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A SHEAF OF R-ALGEBRAS ON THE DUAL SET OF DUAL NUMBERS

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ABSTRACT:

In this paper, we defined presheaf of R-algebras by means of the dual continuous maps on D and we obtained the sheaf of R-algebras which corresponds presheaf and finally we got a general result for the sheaf of R-algebras as follows, for open set $U \subseteq D$,

$$CU \cong \Gamma(U,SD) \cong SD|DU \cong \Gamma\, SD|DU \cong \Gamma(U, \Gamma\, SD).$$

ÖZET:

Bu makalede, D dual sayıları üzerinde tanımlanan dual sürekli fonksiyonlar yardımıyla R-cebirlerinin ön-demeti tanımlanmış ve bu öndemete karşı gelen R-cebirlerinin demeti elde edilmiştir. Sonuç olarak herhangi $U \subset D$ açık altkümesi ve R-cebirlerinin demeti için,

$$CU \cong \Gamma(U,SD) \cong SD|DU \cong \Gamma\, SD|DU \cong \Gamma(U, \Gamma\, SD)$$

genel sonucu elde edildi.

1 INTRODUCTION

The notion of a sheaf has its origins in the analytic continuation of functions, as initiated in the 19th century and then H.Weyl formulated rigorously in his famous book while discussing the *idea* of the Riemann Surface. For several complex variables the study of domains of holomorphy and of the Cousin problems gradually led H.Cartan and K.Oka in the 1940's to study on a domain. J.Leray published the first general and explicit definition of sheaf on a space, described in term of the closed sets of that space. H.Cartan, building an the concept of Leray, rephrased the definition of sheaves in terms of open sets. Lazard introduced the equivalent definition of a sheaf on a space X as an étale map into X. The subtle equivalence between these two notions is a central motivation of topos theory [7]. J.P.Serre and others realized that such sheaves could be used only in algebraic geometry and that the construction of a sheaf on a space X could be proceed from sections s defined on objects U which were not necessary subset of X but simply mappings $U \rightarrow X$ from some other space U into X.

At the present sheaf finds its main applications in topology , analytic manifold and modern algebraic geometry, where it has been used with great success as a tool in the solution of several long-standing problems, see in [1, 4, 6].

The set of dual numbers has a topology which is defined a norm function in paper of Özdamar-Hacisalihoğlu [8]. This topology is obvious and useful rather than others on the set of dual numbers
used in [5] and [8]. Throughout this paper this topology will be considered. In this paper, we build of the foundation of sheaf theory to give a broad definition of real manifold on the set of dual numbers. In order to obtain that, we introduce a sheaf of R-algebras on the set of dual numbers D. For this reason, we define a presheaf of R *-algebras* on D.

2 NOTIONS AND DEFINITIONS

Let R denotes the set of real numbers. Then consider a product set:

$$D = R \times R = \{ (x,x^*) : x, x^* \in R \}$$

which is a commutative ring with identity, together with two binary operations:

$$(x,x^*) \oplus (y,y^*) = (x+y, x^*+y^*) \quad (\textit{addition})$$
$$(x,x^*) \otimes (y,y^*) = (xy, x^*y+y^*x) \quad (\textit{multiplication}).$$

This ring is denoted by D and called dual numbers ring and so for each element of D is called dual number. It has an element ε is called dual identity, we usually write ε for $(0,1)$ such that $\varepsilon^2 = 0$. Moreover the ring of dual numbers D is also a vector space together with a function

$$\Theta \quad : R \times D \to D$$
$$(x,x^*)) \to (\lambda x, \lambda x^*)$$

Thus the ring of dual number D is an R-algebra with those structures; $(D, \oplus, \otimes, \Theta, R, +, .)$ [5],[8]. Furthermore, there is a one-to-one correspondence between the R -algebra D and the set $\{ z=x+\varepsilon x^* \mid x,x^* \in R \ \varepsilon^2=0\}$. That is,

$$D \cong \{z=x+\varepsilon x^* \mid x,x^* \in R \ \varepsilon^2=0\}.$$

So we write $z=x+\varepsilon x^*$ for (x,x^*) and often denoted dual number by z . Some standard functions of $z=x+\varepsilon x^*$ are the real part $Re\ z=x$ and the dual part $Du\ z=x^*$. Özdamar-Hacisalihoğlu defined in [8] a norm on D as follows

$$||z||=|x|+|x|,$$

where $|\ |$ is a usual norm on R. As usual, this norm defines a metric on D; for every $z_1, z_2 \in D$

$$d_D(z_1, z_2) = \|z_1 - z_2\|.$$

Hence D is a topological space with this metric.
A function f with dual-valued is called dual function which is given by

$$f(z) = f(x+\varepsilon x) = f_1\ (x,x) + \varepsilon f_2\,(x,x)$$

for each $z \in D$, where f_1 and f_2 are real-valued functions. This topology with dual-valued function on D serves to define the continuous functions. This can, of course, be done axiomatically but the most important and interesting questions arise from study of subsets of the dual space D.
Now we can define continuous dual function which is given in [8].

Definition 2.1 Let f be the dual function and let $z_0 \in D$. We say that f is continuous at z_0, if for each $z \in D$ and $\varepsilon > 0$, there exists a $\delta > 0$ such that $\|z - z_0\| < \delta$ implies $\|f(z) - f(z_0)\| < \varepsilon$. Further, we call f is continuous dual function if it is continuous at each z point of D.

It is somewhat cumbersome to use this definition in practice, and it is usually more convenient to argue using some of the properties of continuity which are given in [8]. However, the definition of continuity is entirely in terms of neighbourhood we usually take the range of dual function to be D itself, but the definition and result are the same as for the general case, when the range is any subset of D.

In this case, we shall be concerned with the algebraic structure of the set CU of continuous dual functions from an open set U of D to D. For this reason, we have to give the following theorem.

Theorem 2.2 Let f and g be the continuous dual functions.Define addition and multiplication of f, g to be

$$(f+g)(z) = f(z) \oplus g(z)$$
$$(f.g)(z) = f(z) \otimes g(z)$$

for each $z \in D$. Then $f+g$ and $f.g$ are continuous
Proof: The continuity of $f+g$ and $f.g$ were proved in [8].

Let U be an open set in D. Let us consider the set CU of continuous dual functions from U to D, i.e.

$$CU = \{ f \mid f{:}U \to D \ \ \textit{continuous} \ \ U \ \ \textit{is open in} \ \ D \}.$$

If $f,g \in CU$ and $\lambda \in R$, then $f+g$ and $f.g$ in CU by Theorem 2.2. It is clear that CU is a commutative ring with identity, with addition and multiplication defined earlier. Indeed, CU is an R-algebra, since we can identity element of D with those operation for each $z \in U$;

$$(f+g)(z) = f(z) \oplus g(z)$$
$$(f.g)(z) = f(z) \otimes g(z)$$
$$(\lambda f)(z) = \lambda f(z).$$

3 PRESHEAF AND SHEAF ON D

Definition 3.1 [1, 9] Let X be a topological space. A presheaf P of sets on X is given by pieces of information;

(i) for each open set U of X, a set PU,

(ii) for each of open sets $V \subseteq U$ of X, a restriction map $r_{UV}: PU \to PV$ such that

1. $r_{UU} = I_U$ 2. $r_{VW} \circ r_{UV} = r_{UW}$, whenever $W \subseteq V \subseteq U$.

Thus, using functorial terminology we have the following definition. Let X be a topological space. A presheaf P on X is a functor from the category $O(X)^{op}$ of open subset of X and inclusions to the category *Sets* of sets and functions. Then the system $P = \{PU, r_{UV}, X\}$ is said to be presheaf of sets on X.

In general, we define a presheaf with values in an arbitrary category. If the presheaf is satisfying the following properties is said to be presheaf of *R-algebra* :

(i) every PU is an *R -algebra,*

(ii) for $V \subset U$, r_{UV} ,:$PU \to PV$ is an *R -algebra* homomorphism.

Corollary 3. 2 Let U be an open set in D and let $PU=CU$. Then the system $CD = \{CU, r_{UV}, D\}$ is a presheaf of *R-algebra.*

Proof: Let U, V be two open sets in D such that $V \subset U$. We can define a restriction map $r_{UV}: CU \to CV$, $f \mapsto f|V$ which is an *R-algebra* homomorphism. Indeed, for every open set U in D, CU is an *R-algebra.* Hence the system $CD = \{CU, r_{UV}, D\}$ is a presheaf of *R-algebra* on D. In other word, CD is a functor from category of open sets of X and inclusions to category of *R -algebras* and *R -algebras* homomorphisms, i.e.

$$CD: O(X)^{op} \to R\text{-}Alg\,.$$

Definition 3.3 Let S and X be two topological space and let p be a map over X. The trible (S,p,X) is called *sheaf* of sets over X if p is a local homeomorphism.

Let $CD = \{ CU, r_{UV}, D \}$ be a presheaf of *R -algebra* on D. Consider the set M consists of all elements $f \in CU$ for all open sets $U \subseteq D$ with $z \in U$. We define an equivalence relation on M as follows; The elements $f \in CU_1$ and

$g \in CU_2$ of M are equivalent if there is an open neighbourhood U of z in D and $U \subseteq U_1 \cap U_2$ with $r_{U_1\,U}(f) = r_{U_2\,U}(f)$. The equivalence classes of M with this equivalence relation are called the the germs of CD at $z \in U$ and denoted by $(U,f)_z = germ_z f$. Let S_z be the set of all germs. Now combine the various sets S_z of germs in the disjoint union SD (over $z \in D$).

$$SD = \bigcup_{z \in D} S_z = \bigcup_{z \in D} \{(U,f)_z \mid z \in U \subseteq D, open, f \in CU\}$$

and define a canonical projection p: $SD \to D$ as the map sending each $germ_z f = (U,f)_z$ to point z, i.e. $p(S_z) = z$.

SD will be provided with a topology such that p becomes local homeomorphism. Let $U \subset D$ open and let $f \in CU$, then each $f \in CU$ determines a function f^* by

$$f^*: U \to SD \quad f^*(z) = (U,f)_z,\ z \in U.$$

We also define

$$f^*(U) = \bigcup_{z \in U} (U,f)_z .$$

Topoligizing the set SD by taking as a base of open sets all the image $f(U) \subseteq SD$, i.e. the family

$$T = \{f^*(U) \mid U \subseteq D \ \ open,\ f \in DU\}$$

defines a topological base on SD.

Let $f^*_1(U_1), f^*_2(U_2) \in T$. If $f^*_1(U_1) \cap f^*_2(U_2) = \varnothing$, then $\varnothing \in T$, since $f^*(\varnothing) = \bigcup_{z \in \varnothing} (\varnothing,f)_z = \varnothing$. Suppose that $f^*_1(U_1) \cap f^*_2(U_2) \neq \varnothing$, then there is an element $\sigma \in f^*_1(U_1) \cap f_2(U_2)$ such that $p(\sigma) = z \in U_1 \cap U_2$. This gives an open neigbourhood of $z \in U \subseteq U_1 \cap U_2$ such that $\sigma = f^*_2(z) = f^*_1(z)$. For every $z \in U$, since $f^*(z) = f^*_1(z)$, $f^*(U) = f^*_1(U) \subseteq f^*_1(U_1) \cap f^*_2(U_2)$. So $f^*_i(U)$, $i = 1,2$ lies $f^*_1(U_1) \cap f^*_2(U_2)$ and σ is an interior point of $f^*_1(U_1) \cap f^*_2(U_2)$ [2].

Hence SD is a topological space with above topology. Now we have to show that p is a local homeomorphism with this topology.

For each $\sigma = (U,f)_z \in SD$, $z \in D$, there exists an open sets $\sigma \in W \subseteq SD$ and $p(\sigma) = z \in U \subseteq D$ such that $p|_W$; $W \to U$ is a topological map, whereas for $\sigma = (U,f)_z \in SD$, $p(\sigma) = p((U,f))_z = z$. Then, for $z \in U$, and let $f^*: U \to SD$ such that $f^*(z) = (U,f)_z = \sigma \in S_z$.

Let $W = f^*(U)$ and $p|_U = q$.

Firstly we will show that q is bijective. In fact, for $\sigma_1, \sigma_2 \in f^*(U) = W$, there are two element z_1, $z_2 \in U$ such that $\sigma_1 = f^*(z_1)$ and $\sigma_2 = f^*(z_2)$. If $q(\sigma_1) = q(\sigma_2)$, then $q(\sigma_1) = q(f^*(z_1)) = q(\sigma_2) = q(f^*(z_2)) = z_1 = z_2$. This implies $f^*(z_1) = f^*(z_2)$, i.e. $\sigma_1 = \sigma_2$.
The map q is continuous. Choose any point $\sigma \in W = f^*(U)$ such that $q(\sigma) = z \in U$. Then there is an open neighbourhood $z \in V \subseteq U$ such that $f^*(V) \subseteq W = f^*(U)$ is an open neighbourhood of σ and $q(f^*(V)) = V \subseteq U$. So q is continuous.
Now we shall show that $p^{-1} = (p|_W)^{-1} = f^* : U \rightarrow W = f^*(U)$ is continuous. If an arbitrary element $z \in U$, $f^*(z) = \sigma \in W$, $O \subseteq W$ is an open neigbhourhood of σ , then $(p|W)(O) \subseteq U$ is an open neigbhourhood of z in U and $f^*(p|_W) = O$. Hence f^* continuous.
We present here following theorem:

Theorem 3.4 Let $CD=\{CU, r_{UV}, D\}$ be a presheaf on D. Then CD defines a sheaf SD on D.

In the sheaf of germs of dual continuous functions there is no canonical isomorphism S_{z_i}, S_{z_j} for each $i \neq j$, but there is still a relation which tells us then $\sigma_1 \in S_{z_i}$ is the element associated with $\sigma_2 \in S_{z_j}$. Therefore the topology on SD define a neighbourhood of σ_1 to be the set of all elements associated with S_{z_i} and lying in stalks near σ_2.

Definition 3.5 Let SD be the sheaf on D. Let z be an arbitrary point in D and let U be a open neigbourhood of z. Then there is a map $f^*: U \rightarrow SD$ such that $p \circ f^* = DU$. This map f^* is called a section of SD over U. We denote the set of all sections of SD over U by $\Gamma(U, SD)$.

Now we list some elementary consequences of the sheaf SD and the set of sections $\Gamma(U, SD)$, for open set $U \subseteq D$:
(*i*) p is an open map.
(*ii*) Let $f^* : U \rightarrow SD$ be a map with $p \circ f^* = I_U$, for open set $U \subseteq D$. Then $f^* \in \Gamma(U, SD)$ if and only if f^* is open.
(*iii*) Let U be open in D and $f^* \in \Gamma(U, SD)$. Then $p : f^*(U) \rightarrow U$ is a homeomorphism and $f = (p|f^*(U))^{-1}$.
(*iv*) Let σ an arbitrary point in SD. Then there exists an open set $V \subseteq D$ and a section $f^* \in \Gamma(V, SD)$ with $\sigma \in f^*(V)$.

(v) For any two sections $f^*_1 \in\Gamma(U_1, SD)$ and $f^*_2 \in\Gamma(U_2, SD)$, U_1 and U_2 opens, the set U of points $x\in U\subseteq U_1 \cap U_2$ such that $f^*_1(x) = f^*_2(x)$ is open.

Theorem 3.6 The sheaf SD is a sheaf of *R-algebra* on D.

Proof :(1). For every $z\in D$, the stalk is a commutative *R-algebra* with identity ,together with the following operations;

$$(U, f_1)_z+(U, f_2)_z=(U, f_1+f_2)_z$$
$$(U, f_1)_z.(U, f_2)_z=(U, f_1\cdot f_2)_z$$
$$\lambda(U,f)_z=(U,\lambda f)_z$$

These operations are well-defined. It is easy to show that S_z is a commutative *R-algebra* with identity.

(2).Let $SD\times pSD=\{(\sigma_1,\sigma_2)| p(\sigma_1)=p(\sigma_2)\}$. Then the addition $SD\times pSD\to SD$, $(\sigma_1,\sigma_2)\mapsto\sigma_1+\sigma_2$, the multiplication $SD\times pSD\to SD$, $(\sigma_1,\sigma_2)\mapsto\sigma_1.\sigma_2$ and $R\times SD\to SD$, $(\lambda,\sigma)\mapsto\lambda.\sigma$ are continuous if and only if for each open U in D, $\Gamma(U,SD)$ is an R-algebra under point wise addition and multiplication of functions [5]. This will be showed in Corollary 3.7.

(3). Let I be an identity element of CU. Then $z\in D$, I defines a map $I^*:U\to SD$ with $I^*(z)=(U,I)_z\in SD$. Since $p\circ I^*=I_D$, $I^*\in\Gamma(D,SD)$. Hence SD is a sheaf of *R-algebra* on D.

Corollary 3.7 For each open set $U\subseteq D$, the set of sections $\Gamma(U,SD)$ is also an *R-algebra*.

Proof: For each $f^*_1, f^*_2\in\Gamma(U,SD)$, $\lambda\in R$

$(f^*_1+f^*_2)(z)=f^*_1(z)+f^*_2(z)=(U,f_1)_z+(U, f_2)_z=(U,f_1+f_2)_z$

$(f^*_1.f^*_2)(z)=f^*_1(z).f^*_2(z)=(U,f_1)_z.(U, f_2)_z=(U,f_1\cdot f_2)_z$

$\lambda f^*(z)=\lambda(U,f)_z=(U,\lambda f)_z$.

Hence $\Gamma(U,SD)$ is an *R-algebras*.

Also we can obtained a functor $\Gamma D:O(X)^{op}\to R\text{-}Alg$, that means ΓD is a presheaf of *R -algebra*.

Theorem 3.8 Let $CD=\{CU,C_{UV},D\}$ and $\Gamma D=\{\Gamma(U,SD),\Gamma_{UV}, D\}$ be the presheaves of *R-algebra*. Let SD and ΓSD be the corresponding sheaves of R-algebra of CD and ΓD, respectively. For every open $U\subseteq D$

$$CU \cong \Gamma(U,SD) \cong SD|U \cong \Gamma SD|U \cong \Gamma(U,\Gamma SD).$$

Proof: We define a map $\psi:CU\to\Gamma(U,SD)$, with $f\mapsto f^*$. If $\psi(f)=0^*$, we have to show that $f=0$. For every $z\in U$ we have $\psi(f(z))=O_z$, therefore $(U,f)_z=O_z$, that is, there exists a neighbourhood $V\subseteq U$ with $f|V=0$, in particular, $f(z)=0$. Therefore $f=0$.

If $f^*\in\Gamma(U,SD)$ then for every $z\in U$ there exists a neighbourhood $N\subseteq U$ and a continuous function f on U with $(N,f)_z=f^*$. Then there is a neighbourhood $V\subseteq N$ with $\psi(f|V)=f^*|V$.

Also ψ is an R-algebra homomorphism:

$\psi(f+g)=f^*+g^*=\psi(f)+\psi(g)$, $\psi(fg)=f^*g^*=\psi(f).\psi(g)$ and $\psi(\lambda f)=\lambda f^*=\lambda\psi(f)$.

Hence $CU\cong\Gamma(U,SD)$.

We can define a map, for each $z\in D$,

$$germ:CU\to SD|U,\quad f\mapsto germ(f)=germ_z(f).$$

This map is well-defined and it is easy to check the map *germ* is bijective. Now we need to show that *germ* is an R-algebra homomorphism. In fact,

$germ(f+g)=germ_z(f+g)=germ_z(f)+germ_z(g)=germ(f)+germ(g)$,

$germ(f.g)=germ_z(fg)=germ_z(f)germ_z(g)=germ(f)germ(g)$,

$germ(\lambda f)=germ_z(\lambda f)=\lambda germ_z(f)=\lambda germ(f)$.

Hence the map *germ* is an R-algebra isomorphism from CU to $SD|U$, i.e. $CU\cong SD|U$.

Now, we can obtain a sheaf ΓSD which is defined by the canonical presheaf $\Gamma D=\{\Gamma(U,SD),\Gamma_{UV},D\}$. This sheaf is canonically isomorphic to SD [3]. This isomorphism given by map $\phi:\Gamma SD|U\to SD|U$, $germ_z(f^*)=(U,f^*)_z \mapsto f^*(z)$, i.e. $\Gamma SD|U\cong SD|U$. In addition, it is possible to define an *R-algebra* isomorphism between $\Gamma SD|U$ and $\Gamma(U,\Gamma SD)$ as $\Gamma(U,SD)\cong SD|U$. That is,

$$CU \cong \Gamma(U,SD) \cong SD|U \cong \Gamma SD|U \cong \Gamma(U,\Gamma SD).$$

As a result of the theorem, we have;

Corollary 3.9

$$CD \cong \Gamma(D,SD) \cong SD \cong \Gamma SD \cong \Gamma(D,\Gamma SD).$$

Result: Theorem 3.8 can be generalized any sheaf of R–algebra with a topological space.

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