The Reflections On Hyperboloidal Model
of Hyperbolic Plane $H^2$

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ABSTRACT

The purpose of the present paper is to define two types of reflections on $H^2$ and to obtain some geometric results.

ÖZET


1. Introduction. The high point of the prehistory of the model $H^2$ is Lambert’s idea of an "imaginier sphere " in 1766. "Hyperbolic geometry” introduced by Klein in 1871. The parametrization of $H^2$ is used as Weierstrass coordinates by Killing [8],[9]. Meanwhile, in 1881, that parametrization is named "hyperbolic coordinates " by Poincarè [11]. The conformal disc model of hyperbolic space and the quadratic geometries on arbitrary quadratic surfaces in 3-space are developed by Poincarè[12]. Some unpublished works of Poincarè in 1880 related to $H^2$ are discussed by Gans [3]. $H^2$ is studied by Minkowski [10], Sommer-
feld [14], Varičak [18] in special relativity without mentioning hyperboloidal model. In [6], hyperboloidal model is studied by Jansen. Recently, $H^2$ appeared in Reynold's paper [13]. Kinematics of $H^2$ is studied by Garnier [4], Frank [2], Tölke [17] and the other many authors.

Tits has proved [15],[16] every finitely generated Coxeter groups can be represented as a group of projective maps generated by reflections and acting discretely in some domain of projective space. An algebraic description of all representations of this form has been obtained by Vinberg [20], [21]. Among the non-Euclidean Coxeter groups hyperbolic ones has also been studied using projective sphere by Vinberg [22].

The purpose of the present paper is to define two types of reflections on $H^2$ and to obtain some geometric results. We can summarize some important properties as following:

(i) The product of two reflections along $H^1$ is a translation along $H^1$ through twice the $H$-distance between their reflecting lines. One clearly see that this is an anologue of well-known proposition "The product of two Euclidean reflections is a rotation through twice the angle between their reflecting lines".

(ii) The product of two reflections along $H^1$ is a translation through the difference areas of two hyperbolic sectors which are one vertex origin and the other vertices on $H^1$.

(iii) In $H^2$, the order of subgroups generated by two reflections along $H^1$ is infinite.

(iv) Reflection along any $H$-line is determined uniquely by reflection along $H^1$.

2. Minkowski 3-Space and Hyperboloidal Model In this section, we give some definitions and the lemma due to [13].
Let $V$ be real vector spaces with three dimensional and $q$ be the real valued function on it such that

$$q : V \to \mathbb{R},$$

$$q(x) = q\left(\sum_{i=0}^{2} x_i e_i\right)$$

$$= -x_0^2 + x_1^2 + x_2^2$$

for some basis $\{e_0, e_1, e_2\}$ of $V$. Then, the Minkowski 3-space is denoted by $M^3 = (V, q)$.

The set $H^2 = \{x \in M^3 \mid q(x) = -1, x_0 > 0\}$ is called hyperboloidal model of hyperbolic plane. The elements of $H^2$ are called H-points when considered in $H^2$. We consider them as either points or vectors when considered in $M^3$. The H-lines are defined being all nonempty intersections of $H^2$ with two dimensional subspaces of $M^3$. Each pair of distinct H-points $A$ and $B$ lie on a unique H-line, namely the intersection of $H^2$ with the plane OAB of $M^3$ two distinct H-lines have one or zero H-points in common according to the line of intersection of the planes of $M^3$ in which they lie intersects $H^2$ or not. The complement of each H-line in $H^2$ consist of two H-half plane which are called its sides. From now on, we get $M^2 = sp\{e_0, e_1\}$, $H^1 = M^2 \cap H^2$. Let $A, B$ be two elements of $H^1$ such that $A = x_0 e_0 + x_1 e_1, B = y_0 e_0 + y_1 e_1, x_1 < y_1$ and $F$ be one to one smooth mapping of the intervals $x_1 \leq t \leq y_1$, then the distance $A$ and $B$ is defined by

$$d_{H^1}(A, B) = \int_{x_1}^{y_1} \sqrt{q(F'(t))} dt$$

$$= \text{arcsinh} y_1 - \text{arcsinh} x_1$$
The parametrization of $H^1$ by H-length is

$$h(s) = \cosh s \ e_0 + \sinh s \ e_1, \ -\infty < s < +\infty$$  \hspace{1cm} (1)

The isometries of $M^3$ is expressed, in terms of matrices, $O_1(3)$. We denote $G, G_1, G_0$ the subgroups of $O_1(3)$ such that fix $H^2, H^1$ and $e_0$ respectively. Then we see that

$$g = L_s E_1^i E_2^j, \ 0 \leq i, j \leq 1, g \in G_1$$  \hspace{1cm} (2)

and

$$g = R_\theta E_2^j, \ 0 \leq j \leq 1, g \in G_0$$  \hspace{1cm} (3)

where

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$L_s = \begin{pmatrix} \cosh s & \sinh s & 0 \\ \sinh s & \cosh s & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{pmatrix}$$

By (2) and (3), we have $G_1 \cap G_0 = \{ E_1, E_2 \}$.

If the matrix representation of the parametrization (1) is $\tilde{h}(r)$, then

$$R_\theta \tilde{h}(r) = \tilde{H}(r, \theta)$$

and

$$H(s, \theta) = \cosh s \ e_0 + \sinh s \cos \theta \ e_1 + \sinh s \sin \theta \ e_2, \ -\infty < s < +\infty$$  \hspace{1cm} (4)
(4) is the parametrization of $H^2$ and

$$L_r \mathcal{H}(s,0) = \mathcal{H}(s+r,0).$$ \hspace{1cm} (5)

The matrix $L_r$ is called the matrix of H-translation along $H^1$ \cite{1},\cite{13}. Since the matrix $L_r$ plays a role of Euclidean rotation in the motions of $M^2$, it is also called Minkowskian rotation \cite{19}. Since $R_\theta \mathcal{H}(s,\phi) = \mathcal{H}(s,\phi + \theta)$, $R_\theta$ is considered as a matrix of rotation about $e_0$. The H-distance between A and B in $H^2$ is given by

$$d_H(A, B) = \text{arcosh}(-p(A, B))$$ \hspace{1cm} (6)

where

$$p(x, y) = \frac{1}{2}[q(x+y) - q(x) - q(y)].$$

The H-angle $\angle BAC$ is $\overline{AB} \cup \overline{AC}$ where $\overline{AB} \not= \overline{AC}$. Its measure $m(\angle BAC)$ is defined as follows: Let $V, W$ be the vectors in $M^3$ tangent to $\overline{AB}, \overline{AC}$ respectively at A such that $q(V) = q(W) = 1$, then

$$m(\angle BAC) = \text{arccos}(p(V, W)).$$ \hspace{1cm} (7)

**Lemma 2.1** Let $l_j A \bar{B} \not= \overline{AC}$ be H-line, $A_j \bar{B}_j$ a ray on $l_j, 1 \leq j \leq 2$ and $S_j$ a side of $l_j$. Then there exists exactly one $T \in G$ such that $T(A_1) = A_2, T(l_1) = l_2, T(A_1 \bar{B}_1) = A_2 \bar{B}_2$ and $T(S_1) = S_2$.

**Proof.** It follows from Theorem 8 \cite{13}.

**3. Reflections along $H^1$** Now, we define a reflection along $H^1$. Since it is an analogue of the half turn in Euclidean Kinematics \cite{1}, it may be called half translation along $H^1$ in $H^2$. 

Let \( t, \dot{t}, \ddot{t} \) be lines which are tangent to \( H^1 \) at \( h(s_0) \) in (1), passing through \( O \) and parallel to \( t \), passing through \( x \) and parallel to \( t, \dot{t} \) respectively. Let \( \ell \) be the line passing through \( O, h(s_0) \) and \( \mathcal{H} \) be the intersection point of \( \dot{t}, \ddot{t} \). If \( \tilde{S}_{h(s_0)}(x) \) is a point such that it has distance \( \sqrt{q(h - x)} \) to \( x \) and on different side with \( x \) with respect to \( \ell \) in \( \mathcal{H} \) which is a plane determined by \( \ell, x \), then

\[
\tilde{S}_{h(s_0)}(x) = x - 2p(x, h(s_0))h(s_0)
\]
or

\[
S_{h(s_0)} = \begin{pmatrix}
\cosh 2s_0 & -\sinh 2s_0 & 0 \\
\sinh 2s_0 & -\cosh 2s_0 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

The above matrix \( S_{h(s_0)} \) is called reflection matrix along \( H^1 \). From now on, we mean dot derivative of.

**Theorem 3.1** Let \( S_{h(s)}, S_{h(t)} \) be two reflection along \( H^1 \) and \( \lambda \in \mathbb{R}, g \in O_1(3) \). Then

(i) \( S_{h(s)}^2 = I_3 \),

(ii) \( S_{h(s)} S_{h(t)} S_{h(s)} = S_{S_{h(s)} h(t)} \),

(iii) \( det S_{h(s)} = -1 \),

(iv) \( S_{\lambda h(s)} = S_{h(s)} \),

(v) \( S_{h(s)} \in G_1 \),

(vi) \( g S_{h(s)} g^{-1} = S_{g h(s)} \).
(vii) $S_{h(s)}(h(t)) = h(2s - t)$,

(viii) $S_{h(s)}'(h(t)) = -h'(2s - t)$

Proof. (i), (ii), (iii), (iv), (vi) are the same as Semi-Euclidean reflections and may be seen in [7]. (v) is evident by (3) and $S_{h(s)} = L_{2s}E_1$. (vii), (viii) are obtained by matrix calculation.

**Theorem 3.2** The product of two reflections along $H^1$ is a translation along $H^1$ through twice the H-distance between their reflecting lines.

Proof. It is obvious by definition of reflection along $H^1$ and translation along $H^1$.

Then we have the following result.

**Corollary 3.3** Let $A_1, A_2$ be areas of hyperbolic sectors in $M^3$ which are one vertex origin and the other vertices on $H^1$, then the product of two reflections along $H^1$ is a translation through the $A_2 - A_1$.

**Theorem 3.4** Let $S_{h(s)}, S_{h(t)}$ be two reflection along $H^1$. Then

(i) $S_{h(s)}L_{2t} = S_{h(s-t)}$

(ii) $L_{2t}S_{h(s)} = S_{h(s+t)}$

(iii) $S_{h(s)}L_{-t}S_{h(s)} = L_{-t}$

(iv) $(S_{h(s)}S_{h(t)})^p = S_{h(ps)}S_{h(pt)}, p ∈ R$

Proof. It follows from Theorem 3.2 and $L_{t}L_{s} = L_{s+t}$.

Now we can say that the product of a translation and a reflection along $H^1$ is another reflection along $H^1$. 
Corollary 3.5 Let \( S_{h(s)}, S_{h(t)} \) be two reflection along \( H^1 \). Then

\[
(S_{h(s)} S_{h(t)})^m = L_{2m(s-t)}, m \in \mathbb{R}
\]

Proof. It can be seen from Theorem 3.2 and the definition of translation along \( H^1 \).

Theorem 3.6 If we take \( l_1 = H^1 \) in Lemma 2.1, then reflection along any \( H \)-line is determined uniquely by reflection along \( H^1 \).

Proof. By Lemma 2.1, there is a unique \( g \in G \) such that \( g(H^1) = H \). By (iv) of Theorem 3.1, \( gS_{h(s)}g^{-1} = S_{gh(s)} \). The proof is complete. \( \blacksquare \)

4. Reflections Along the s-Curves In this section we give another reflections in \( H^2 \). This kind of reflections in \( H^2 \) are obtained by the curve \( s = \text{constant} \) in (4). Therefore, we call such reflections as reflections along the s-curves.

If we take \( \alpha(\theta) = \cosh s \ e_0 + \sinh s \cos \theta \ e_1 + \sinh s \sin \theta \ e_2 \) for \( s = \text{constant} \), then, we have

\[
\overline{S}_\alpha(\theta) = \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos 2\theta & \sin 2\theta \\
0 & \sin 2\theta & -\cos 2\theta
\end{pmatrix}
\]  

(9)

We call the reflection matrix along the s-curves to (9).

Theorem 4.1 Let \( \overline{S}_{\alpha(\theta_1)}, \overline{S}_{\alpha(\theta_2)} \) be two reflections along the s-curves. Then

(i) \( \overline{S}_{\alpha(\theta)}^2 = I_3 \),

(ii) \( \overline{S}_{\alpha(\theta_1)} \overline{S}_{\alpha(\theta_2)} \overline{S}_{\alpha(\theta_1)} = \overline{S}_{\overline{S}_{\alpha(\theta_1)}}(\alpha(\theta_2)) \),

(iii) \( \det \overline{S}_{\alpha(\theta)} = -1 \),
(iv) $\overline{S}_{\lambda \dot{\alpha}(0)} = \overline{S}_{\dot{\alpha}(0)}, \lambda \in R,$

(v) $\overline{S}_{\dot{\alpha}(0)} \in G_0$

(vi) $g \overline{S}_{\dot{\alpha}(0)} g^{-1} = \overline{S}_{y \dot{\alpha}(0)}, g \in O_1(3),$

(vii) $\overline{S}_{\dot{\alpha}(\theta_1)} \dot{\alpha}(\theta_2) = -\dot{\alpha}(2\theta_1 - \theta_2)$

(viii) $\overline{S}_{\dot{\alpha}(\theta_1)}(\alpha(\theta_2)) = \alpha(2\theta_1 - \theta_2)$

Proof. (i), (ii), (iii), (iv), (vi) is same with Euclidean reflections which is seen in [5]. (v) is evident by (9) and $\overline{S}_{\dot{\alpha}(0)} = R_{2\theta} E_2.$ the others are followed by matrix calculations $\blacksquare$

**Theorem 4.2** Let $\overline{S}_{\dot{\alpha}(\theta_1)}, \overline{S}_{\dot{\alpha}(\theta_2)}$ be two reflections along the s-curves. Then

$$\overline{S}_{\dot{\alpha}(\theta_1)} \overline{S}_{\dot{\alpha}(\theta_2)} = R_{2(\theta_1 - \theta_2)}$$

Proof. It follows from (9) $\blacksquare$

**Theorem 4.3**

$$(S_{h(s)} \overline{S}_{\dot{\alpha}(0)})^m = I_3$$ if and only if $s = 0$ and $\theta = \frac{m-2}{2m} \pi.$

Proof. If $(S_{h(s)} \overline{S}_{\dot{\alpha}(0)})^m = I_3,$ using matrix calculations, we see that $s$ must be zero. Since

$$S_{h(0)} \overline{S}_{\dot{\alpha}(0)} = R_{\pi - 2\theta}$$

we find

$$(S_{h(0)} \overline{S}_{\dot{\alpha}(0)})^m = R_{m(\pi - 2\theta)}$$ (10)
By (10), we see that $\theta$ must be $\frac{m-2}{2m}\pi$. Reverse of theorem is evident.

**Corollary 4.4** (i) If $m$ is even integer, then,

\[ (S_{h(0)}S_{\theta(\pi/m)})^m = I_3 \]

(ii) If $m$ is odd integer, then

\[ (S_{h(0)}S_{\theta(\pi/2m)})^m = I_3 \]

Proof. It is evident by Theorem 4.3

**Corollary 4.5**

(i) The order of the product any two reflections along $H^1$ is infinite

(ii) The order of the product any reflection along the $s$-curves $s \neq 0$ with $S_{h(0)}$ is an integer if $\theta = \pi/m$ or $\theta = \pi/2m$ where $m$ is order of product.

(iii) The order of the product any two reflections along the $s$-curves is an integer if $\theta_2 = \theta_1 + \pi/m$ where $m$ is order of product.

Proof. It follows from Theorem 3.3, Corollary 4.4 and Theorem 4.2

**References**


