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SOME RELATIONS CONNECTED WITH THE ALGEBRAIC INVARYANT OF THE SURFACES CONNECTED WITH THE EXPONENTIAL FUNCTION OF REVOLUTION AROUND THE X_3 - AXIS IN E^3 AND L^3

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ABSTRACT

In this study, the shape operator of $(t\mathrm{COS}\theta, t\mathrm{SIN}\theta, e^{tA})$ surface is calculated where t is real parameters, $A \in \mathbb{R}_3^3$ is an anti - symmetric matrix, $\mathrm{COS}\theta = \cos\theta I_3$, $\mathrm{SIN}\theta = \sin\theta . I_3$, and $g(t) = e^{tA}$ is an (orthogonal) matrix which corresponds to an orthogonal mapping [6].

In addition the principal curvatures, the mean curvature, Gaussian curvature of this surface are obtained by making use of the matrix of this shape operator. The Gaussian curvature obtained by zero.

Further more, some relations are considered in the space L^3 (Lorentzian) which correspond to the curvatures defined in the space E^3 (Euclidean), and we obtain the relations of curvatures between the space L^3.

E^3 VE L^3 DE X EKSENİ ETRAFINDA DÖNEN ÜSTEL FONKSİYON İLE BAĞLANTILI YÜZEYLERİN CEBİRSEL İNVARYANTLARI İLE İLGİLİ BAZI BAGINTILAR

ÖZET

Bu çalışmada $(tCOS\theta, tSIN\theta, e^{tA})$ yüzeyinin şekil operatörü hesaplandı. Burada t reel parametre, $A \in \mathbb{R}^3_3$ bir antisimetrik matris, $COS\theta = \cos\theta I_3$, $SIN\theta = \sin\theta . I_3$, ve $g(t) = e^{tA}$ üstel dönüşüme karşılık gelen ortogonal bir matristir.

Bu şekil operatörünün matrisi kullanılarak yüzeyin Gaus eğriliği, ortalama eğriliği ve esas eğrilikleri hesaplandı. Sonuçta Gaus eğriliğinin sıfıra eşit olduğu elde edildi.

Bundan başka, E^3 uzayında tanımlanan eğriliklere karşılık gelen bazı bağıntılar L^3 uzayında göz önüne alınarak bu iki uzay arasında eğrilik için bazı bağıntılar elde edildi.

1. INTRODUCTION

We denote by O^2_n the 2-dimensional Riemannian manifold defined on the unit disk $D^2 : u^2+v^2 < 1$ in the uv - plane with the following metric:

$$ds^2 = (1 - u^2 - v^2)^{n-2} \{(1 - v^2)\, du^2 + 2\, uvdudv + (1 - u^2)\, dv^2\} \qquad (1.1)$$

which is called Otsuki manifold.

On the other hand, O^2_o is the hyperbolic plane H^2 of curvature -1, and (1.1) is the metric described in the Cayley - Kleins model of H^2. O^2_1 is the semi - sphere; $u^2 + v^2 + w^2 = 1$ and $w > 0$ and (1.1) is the metric described in the plane of the equator; $w = 0$ through the orthogonal projecttion.

As it is well known, some part of H^2 but not whole plane can be represented as a surface of revolution in Euclidean 3 - Space E^3.

Let IR^3 denote the Cartesian product IR x IR x IR where IR is the set of real numbers. On IR^3 with the canonical coordinates, x_1, x_2, x_3 the Euclidean 3 - space E^3 and Lorentzian 3 - space L^3 are defined by the metrices.

$$E^3 : ds^2 = dx_1^2 + dx_2^2 + dx_3^2 \quad , \quad L^3 : ds^2 = dx_1^2 + dx_2^2 - dx_3^2 \qquad (1.2)$$

respectively.

We denote the inner products, in E^3 and L^3 of any two vectors

$$X = \sum_{i=1}^{3} X_i \frac{\partial}{\partial x_i} \text{ and } Y = \sum_{i=1}^{3} Y_i \frac{\partial}{\partial x_i} \text{ by}$$

$$\langle X, Y\rangle = X_1Y_1 + X_2Y_2 + X_3Y_3$$

$$\langle X, Y\rangle_L = X_1Y_1 + X_2Y_2 - X_3Y_3 \qquad (1.3)$$

respectively, denote the symmetry of E^3 with respect to the x_2y_2 - plane by φ, and extend φ to vectors as follows:

$$\varphi(X) = X_1 \frac{\partial}{\partial x_1} + X_2 \frac{\partial}{\partial x_2} - X_3 \frac{\partial}{\partial x_3} \qquad (1.4)$$

Then we have

$$\langle X, Y\rangle_L = \langle X, \varphi(Y)\rangle = \langle \varphi(X), Y\rangle \quad [5] \qquad (1.5)$$

Proposition 1.1: For any X, Y $\in$ the set Γ (T(M)) of smooth cross sections of the tangent bundle T(M) of M at any regular point of M in L^3, we have

$$\widetilde{\nabla}_X Y = \nabla_X Y - \frac{A\,(X,\,Y)}{<e_3,\,e_3>} \mathrm{Proj}\,\varphi\,(e_3)$$

$$\widetilde{T}_X Y = \frac{A\,(X,\,Y)}{<e_3,\,e_3>} \varphi\,(e_3) \qquad (1.6)$$

$$\mathrm{Proj}\,(e_3) = < e_1,\, e_3 > e_1 + < e_2,\, e_3 > e_2.$$

where A(X, Y) is the 2nd fundamental form of M in E^3. [5]

Definition 1.1: $\exp : IR \times IR^n_n \rightarrow GL\,(n,\, IR) \subset IR^n_n$

$$(t,\, A) \rightarrow \exp\,(tA) = g\,(t)$$

or

$$g(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k$$

is called on exponential motion [3].

Definition 1.2: $<A,\, B> = tr\,(AB^T)$ is called an inner product where A, B $\in IR^n_n$ matrices [2].

Theorem 1.1: Let A be an anti - symmetric matrix and $n \in IN^+$. Then

i) If n is odd, A^n is an anti - symmetric matrix

ii) If n is even, A^n is a symmetric matrix

iii) The trace of ani symmetric matrix is zero [6].

2. The algebraic invaryant of the surface connected with the exponential function of revolution around the x3 - axis in E^3.

$\phi\,(t,\,\theta) = (t\,COS\theta,\, t\,SIN\theta,\, e^{tA})$ is represent a surface connected with the exponential functional of revolution around the x_3 - axis in E^3. Then,

Theorem 2.1: The shape operator of the surface connected to the exponential function of $M \subset E^3$ is

$$S = \begin{bmatrix} \dfrac{-\sqrt{3}\, A^2 g}{(3 - trA^2)^{3/2}} & 0 \\ 0 & \dfrac{-Ag}{t\sqrt{3}\,(3 - trA^2)^{1/2}} \end{bmatrix} \qquad (2.1)$$

Proof: The derivatives of the surface with respect to the parameters t, θ as follows:

$$V_1 = \frac{\Phi_t}{|\Phi_t|} = \frac{(COS\theta, SIN\,\theta, Ag)}{(3 - trA^2)^{1/2}}$$

and

$$V_2 = \frac{\Phi_\theta}{|\Phi_\theta|} = \frac{(-\,tSIN\theta, tCOS\theta, 0)}{t\sqrt{3}} = \frac{1}{\sqrt{3}}\ (-\,SIN\theta, COS\theta, 0),\ 0 \in IR^3_3$$

are obtained.

Here the system $\{V_1, V_2\}$ is an orthonormal basis of X (M) is the normal vector field of M.

The matrix of map weingarten for surfaces with respect to $\{V_1, V_2\}$ denoted as follows:

$$S = \begin{bmatrix} \dfrac{-\det(\Phi_{tt}, \Phi_t, \Phi_\theta)}{|\Phi_t|^3\, |\Phi_\theta|} & 0 \\ 0 & \dfrac{-\det(\Phi_{\theta\theta}, \Phi_t, \Phi_\theta)}{|\Phi_\theta|\, |\Phi_t|^3} \end{bmatrix} \qquad (2.2)$$

By considering (2.2) and definition 1.2 together, (2.1) is obtained.

Definition 2.1: The curvatures on surfaces are vector - valued functionals.

Conclusion 2.1: The fundamental curvatures of surface are

$$k_1(t, \theta) = \frac{-\sqrt{3}\, A^2 g}{(3 - trA^2)^{3/2}} \quad (2.3)$$

and

$$k_2(t, \theta) = - \frac{Ag}{t\sqrt{3}\,(3 - trA^2)^{1/2}} \quad (2.4)$$

Conclusion 2.2: The mean curvature of the surface corresponds to matrix.

Proof: Since H = tarece S, then

$$H = \text{trace } S = k_1 + k_2 = -\left[\frac{\sqrt{3}\, A^2 g}{(3 - trA^2)^{3/2}} + \frac{Ag}{\sqrt{3}\,(3 - trA^2)^{1/2}\, t}\right] \quad (2.5)$$

Conclusion 2.3: The Gauss curvature K of the surface corresponds to a matrix and K=0.

Proof: The Gauss curvature K of the surface is

$$K = k_1\, k_2 = \frac{\langle A^2 g\ Ag \rangle}{(3 - trA^2)^2 t} \quad (2.6)$$

where $\langle A^2 g, Ag \rangle$ from definition 1.2 and Teorem 1.1,

$$\begin{aligned} \langle A^2 g, Ag \rangle &= tr\left[(A^2 g)\,(Ag)^T\right] \quad , (Ag)^T = g^T.A^T \ , A^T = - A \\ &= tr\left[A^2 g\, g^T (- A)\right] \quad , g \in 0_{(n)} \\ &= tr\,(A^3) \\ &= 0 \end{aligned}$$

So, K is zero.

Theorem 2.2: The only surface of revolution with K = 0 are the right circular cylinder, the right circular cone, and the plane [4].

Now let us consider a surface connected with the exponential function of revolution around the x_3 - axis in L^3 given by

$\varnothing (t, \theta) = (t \, COS\theta, t \, SIN\theta, e^{tA})$ (2.7)

Take the orthonormal frame ($\varnothing (t, \theta)$, e_1, e_2, e_3^*) of E^3 given by

$$e_1 = \frac{1}{(3 - trA^2)^{1/2}} (COS\theta, SIN\theta, Ag)$$

$$e_2 = \frac{1}{\sqrt{3}} (- SIN\theta, COS\theta, 0) = \varphi(e_2)$$

$$e_3 = \frac{1}{\sqrt{3}\,(3 - trA^2)^{1/2}} (- Ag \, COS\theta, - Ag \, SIN\theta, I_3)$$

$$e_3^* = \sqrt{3}\, e_3$$

$$\varphi(e_3^*) = \frac{1}{(3 - trA^2)^{1/2}} (- Ag \, COS\theta, - Ag \, SIN\theta, -I_3)$$

from which we obtain

$$<e_3^*, e_3^*> = -1 / \mu = - <e_1, e_1>, \quad <e_2, e_2> = 1 \qquad (2.8)$$

where

$$\mu = (3 - trA^2) / (3 + trA^2)$$

Then putting

$$\widetilde{e_1} = \sqrt{\mu}\, e_1 , \ \widetilde{e_2} = e_2 , \ \widetilde{e_3} = \sqrt{\mu}\, \varphi(e_3^*) \qquad (2.9)$$

We see that ($\varnothing (t, \theta)$, $\widetilde{e_1}$, $\widetilde{e_2}$, $\widetilde{e_3}$) is an orthonormal frame of L^3 in the following sense;

$$< \widetilde{e_1}, \widetilde{e_1} > = < \widetilde{e_2}, \widetilde{e_2} > = - < \widetilde{e_3}, \widetilde{e_3} > = 1$$
$$< \widetilde{e_1}, \widetilde{e_2} > = < \widetilde{e_1}, \widetilde{e_3} > = < \widetilde{e_2}, \widetilde{e_3} > = 0$$

Proposition 2.1: The relation of curvtures between of surface connected with the exponential function of revolution around the x_3 - axis in L^3 and E^3 is given by.

$$\widetilde{k_1} = \varphi(\mu)\, k_1 \quad , \widetilde{k_2} = \psi(\mu)\, k_2$$

Where $\widetilde{k_1}$, $\widetilde{k_2}$ and k_1, k_2 respectively of surfaces in L^3, E^3 are curvatures.

Proof: Let us compute in principal curvatures $\widetilde{k_1}$ and $\widetilde{k_2}$ of this surface in L^3 by means of the frame $(\phi(t, \theta), \widetilde{e_1}, \widetilde{e_2}, \widetilde{e_3})$ stated above. Define the 2nd fundamental from $\widetilde{A}(X, Y)$ of this surface in L^3 by,

$$\widetilde{T_X} Y = \widetilde{A}(X, Y)\, \widetilde{e_3} \;,\; X, Y \in \Gamma(T(M)) \qquad (2.10)$$

From (1.6), (2.8), (2.9), (2.10) we can easily obtain.

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