdots + T^p\mathcal{M}_V^U(x_n).$$

If follows that N^* is a graded subpresheaf of $\mathcal{M}[T]$.

Lemma 2.2: In the situation of remark 2.1. if $N \in \sigma(R)$, then $\mathsf{N}^* \in g\sigma(R[T])$.

Proof :

(i): Let $U \in \text{Open}(X)$ and $\{U_i : i \in I\}$ and open covering of U. Suppose $n \in \mathsf{N}(U)$ and $(\mathsf{N}^*)_{U_i}^U(n^*) = 0$ for all $i \in I$, then;

$0 = (\mathsf{N}^*)_{U_i}^U(n^*) = (\mathcal{M}^*)_{U_i}^U(n^*) = \mathcal{M}_{U_i}^U(n)^* = \mathsf{N}_{U_i}^U(n))^*$ and from $(x^*)_* = x$, it follows that $\mathsf{N}_{U_i}^U(n) = 0$ for all $i \in I$. Because N is a sheaf we obtain n=0, thus $n^* = 0$.

(ii) : Let $n_i' \in h(\mathcal{M}^*(U_i))$ with compatibility conditions :

$$(\mathsf{N}^*)_{U_i \cap U_j}^{U_i}(n_i') = (\mathcal{M}^*)_{U_j \cap U_i}^{U_j}(n_j')$$

Every n_i' is of the form $n_i' = T^{ki}(n_i^*)$ with $n_i \in \mathsf{N}(U_i)$. For all i, j we have :

$T^{ki}(N^*)^{U_i}_{U_i\cap U_j}(n_i))^* = (N^*)^{U_i}_{U_i\cap U_j}(T^{ki}n_i^*) = (M)^{*U_j}_{U_i\cap U_j}(T^{kj}n_i^*)$
$= T^{kj}(M^{U_j}_{U_i\cap U_j}(n_j))^*$.

Dehomogenizing both sides yields: $N^{U_i}_{U_i\cap U_j}(n_i) = M^{U_j}_{U_i\cap U_j}(n_j)$ for all $i, j \in I$. Using the second sheaf condition for N we find an $n \in N(U)$ such that $N^{U}_{U_i}(n) = n_i$ for all i.

From the equalities above, it follow that $k_i = k_j = k$ for all i, j. $n' = T^k n^*$.

Remark 2.3 : Let N be graded subpresheaf of $M[T]$ with $M \in g\pi R$. For every $U \in \text{Open}(X)$ we put: $N_*(U) = (N(U))_*$. For $V \subset U \in \text{Open}(X)$ the restriction morfizm $(N_*)^U_V$ is given in the following way : if $x \in M_*(U)$, then there exists an $y \in h(M(U))$ such that $y_* = x$, put:$(N_*)^U_V(x) = (M^U_V(y))_*$. One easily checks that this definition is independent of the choice of y.

Lemma 2.4: In the situation of remark 2.3: if X is a compact topological space and $N \in g\sigma(R[T])$, then $N_* \in \sigma(R)$.

Proof:

(i) : Let $U \in \text{Open}(X)$ and let $\{U_i : i \in I\}$ be an open covering of U. Suppose $x \in N_*(U)$ such that $(N_*)^U_{U_i}(x) = 0$ for will $i \in I$. There exists an $y \in h(N(U))$ such that $y_* = x$.

Now, $(N^U_{U_i}(y))_* = (N_*)^U_{U_i}(x) = 0$, thus $N^U_{U_i}(y) = 0$ for all $i \in I$ and the fact that N is a sheaf, yields y=0, hence x=0. Compactness is not necessary for this part of proof.

(ii) : The compactness hypothesis allows us to restrict to a finite covering $\{U_i; i = 1,\dots,n\}$. Take x_i in $N_*(U_i)$ such that

$(N_*)^{U_i}_{U_i \cap U_j}(x_i) = (N_*)^{U_j}_{U_i \cap U_j}(x_j)$ for all $i, j = 1, \ldots, n$

There exist $y_i \in h(U_i)$ such that $(y_i)_* = x_i$. Put deg $(y_i) = d_i$ and $m = \max d_i$; i=1,...,n. Replace y_i by $y'_i = T^{m-d_i} y_i$, then $(y'_i)_* = x_i$ and :

$$(N^{U_i}_{U_i \cap U_j}(y'))_* = (N_*)^{U_i}_{U_i \cap U_j}(x_i) = (N_*)^{U_j}_{U_j \cap U_i}(x_j) = (N^{U_j}_{U_i \cap U_j}(y'_j))_*$$

Because deg $(N^{U_i}_{U_i \cap U_j}(y_i')) = \deg(N^{U_j}_{U_j \cap U_i}(y'_j))$ it follows that $N^{U_i}_{U_i \cap U_j}(y_i') = N^{U_j}_{U_i \cap U_j}(y'_j)$ and therefore there exists an $y \in h(N(U))$ such that $N^{U}_{U_i}(y) = y_i$ for all i. y_* is the required element in $N_*(U)$.

Corollary : The compactness condition cannot be dropped :

Example : Take N with the discrete topology and R the constant sheaf of ring Z over it. Now, take N to be the graded subsheaf of Z[T] as follows: if U is a finite open set of N, the $N(U) = T^n Z[T]$ with n the maximal element in U. If U is infinite, $N(U) = 0$.

If $U = \varnothing$, $N(U) = Z[T]$. Restriction morphisms are inclusions or the zero map. It is easily checked that N is a sheaf. N_* is the presheaf with $N_*(U) = Z$ if U is finite, $N_*(U) = 0$ if U is infinite and inclusion or zero map for the restriction morphisms. (ii) fails, for, take U=N, $U_i = \{i, i+1\}$ and $x_i = 1 \in N(U_i)$ There exist no element x in $N_*(N)$ such that $N^{N}_{U_i}(x) = 1$.

3. COMPATIBILITY WITH THE SHEAFIFICATION FUNCTOR

We recall the construction of the reflector $\underline{\underline{a}}$ for the inclusion $\sigma(R) \to \pi(R)$, usually called the sheafification functor, cfr.e.g. [5].

First, define a functor $L:\pi(R) \to \pi(R)$ as follows. Let $U \in Open(X)$ we give $Cov_X(U)$, i.e. the set of all open coverings of U, the structure of a category: if $U = \{U_i ; i \in I\}$, $V = \{V_j ; j \in I\}$ are in $Cov_X(U)$, a morphism $U \to V$ is given by a map $\varepsilon : I \to J$ such that $U_i \subset V_{\varepsilon(i)}$ for all $i \in I$. Let $M \in \pi( )$ and define $[M,U]$ $U \in Open(X)$, by its action on a covering $U = \{U_i ; i \in I\}$ of U:

$$[M,U](U) = \mathrm{Ker}\,(\prod_{i \in I} M(U_i) \overset{p}{\underset{q}{\rightrightarrows}} \prod_{(j,k) \in I \times I} M(U_j \cap U_k))$$

Where the (j, k) - component of p is $M^{U_j}_{U_j \cap U_k}(m_j)$ and the (j,k)component of q is $M^{U_k}_{U_j \cap U_k}(m_k)$; with $m_i : \prod_{j \in I} M(U_j) \to M(U_i)$ be the restriction morphism. Note that $[M,U]:Cov_X(U) \to R(U)$-mod is a contravariant functor. Hence we can define an object LM of $\pi(R)$ by :

$$LM : Open(X)^{opp} \to set \quad , \quad U \to \lim_{U \in Cov_X(U)} [M,U](U)$$

Note that $LM(U) = \varinjlim_{U \in Cov_X(U)} \varprojlim_{V \in U} (V)$

The assigment $M \to LM$ defines a left exact endofunctor of $\pi(R)$ Satisfying : 1. If $M \in \sigma(R)$, i.e. the class of separated objects in $\pi(R)$ (satisfying (i)), then the canonical morphism $M \to LM$ is a monomorphism and $LM \in \sigma(R)$.

2. If $\mathcal{M} \in \pi(R)$, the $L\mathcal{M} \in \sigma(R)$

3. If $\mathcal{M} \in \sigma(R)$ then $L\mathcal{M} \cong \mathcal{M}$ an Conversely.

Finally, define $i.\underline{\underline{a}} = L \circ L$ where $i:\sigma(R) \to \pi(R)$ is the cannonical inclusion, then $\underline{\underline{a}}$ is a left adjoint of i and is called sheafification functor.

Let us denote $\underline{\underline{a}}'$ for sheafification in $\pi(R[T])$, then we have :

Lemma 3.1. If $\mathcal{M} \in g\pi(R[T])$, then $\underline{\underline{a}}'(\mathcal{M}) \in g\sigma(R[T])$.

Proof: For all $U \in \mathrm{Open}(X)$: R(U)-gr is closed under direct and inverse limits, hence we are done by the remaks preceding the Lemma.

Theorem 3.2. Let $\mathcal{N} \in \pi(R)$ be a subpresheaf of an $\mathcal{M} \in g\pi(R)$, then: $\underline{\underline{a}}'(\mathcal{N}^*) = \underline{\underline{a}}(\mathcal{N})^*$, i.e. the following diagnom "Commutes" for suitable $\mathcal{N}$.

$$\begin{array}{ccc} \pi(R) & \xrightarrow{()^*} & g\pi(R[T]) \\ \underline{\underline{a}}\downarrow & & \downarrow\underline{\underline{a}}' \\ \sigma(R) & \xrightarrow{()^*} & g\sigma(R[T]) \end{array}$$

Proof: First step: for every $x \in X: S_x(\mathcal{N}^*) \cong (S_x(\mathcal{N}))^*$ where $S_x(-)$ is the stalk at x.

Let $a \in h(S(\mathcal{N}^*))$, then we can find a neighborhood U of x and an element $y \in h(\mathcal{M}^*(U))$ representing a. Let $y = y_{-m}T^{m+n+p} + \cdots + y_n T^p$ with $y_i \in \mathcal{M}(U)_i$. We consider the morphism :

$$\begin{aligned} f: S_x(\mathcal{N}^*) &\longrightarrow (S_x(\mathcal{N}))^* \\ a &\longrightarrow \mathcal{M}_x^U(y_{-m})T^{m+n+p} + \cdots + \mathcal{M}_x^U(y_o)T^{n+p} + \cdots + \mathcal{M}_x^U(y_n)T^p \end{aligned}$$

f does not depend on the choice of U and y, for, if V is another neighborhood of x and

$y' = y_{-r}T^{r+q+p'} + \cdots + y_o T^{q+p'} + \cdots + y_q T^{p'} \in h(\mathcal{N}^*(V))$ representing a. Then, there is a neighborhood $W \subset U \cap V$ of x such that $(\mathcal{N}^*)^U_W(y) = (\mathcal{N}^*)^V_W(y')$. Hence

$$\mathcal{M}^U_x(y_{-m})T^{m+n+p} + \cdots + \mathcal{M}^U_x(y_o)T^{n+p} + \cdots + \mathcal{M}^U_x(y_n)T^p =$$
$$(\mathcal{N}^*)^W_x((\mathcal{N}^*)^U_W(y)) = (\mathcal{N}^*)^W_x((\mathcal{N}^*)^V_W(y')) = \mathcal{M}^V_x(y'_{-r})T^{r+q+p'} + \cdots$$
$$\mathcal{M}^V_x(y'_o)T^{q+p'} + \cdots + \mathcal{M}^V_x(y'_q)T^{p'}.$$

Moreover, f is injective, for f(a)=0, then $\mathcal{M}^U_x(y_i) = 0$ for all i, hence we can find a neighborhood $W \subset U$ of x such that $\mathcal{M}^U_W(y_i) = 0$ for all i. Because the definition of f does not depend on U and y, x=0 follows. Also, f is surjective. Indeed, if $y \in h(S_x(\mathcal{N})^*)$, then y is of the form $y = T^p((y')^*)$ with $y' \in S_x(\mathcal{N})$, letting $y' = y_{-m} + \cdots + y_n$, then for all i, we can find a neighborhood U_i of x and an $x_i \in h(\mathcal{M}(U_i))$ representing y_i. Take $U = \bigcap U_i$ and Consider :

$$b' = \mathcal{M}^{U_{-m}}_U(y_{-m}) + \cdots + \mathcal{M}^{U_o}_U(y_o) + \cdots + \mathcal{M}^{U_n}_U(y_n), \text{ then;}$$
$$b = T^p(b')^* \in h(\mathcal{N}^*(U)) \text{ represents y.}$$

Second step : in view of lemma 2.3., both $(\underline{\underline{a}}(\mathcal{N}))^*$ and $\underline{\underline{a}}'(\mathcal{N}^*)$ are in $g\sigma(R[T])$. In order to establish to isomorphism it will be sufficient to establish isomorphisms between the stalks. Now, for all $x \in X$:

$$S_x(\underline{\underline{a}}'(\mathcal{N}^*)) = S_x(\mathcal{N}^*) = (S_x(\mathcal{N}))^* = (S_x(\underline{\underline{a}}(\mathcal{N})))^* = S_x(\underline{\underline{a}}(\mathcal{N})^*)$$

Theorem 3.3: Let X be a compact topological space and $\mathsf{N} \in g\pi(R[T])$ a graded subpresheaf of $\mathsf{M}[T]$ with $\mathsf{M} \in \pi(R)$, then: $(\underline{\underline{a}}'(\mathsf{N}))_* = \underline{\underline{a}}(\mathsf{N}_*)$, i.e. the following diagram "Commutes" for suitable as above :

$$\begin{array}{ccc} g\pi(R[T]) & \xrightarrow{(-)_*} & \pi(R) \\ \underline{\underline{a}}'\downarrow & & \downarrow\underline{\underline{a}} \\ g\sigma(R[T]) & \xrightarrow{(-)_*} & \sigma(R) \end{array}$$

Proof: First step: for every $x \in X$: $S_x(\mathsf{N}_*) = (S_x(\mathsf{N}))_*$. Let $a \in S_x(\mathsf{N}_*)$, then there is a neighborhood U of x and an element $y \in \mathsf{N}_*(U)$ representing a. Pick $z \in h(\mathsf{N}(U))$ such that $z_* = y$, and define

$$f{:}S_x(\mathsf{N}_*) \to (S_x(\mathsf{N}))_*$$
$$a \to (\mathsf{N}_x^U(z))_*$$

This definition is independent of the choices made. For, let V be another neighborhood of x and y' (with corresponding z') in $\mathsf{N}_*(V)$ (resp. in $h(\mathsf{N}(V))$) representing a, then we can find an open $x \in W \subset U \cap V$ such that

$$(\mathsf{N}_*)_W^U(y) = (\mathsf{N}_*)_W^V(y')$$

Hence $(\mathsf{N}_W^U(z))_* = (\mathsf{N}_W^V(z'))_*$ and thus there exists a natural number k such that, $\mathsf{N}_W^U(z) = \mathsf{N}_W^V(z')T^k$.

Finally :

$$(\mathsf{N}_x^U(z))_* = (\mathsf{N}_x^W(\mathsf{N}_W^U(z)))_* = (\mathsf{N}_x^W(\mathsf{N}_W^V(z')T^k))_* = (\mathsf{N}_x^V(z'))_*$$

Now, f is injective; for if f(a)=0, then $N_x^U(z)=0$ and z (hence y) represents the zero morphism, thus a=0.

Also, f is surjective; for if $y \in (S_x(N))_*$ then there exists an element $z \in h(S_x(N))$ with $z_* = y$. Take an element $v \in h(N(U))$ representing z, then put $a = (N_*)_x^U(v_*)$ and one easily checks that f(a)=y.

Second step: in view of lemma 2.4. $(\underline{\underline{a}}'(N))_*$ and $\underline{\underline{a}}(N_*)$ are both in $\sigma(R)$. Isomorphism will follow from the stalkwise isomorphisms. For every $x \in X$:

$$S_x(\underline{\underline{a}}'(N)_*) \cong S_x\underline{\underline{a}}'(N))_* \cong S_x(N)_* \cong S_x(N_*) \cong S_x(\underline{\underline{a}}(N_*))$$

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