Erc. Ünv. Fen Bil. Derg. 13, 1-2 (1997) 78-85

On The Tzitzéica's Submanifolds

Baki KARLIĞA

Department of Mathematics, Arts and Sciences Faculty, Gazi University
06500 Ankara, TURKEY.

ABSTRACT

Let M be an m-dimensional submanifold in E^n .We define C^∞ function $\tau_{M\cap W}$ by using the idea of Putinar [2].We show that for $m < n$, $\tau_{M\cap W}$ is invariant under the orthogonal group O(n) and it is invariant under absolute unimodular group for m = n-1.We prove that

$$\tau_{M\cap W}\circ F = (-1)^m \frac{G(.,\ N)\circ F}{(d^{m+2})\circ F}$$

where F is parametrization of M, G(., N) is Lipschitz-Killing curvature, N is unit normal of M and d is a distance between origin and tangent plane on a point of M.

ÖZET

M, E^n de m-boyutlu altmanifold olsun. Putinar'ın düşüncesini kullanarak bir $\tau_{M\cap W}$ fonksiyonu tanımladık. $\tau_{M\cap W}$ nın $m < n$ için,O(n) ortogonal grubu altında, $m = n-1$ için de mutlak unimodular grup altında değişmez kaldığını gösterdik. F, M nın parametrizasyonu ; G(., N) Lipschitz-Killing Eğriliği; N, M'nin birim normali; d de M'nin bir noktasındaki teğet düzleminin orjine uzaklığı olmak üzere

$$\tau_{M\cap W}\circ F = (-1)^m \frac{G(.,\ N)\circ F}{(d^{m+2})\circ F}$$

olduğunu gördük.

1. Introduction. M.G. Tzitzèica, Rumanian mathematician, has shown that $\frac{K}{d^4}$ is constant for some class of surfaces $M \subset E^3$ when he was studying regular tetrahedron, where K, d are Gaussian curvature and the distance of a point on M to the origin, respectively.He has also proved it [1] for some hypersurface in $M \subset E^n$. If $\frac{K}{d^{n+1}}$ is constant for any hypersurface M then, M is called Tzitzéica's hypersurface and the ratio $\frac{K}{d^{n+1}}$ is called Tzitzéica's invariant. The equivalency of any two Tzitzéica's invariants are given by Putinar [2]. The aim of this present work is to generalize the Tzitzéica's invariant [2] to any m-dimensional submanifold in E^n.

Firstly, we give some basic concepts in order to understandant ℓ - vector space, a vector field over C^∞ -map and an affine connexion on C^∞ - map. Afterwards, we define a C^∞ function $\tau_{M\cap W}$ by using the idea of Putinar [2] and we show that $\tau_{M\cap W}$ is invariant under the orthogonal group $O(n)$ in the case $m < n$ and it is invariant under absolute unimodular group in the case $m = n-1$. Finally, we show that

$$\tau_{M\cap W} = (-1)^m \frac{G(.,\ N)}{d^{m+2}} \quad .$$

2. Background and Notations.

Definition 2.1. A subset $M \subset E^n$ is called a smooth manifold of dimension m if each $x \in M$ has a neighborhood W in M which is diffeomorphic to an open subset U of Euclidean space E^m.In addition, if $m < n$ then, M is called an *m-dimensional submanifold of* E^n [3].

Definition 2.2. Let M be m-dimensional submanifold of E^n. Then the C^∞-diffeomorphism

$$F : U \longrightarrow M \cap W$$

is called a *local parametrization of the region* $M \cap W$ [3]. The following definitions 2.3, 2.4 and 2.5 are due to [4].

Definition 2.3. Let $\chi(F)$ be $C^\infty(U)$ -module of vectorfield over F. The set

$$\Lambda^\ell \chi(F) = \{\alpha_1 \wedge ... \wedge \alpha_\ell \mid \alpha_j \in \chi(F),\ j = 1,\ ...,\ \ell\}$$

is called a $C^\infty(U)$ *-module of* ℓ *-vector field.*

Definition 2.4. The form, given by

$$<,\ >_\ell \circ F : \Lambda^\ell \chi(F) x \Lambda^\ell \chi(F) \longrightarrow C^\infty(U)$$

$$<\alpha,\ \beta>_\ell \circ F = det[<\alpha_i,\ \beta_j> \circ F]$$

is called a *metric tensor of ℓ-vector fields*, where $\alpha = \alpha_1 \wedge \alpha_2 \wedge ... \wedge \alpha_\ell$, $\beta = \beta_1 \wedge ... \wedge \beta_\ell$, α_i, $\beta_j \in \chi(F)$.

Definition 2.5. Let

$$A : \chi(F) \longrightarrow \chi(F)$$

be a linear transformation.Then the transformation

$$\Lambda^\ell A(\alpha_1 \wedge ... \wedge \alpha_\ell) = A(\alpha_1) \wedge ... \wedge A(\alpha_\ell),\ \alpha_i \in \chi(F),\ 1 \leq i \leq \ell$$

is called $\ell - th$ *exterior power of* A.

3.Tzitzéica's invariant for Submanifolds

Lemma 3.1. *Let M be C^∞ -submanifold of E^n and $M \cap W$ be a neigehborhood of $p \in M$. if $\alpha \neq 0$ on U, there is a unique normal vector field $\vec{\xi} \in \chi(F)$ such that*

$$< \vec{F},\ \vec{\xi} > \circ F = 1, \qquad < x_i \circ F,\ \vec{\xi} > \circ F\ =\ 0,\ 1 \leq i \leq m \qquad (1)$$

hold, where $\alpha = \vec{F} \wedge x_1 \circ F \wedge ... \wedge x_m \circ F \in \Lambda^{m+1}\chi(F)$ and $x_i \circ F = F_*(\frac{\partial}{\partial u_i}) \circ F$.

Proof. Consider $< \alpha,\ \alpha > > 0$ and the matrix $B = X.X^t$ for $X = [F\ x_1 \circ F ...\ x_m \circ F]$, then we find a vector field

$$\vec{\xi} = (\frac{B^{oo}}{detB})\vec{F} + \Sigma_{i=1}^m (\frac{B^{oi}}{detB})(x_i \circ F),$$

where $B^{oo} = cofac\ < \vec{F},\ \vec{F} > \circ F$ and $B^{oi} = cofac < \vec{F},\ x_i \circ F > \circ F$. It is known that $\vec{\xi}$ is a C^∞ -vector field over F, since $detB \neq 0$, $B^{oi} \in C^\infty(U)$, $1 \leq i \leq m$. $\vec{\xi}$ is related to F and so $\vec{\xi}$ is a unique vector field over F. If we consider the cofactor of B and the properties of a determinant function, we obtain $\vec{\xi}$ such as in (1) .By the last equalities of (1), $\vec{\xi}$ is a normal vector field over F

Lemma 3.2. *Let U, V be open subsets of E^m and ∇, ∇' affine connexions on the diffeomorphisms F, $F \circ \Phi^{-1}$, respectively.Then, there exists a unique $\vec{\xi'} \in \chi(F \circ \Phi^{-1})$ for the $\vec{\xi} \in \chi(F)$ such that $\vec{\xi'} = \xi \circ \vec{\Phi}^{-1}$. Furthermore, we have the following relation*

$$\nabla_{\frac{\partial}{\partial u_i}} \vec{\xi'} = (\ \nabla_{\Phi_*(\frac{\partial}{\partial u_i})}\ \vec{\xi'}) \circ \Phi$$

Proof. If we consider the following diagrams

$$\begin{array}{ccc} & F & \\ U & \longrightarrow & E^n \\ \Phi \downarrow & \nearrow & \\ & & F \circ \Phi^{-1} \\ V & & \end{array}$$

$$\begin{array}{ccc} & \vec{\xi} & \\ U & \longrightarrow & TE^n \\ \Phi^{-1} \uparrow & \nearrow & \\ & & \xi \circ \vec{\Phi}^{-1} \\ V & & \end{array}$$

the proof of the Lemma is clear.

Theorem 3.3. *If M is an m-dimensional submanifold of E^n and $\alpha \neq 0$ on U then, there is a unique function $\tau_{M\cap W} \in C^\infty(M \cap W,\ R)$ such that $\tau_{M\cap W} \circ F = \frac{<\alpha,\ \beta>_{m+1} \circ F}{<\alpha,\ \alpha>_{m+1} \circ F}$ for $\beta = \vec{\xi} \wedge \nabla_{\frac{\partial}{\partial u_1}} \vec{\xi} \wedge \nabla_{\frac{\partial}{\partial u_2}} \vec{\xi} \wedge \ldots \wedge \nabla_{\frac{\partial}{\partial u_m}} \vec{\xi}$.*

Proof. Firstly, if we show that $M \cap W$ has the parametrization F such that $\alpha \neq 0$ on U then, every parametrization of $M \cap W$ has the same property.To this end we choose another parametrization $\mathcal{G}$ of $M \cap W$. Since $\alpha \neq 0$ on U, we have the diffeomorphism Φ as in the diagram

$$\begin{array}{ccc} & F & \\ U & \longrightarrow & M \cap W \\ \Phi \uparrow & \nearrow & \\ & & \mathcal{G} \\ V & & \end{array}$$

It is easy to see that

$$\vec{\mathcal{G}} \wedge (\mathcal{G}_*(\frac{\partial}{\partial v_1}) \circ \mathcal{G}) \wedge \ldots \wedge (\mathcal{G}_*(\frac{\partial}{\partial v_m}) \circ \mathcal{G}) \neq 0,$$

since $x_i \circ F = (\mathcal{G}_*(\frac{\partial}{\partial v_m}) \circ \mathcal{G})$ for $\Phi_*(\frac{\partial}{\partial u_i}) = \frac{\partial}{\partial v_i}$. Now by using Lemma 3.2, we obtain the equality

$$\tau_{M\cap W} \circ F = \frac{< \alpha',\ \beta' >_{m+1} \circ (\mathcal{G} \circ \Phi)}{< \alpha',\ \alpha' >_{m+1} \circ (\mathcal{G} \circ \Phi)} = \tau_{M\cap W} \circ (\mathcal{G} \circ \Phi),$$

where $\alpha' = \vec{\mathcal{G}} \wedge (\mathcal{G}_*(\frac{\partial}{\partial v_1}) \circ \mathcal{G}) \wedge ... \wedge (\mathcal{G}_*(\frac{\partial}{\partial v_m}) \circ \mathcal{G})$, $\beta' = \vec{\xi'} \wedge \nabla'_{\frac{\partial}{\partial v_1}} \vec{\xi'} \wedge \nabla'_{\frac{\partial}{\partial v_2}} \vec{\xi'} \wedge ... \wedge \nabla'_{\frac{\partial}{\partial v_m}} \vec{\xi'}$.

From definition 2.3 and $\alpha \neq 0$ on U, it is routine to check that $\tau_{M \cap W} \in C^{\infty}(M \cap W,\ R)$.This completes the proof of theorem.

Theorem 3.4. *The function $\tau_{M \cap W}$ is an invariant of $O(n)$ in the case of $m < n$, and it is an invariant of the absolute unimodular group in the case $m = n\text{-}1$.*

Proof. Let Ψ be a linear transformation which corresponds to $g \in GL(n,\ R)$.Then we get a C^{∞}-map

$$\tilde{\Psi} \circ F : U \to \tilde{\Psi}(M \cap W),$$

according the diagram

$$\begin{array}{ccc} & \tilde{\Psi} \circ F & \\ U & \longrightarrow & M \cap W \\ F \downarrow & \nearrow & \\ & & \tilde{\Psi} \\ M \cap W & & \end{array}$$

where $\tilde{\Psi}$ is a restriction of Ψ on $M \cap W$. By the equalities $\tilde{\Psi}_* = g,\ g(x_i \circ F) = (gx_i) \circ F,\ \ \tilde{\Psi}_*(\vec{F}) = g(\vec{F})$, we find a unique $\vec{\xi'} \in \chi(\tilde{\Psi} \circ F)$ such that

$$< g(\vec{F}),\ \vec{\xi'} > \circ F = 1\ ,\ < g(x_i \circ F),\ \vec{\xi'} > \circ F = \ 0,\ 1 \leq i \leq m \quad (2)$$

Thus we may take the following system instead of (2).

$$< \vec{F},\ g^t\vec{\xi'} > \circ F = \ 1\ ,\ < x_i \circ F,\ g^t\vec{\xi'} > \circ F = \ 0,\ 1 \leq i \leq m \quad (3)$$

Taking into account Lemma 3.1 and (3), we realize that

$$\vec{\xi} = g^t\vec{\xi'}$$

holds.By getting help from the Definition 2.5 and $(g^t)^{-1}(\nabla_{\frac{\partial}{\partial u_i}} \vec{\xi}) = (\nabla_{\frac{\partial}{\partial u_i}} \vec{\xi'})$, we have

$$\tau_{\tilde{\Psi} M \cap W} \tilde{\Psi} \circ F = \frac{< \alpha,\ \beta >_{m+1} \circ F}{< \alpha,\ c\alpha >_{m+1} \circ F},$$

where $c = g^t g$. If $g \in O(n)$, then we see that the equality

$$\tau_{\tilde{\Psi} M \cap W} \tilde{\Psi} \circ F = \tau_{M \cap W} \circ F$$

holds.Thus $\tau_{M \cap W}$ is the invariant of O(n). Since $\Lambda^n c(\vec{F} \wedge x_1 \circ F \wedge \ldots \wedge x_m \circ F) = detc\alpha$, for m = n-1, we obtain

$$\tau_{\tilde{\Psi} M \cap W} \tilde{\Psi} \circ F = \frac{< \alpha,\ \beta >_n \circ F}{detc < \alpha,\ \alpha >_n \circ F} = \frac{\tau_{M \cap W} \circ F}{(detg)^2} \tag{4}$$

If $| detg |= 1$ in (4) then we have $\tau_{\tilde{\Psi} M \cap W} \tilde{\Psi} \circ F = \tau_{M \cap W} \circ F$, that is, $\tau_{M \cap W}$ is an invariant of absolute unimodular group, as required.

Let M be an m-dimensional submanifold of E^n.Then, we call the function $\tau_{M \cap W}$ in Theorem 3.3 , as *Tzitzéica's invariant.*We also call M as *Tzitzéica's submanifold* if $\tau_{M \cap W}$ is constant for all region $M \cap W$ of M.

Theorem 3.5. *Let M be an m-dimensional submanifold in E^n and $\alpha \neq 0$ on U.Then the equality*

$$\tau_{M \cap W} \circ F = (-1)^m \frac{G(.,\ N) \circ F}{d^{m+2} \circ F}$$

is valid, where N is the unit normal in direction $\vec{\xi}$ of affine subspace and G(., N) is the function of Lipschitz-Killing curvature in direction N.

Proof. Let D be a connexion of E^n and ∇ be an affine connexion of F.Then we have the following three relation

$$(D_{x_i} N) \circ F = \nabla_{(\frac{\partial}{\partial u_i})} N \circ F \tag{5}$$

$$\nabla_{(\frac{\partial}{\partial u_i})} (\frac{\vec{\xi}}{\| \vec{\xi} \|}) = (\frac{\partial}{\partial u_i} [\frac{1}{\| \vec{\xi} \|}]) \vec{\xi} + \frac{1}{\| \vec{\xi} \|} \nabla_{(\frac{\partial}{\partial u_i})} \vec{\xi} \tag{6}$$

$$(D_{x_i} N) \circ F = -S_N(F)(x_i \circ F) + (D^{\perp}{}_{x_i} N) \circ F \tag{7}$$

By using (5), (6), we obtain

$$\langle (D_{x_i} N) \circ F,\ x_j \circ F \rangle \circ F = -\langle S_N(F)(x_i \circ F),\ x_j \circ F \rangle \circ F, \tag{8}$$

$$\langle (D_{x_i}N)\circ F,\ x_j\circ F\rangle\circ F=\frac{1}{\|\vec{\xi}\|}\langle\nabla_{(\frac{\partial}{\partial u_i})}\vec{\xi},\ x_j\circ F\rangle\circ F \tag{9}$$

By following (8), (9), we find that

$$-\|\vec{\xi}\|\langle S_N(F)(x_i\circ F),\ x_j\circ F\rangle\circ F=\langle\nabla_{(\frac{\partial}{\partial u_i})}\vec{\xi},\ x_j\circ F\rangle\circ F \tag{10}$$

holds.By considering the definition of $\tau_{M\cap W}\circ F$, we get

$$\tau_{M\cap W}\circ F=\frac{det[<\nabla_{(\frac{\partial}{\partial u_i})}\vec{\xi},\ x_j\circ F>\circ F]}{<\alpha,\ \alpha>_{m+1}\circ F} \tag{11}$$

If we put (10) in (11), we find that

$$\tau_{M\cap W}\circ F=(-1)^m(\|\vec{\xi}\|)^m\frac{det[<S_N(F)(x_i\circ F),\ x_j\circ F>\circ F]}{<\alpha,\ \alpha>_{m+1}\circ F} \tag{12}$$

If we use the Definition 2.3, 2.4 and 2.5 in (12), we obtain that

$$\tau_{M\cap W}\circ F=(-1)^m(\|\vec{\xi}\|)^m detS_N(F)\frac{<\Omega,\ \Omega>_m\circ F}{<\alpha,\ \alpha>_{m+1}\circ F} \tag{13}$$

where $\Omega=x_1\circ F\wedge\ldots\wedge x_m\circ F$. By getting help from Lemma 3.1 and Theorem 3.3, we reach

$$\tau_{M\cap W}\circ F=(-1)^m(\|\vec{\xi}\|)^{m+2}detS_N(F) \tag{14}$$

On the other hand, it is not difficult to calculate that

$$d\circ F=\frac{1}{\|\vec{\xi}\|} \tag{15}$$

Finally, by using (15) in (14), we have

$$\tau_{M\cap W}\circ F=(-1)^m\frac{G(.,\ N)\circ F}{(d^{m+2})\circ F}$$

which makes end the proof of theorem.

References

[1] Tzitzéica, M.G., Sur quelques propriétés affines, Bull.Math.-Physique, E.P.Bucarest VI, 1935.

[2] Putinar, M., Sur les courbes et hypersurfaces de Tzitzéica, Bull.Math.de la Soc.Sci.Math.de la R.S, de Roumanie, Tome 29(71).nr.4, 1979.

[3] Milnor, W.J., Topology from the Differentiable Viewpoint, The University Press of Virginia Charlotesville, 1981

[4] Flanders, H., Differential Forms with Applications to the Physical Sciences, Academic Press, New York, 1963.