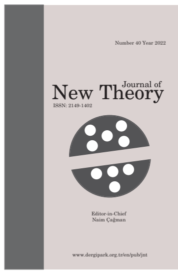
Journal of New Theory
ISSN: 2149-1402

40 (2022) 12-26
Journal of New Theory
https://dergipark.org.tr/en/pub/jnt
Open Access



Salkowski Curves and Their Modified Orthogonal Frames in $\mathbb{E}^3$

Sümeyye Gür Mazlum[1], **Süleyman Şenyurt**[2], **Mehmet Bektaş**[3]

Article Info
Received: 05 Jul 2022
Accepted: 21 Sep 2022
Published: 30 Sep 2022
doi:10.53570/jnt.1140546
Research Article

Abstract — In this study, we examine some properties of Salkowski curves in $\mathbb{E}^3$. We then make sense of the angle (nt) in the parametric equation of the Salkowski curves. We provide the relationship between this angle and the angle between the binormal vector and the Darboux vector of the Salkowski curves. Through this angle, we obtain the unit vector in the direction of the Darboux vector of the curve. Finally, we calculate the modified orthogonal frames with both the curvature and the torsion and give the relationships between the Frenet frame and the modified orthogonal frames of the curve.

Keywords — *Salkowski curves, modified orthogonal frame, Frenet frame*

Mathematics Subject Classification (2020) — 53A04, 53A55

1. Introduction

In differential geometry, the Frenet frame of a continuous differentiable regular curve in Euclidean space $\mathbb{E}^3$ describe the geometric properties of any point moving along the curve. If $\{T(t), N(t), B(t)\}$ is the Frenet frame of any regular curve α in Euclidean space $\mathbb{E}^3$, here, the vector $T(t)$ (resp. $N(t)$ and $B(t)$) is called tangent vector (normal vector and binormal vector) [1, 2]. In Euclidean 3-space, the curves whose principal normal vector makes a constant angle with a fixed direction are called slant helices. Kula et al. [3] studied on the slant helices in Euclidean space $\mathbb{E}^3$. Ali [4] obtained the position vectors of slant helices Euclidean space $\mathbb{E}^3$. In a paper published by Salkowski [5], a curve family, whose curvature is constant but torsion is not, has been defined. In a study done by Monterde [6], it is found out that base normal vectors of Salkowski curves make a constant angle with a constant line. The Frenet vectors and the geodesic curvatures of the spherical indicatrix curves belong to Salkowski curves in $\mathbb{E}^3$ are calculated in [7]. The Smarandache curves according to Frenet frame of Salkowski curves are studied in [8]. Moreover, the modified orthogonal frame of a space curve in Euclidean space $\mathbb{E}^3$ was described by Sasai [9, 10] as a useful tool for investigating analytic curves with singular points where the Frenet frame does not work. The authors in [11–14] obtained the modified orthogonal frame of a space curve and its spherical curves in Euclidean and Minkowski 3-space. Arıkan and Nurkan [15] had a paper on adjoint curve according to modified orthogonal frame with torsion in Euclidean 3-space. Furthermore, there are many studies on the frames of various curves or surfaces in Euclidean 3-space, [16–25]. In another study, we [26] examined the modified frames with both the non-zero curvature and the torsion of the non-unit speed curves in Euclidean space $\mathbb{E}^3$. In

[1]sumeyyegur@gumushane.edu.tr (Corresponding Author); [2]ssenyurt@odu.edu.tr; [3]mbektas@firat.edu.tr
[1]Department of Computer Technology, Kelkit Aydın Doğan Vocational School, Gümüşhane University, Gümüşhane, Türkiye
[2]Department of Mathematics, Faculty of Arts and Sciences, Ordu University, Ordu, Türkiye
[3]Department of Mathematics, Faculty of Science, Fırat University, Elazığ, Türkiye

this study, it has been shown that Salkowski curves in $\mathbb{E}^3$ is not a unit speed curve. Therefore, the torsion of Salkowski curves is investigated. The Frenet derivative formulas are obtained for this curve. Besides, the angle (nt) is made sense and the relationship between this angle and the angle between the binormal vector and the Darboux vector of Salkowski curves is given. By means of this angle, the unit vector in the direction of this vector and the Darboux vector of the curve is calculated. Finally, based on the work [26] done on calculating the modified orthogonal frame of a non-unit speed curve, the modified orthogonal frames with the both curvature and torsion of Salkowski curves are calculated. And relations between the modified orthogonal frames and the Frenet frame of Salkowski curves in Euclidean 3-space are given.

2. Preliminary

Let the curve $\alpha(t)$ be a differentiable space curve in $\mathbb{E}^3$. Frenet vectors (the tangent, binormal and principal normal vectors), the curvature and the torsion of this curve are given as follows:

$$T(t) = \frac{\alpha'(t)}{v(t)}, \quad B(t) = \frac{\alpha'(t) \wedge \alpha''(t)}{\|\alpha'(t) \wedge \alpha''(t)\|}, \quad N(t) = B(t) \wedge T(t) \tag{1}$$

$$\kappa(t) = \frac{\|\alpha'(t) \wedge \alpha''(t)\|}{v^3(t)}, \ \ and\ \tau(t) = \frac{\det\left(\alpha'(t), \alpha''(t), \alpha'''(t)\right)}{\|\alpha'(t) \wedge \alpha''(t)\|^2} \tag{2}$$

respectively. Here, $v(t) = \|\alpha'(t)\|$. Frenet derivative formulas of this curve [2] are as follows:

$$\begin{bmatrix} T'(t) \\ N'(t) \\ B'(t) \end{bmatrix} = \begin{bmatrix} 0 & v(t)\kappa(t) & 0 \\ -v(t)\kappa(t) & 0 & v(t)\tau(t) \\ 0 & -v(t)\tau(t) & 0 \end{bmatrix} \begin{bmatrix} T(t) \\ N(t) \\ B(t) \end{bmatrix} \tag{3}$$

The Darboux vector $W(t)$ of the non-unit speed curve $\alpha(t)$ is as follows:

$$W(t) = N(t) \wedge N'(t) = v(t)\left(\tau(t)T(t) + \kappa(t)B(t)\right) \tag{4}$$

The unit vector $C(t)$ in direction of the vector $W(t)$ of the non-unit speed curve $\alpha(t)$ is

$$C(t) = v(t)\left(\frac{\tau(t)}{\sqrt{\kappa^2(t) + \tau^2(t)}}T(t) + \frac{\kappa(t)}{\sqrt{\kappa^2(t) + \tau^2(t)}}B(t)\right) \tag{5}$$

or if the angle between of the vectors $B(t)$ and $W(t)$ of the curve $\alpha(t)$ is $\varphi(t)$, then the unit vector [27] is

$$C(t) = \sin\varphi T(t) + \cos\varphi B(t) \tag{6}$$

If $v(t) = 1$, then the curve $\alpha(t)$ is called unit speed curve. Let us create the following vectors for the curve $\alpha(t)$:

$$E_1(t) = \alpha'(t), \quad E_2(t) = E_1'(t), \quad E_3(t) = E_1(t) \wedge E_2(t) \tag{7}$$

These vectors form an orthogonal frame. If $\alpha(t)$ is a unit speed curve and its curvature is non-zero, we can write these vectors in terms of Frenet vectors of the curve as following [9]:

$$E_1(t) = T(t), \quad E_2(t) = \kappa(t)N(t), \quad E_3(t) = \kappa(t)B(t) \tag{8}$$

where,

$$\langle E_1(t), E_1(t)\rangle = 1, \quad \langle E_2(t), E_2(t)\rangle = \langle E_3(t), E_3(t)\rangle = \kappa^2(t) \tag{9}$$

In this case, this frame obtained for the curve is called a modified orthogonal frame with the curvature $\kappa(t)$ of the curve $\alpha(t)$. There are the following relationships between these vectors and them derivative vectors:

$$\begin{bmatrix} E_1'(t) \\ E_2'(t) \\ E_3'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\kappa^2(t) & \dfrac{\kappa'(t)}{\kappa(t)} & \tau(t) \\ 0 & -\tau(t) & \dfrac{\kappa'(t)}{\kappa(t)} \end{bmatrix} \begin{bmatrix} E_1(t) \\ E_2(t) \\ E_3(t) \end{bmatrix} \tag{10}$$

The parametric equation of Salkowski curves γ_m are as follows:

$$\begin{aligned} \gamma_m(t) \;=\; & \frac{1}{\sqrt{1+m^2}}\left(-\frac{1-n}{4(1+2n)}\sin\left((1+2n)t\right) - \frac{1+n}{4(1-2n)}\sin\left((1-2n)t\right) - \frac{1}{2}\sin t\,,\right. \\ & \frac{1-n}{4(1+2n)}\cos\left((1+2n)t\right) + \frac{1+n}{4(1-2n)}\cos\left((1-2n)t\right) + \frac{1}{2}\cos t, \\ & \left.\frac{1}{4m}\cos(2nt)\right) \end{aligned}$$

Figure 1. Here, $m \neq \pm\frac{1}{\sqrt{3}}, 0 \in \mathbb{R}$ and $n = \frac{m}{\sqrt{m^2+1}}$. Moreover, $\left\|\gamma_m'(t)\right\| = \frac{\cos(nt)}{\sqrt{m^2+1}}$; therefore, the curve is regular in the interval of $\left]-\frac{\pi}{2n}, \frac{\pi}{2n}\right[$. The curvature, the torsion and the Frenet vectors of the curve provided in [5,6] are as follows:

$$\begin{aligned} \kappa(t) &= 1 \\ \tau(t) &= -\tan(nt) \\ T(t) &= -\left(\cos t\cos(nt) + n\sin t\sin(nt)\,, \cos(nt)\sin t - n\cos t\sin(nt)\,, \frac{n}{m}\sin(nt)\right) \\ N(t) &= \frac{n}{m}(\sin t, -\cos t, -m) \\ B(t) &= -\left(\cos t\sin(nt) - n\cos(nt)\sin t, \sin t\sin(nt) + n\cos t\cos(nt)\,, -\frac{n}{m}\cos(nt)\right) \end{aligned}$$

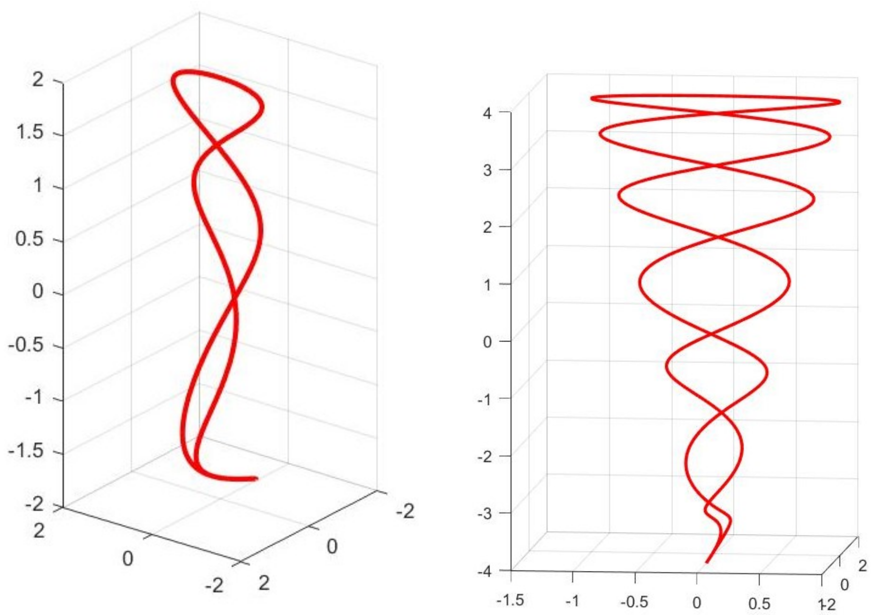


Fig. 1. Salkowski curves for $m = \frac{1}{8}$ and $m = \frac{1}{16}$

3. The Some Properties of Salkowski Curves in $\mathbb{E}^3$

In this section, we show whether Salkowski curves in $\mathbb{E}^3$ are a unit speed curve. We obtain the its torsion and the Frenet derivative formulas. Also, we make sense the angle (nt) and give the relationship between this angle and the angle between the binormal vector and the Darboux vector of Salkowski curves. By means of this angle, we get the Darboux vector of the curve and the unit vector in the direction of this vector.

3.1. Are Salkowski Curves in $\mathbb{E}^3$ Unit Speed Curves?

First, let us examine whether Salkowski curves are a unit speed curve. If we take first derivative according to the parameter t of Salkowski curves $\gamma_m(t)$, we get

$$\gamma_m'(t) = -\frac{n}{m}\cos(nt)\Big(\cos t\cos(nt) + n\sin t\sin(nt)\,, \cos(nt)\sin t - n\cos t\sin(nt)\,, \frac{n}{m}\sin(nt)\Big) \tag{11}$$

The norm of the Equation 11 is

$$\left\|\gamma_m'(t)\right\| = v(t) = \frac{n}{m}\cos(nt) = \frac{\cos(nt)}{\sqrt{m^2+1}} \tag{12}$$

If Salkowski curves were a unit speed curve, then we could write

$$\left\|\gamma_m'(t)\right\| = \frac{\cos(nt)}{\sqrt{m^2+1}} = 1$$

Hence, we get

$$\cos(nt) = \sqrt{m^2+1}$$

For the cosine function, since $-1 \leqslant \cos(nt) \leqslant 1$, we have

$$-1 \leqslant \sqrt{m^2+1} \leqslant 1$$

And thus, we obtain

$$-1 \leqslant m^2 \leqslant 0, \quad m^2 = 0,\ \textit{and}\ m = 0$$

But in the definition of Salkowski curves, $m \neq 0$, this is a contradiction.

Corollary 3.1. Salkowski curves are not a unit speed curve.

3.2. The Torsion of Salkowski Curves in $\mathbb{E}^3$

The sign of torsion of Salkowski curves in [6] is regarded as $\tau(t) = \langle B'(t), N(t)\rangle$, thus the torsion of Salkowski curves is stated as $\tau(t) = \tan(nt)$. However, in [5], the torsion of Salkowski curves is given as $\tau(t) = -\tan(nt)$. We know the torsion of the a unit speed curve is found as $\tau(t) = \langle N'(t), B(t)\rangle = -\langle B'(t), N(t)\rangle$. But, we said that Salkowski curves are not a unit speed curve. Thus, to find the torsion of Salkowski curves, we use the following equation:

$$\tau(t) = \frac{\det\left(\gamma_m'(t), \gamma_m''(t), \gamma_m'''(t)\right)}{\|\gamma_m'(t)\wedge\gamma_m''(t)\|^2} \tag{13}$$

If we take second and third derivatives according to the parameter t of Salkowski curves $\gamma_m(t)$, we get

$$\begin{aligned}\gamma_m''(t) \;=\;& \frac{n}{m}\cos(nt)\Big(\cos(nt)\sin t + n\cos t\sin(nt) + n^2\sin t\left(\sin^2(nt) - \cos^2(nt)\right) \\ & -\cos t\cos(nt) + n\sin t\sin(nt) - n^2\cos t\left(\sin^2(nt) - \cos^2(nt)\right) \\ & +\frac{n^2}{m\cos(nt)}\left(\sin^2(nt) - \cos^2(nt)\right)\Big)\end{aligned} \tag{14}$$

$$\begin{aligned}\gamma_m'''(t) &= \frac{n}{m}\cos(nt)\big(\cos t\cos(nt) - 3n\sin t\sin(nt) + 4n^3\sin t\sin(nt)\\ &\quad \cos(nt)\sin t + 3n\cos t\sin(nt) - 4n^3\cos t\sin(nt),\frac{4n^3}{m}\sin(nt)\big)\end{aligned} \tag{15}$$

If we apply the vector product to the vectors 11 and 14, we have

$$\begin{aligned}\gamma_m'(t)\wedge\gamma_m''(t) &= -\frac{n^3}{m^3}\cos^3(nt)\big(\cos t\sin(nt) - n\cos(nt)\sin t, \sin t\sin(nt) + n\cos t\cos(nt),\\ &\quad -\frac{n}{m}\cos(nt)\big)\\ &= -\frac{n^3}{m^3}\cos^3(nt)\,B(t)\end{aligned} \tag{16}$$

Then, the norm of the Equation 16 is

$$\left\|\gamma_m'(t)\wedge\gamma_m''(t)\right\| = \frac{n^3}{m^3}\cos^3(nt) \tag{17}$$

If we apply the cross product the vectors 15 and 16, we get

$$\begin{aligned}\left\langle\gamma_m'(t)\wedge\gamma_m''(t),\gamma_m'''(t)\right\rangle &= \frac{n^3}{m^3}\Big(-\frac{n}{m}\cos^2 t\cos^5(nt)\sin(nt) + \frac{n^2}{m}\cos t\sin t\cos^6(nt)\\ &\quad +\frac{3n^2}{m}\cos t\sin t\cos^4(nt)\sin^2(nt) - \frac{3n^3}{m}\sin^2 t\cos^5(nt)\sin(nt)\\ &\quad -\frac{4n^4}{m}\cos t\sin t\cos^4(nt)\sin^2(nt) + \frac{4n^5}{m}\sin^2 t\cos^5(nt)\sin(nt)\\ &\quad -\frac{n}{m}\sin^2 t\cos^5(nt)\sin(nt) - \frac{n^2}{m}\cos t\sin t\cos^6(nt)\\ &\quad -\frac{3n^2}{m}\cos t\sin t\cos^4(nt)\sin^2(nt) - \frac{3n^3}{m}\cos^2 t\cos^5(nt)\sin(nt)\\ &\quad +\frac{4n^4}{m}\cos t\sin t\cos^4(nt)\sin^2(nt) + \frac{4n^5}{m}\cos^2 t\cos^5(nt)\sin(nt)\\ &\quad +\frac{4n^5}{m^3}\cos^5(nt)\sin(nt)\Big)\end{aligned}$$

$$\begin{aligned}\left\langle\gamma_m'(t)\wedge\gamma_m''(t),\gamma_m'''(t)\right\rangle &= \frac{n^3}{m^3}\Big(-\frac{n}{m}\cos^5(nt)\sin(nt) - \frac{3n^3}{m}\cos^5(nt)\sin(nt)\\ &\quad +\frac{4n^5}{m}\cos^5(nt)\sin(nt) + \frac{4n^5}{m^3}\cos^5(nt)\sin(nt)\Big)\\ &= -\frac{n^6}{m^6}\cos^5(nt)\sin(nt)\end{aligned} \tag{18}$$

Then, if the Equations 17 and 18 are substituted in the Equation 13, we obtain the torsion of Salkowski curves as follows:

$$\tau(t) = -\tan(nt) \tag{19}$$

[5]. But numerous authors take $\tau(t) = \tan(nt)$ in the their studies. In this study, we will use the equality 19 for the torsion of Salkowski curves.

3.3. The Frenet Derivative Formulas of Salkowski Curves in $\mathbb{E}^3$

By using the Equations 12 and 19, the derivative vectors of the Frenet vectors $T(t), N(t), B(t)$ of Salkowski curves are obtained as follows:

$$\begin{aligned}T'(t) &= \frac{n^2}{m^2}\cos(nt)(\sin t, -\cos t, -m)\\ &= \frac{n}{m}\cos(nt)\,N(t)\end{aligned} \tag{20}$$

$$\begin{aligned}
N'(t) &= \frac{n}{m}(\cos t, \sin t, 0)\\
&= \frac{n}{m}\Big(\cos t\cos^2(nt) + \cos t\sin^2(nt) + n\cos(nt)\sin t\sin(nt) - n\cos(nt)\sin t\sin(nt),\\
&\quad \cos^2(nt)\sin t + \sin t\sin^2(nt) - n\cos t\cos(nt)\sin(nt) + n\cos t\cos(nt)\sin(nt),\\
&\quad \frac{n}{m}\cos(nt)\sin(nt) - \frac{n}{m}\cos(nt)\sin(nt)\Big)\\
&= \frac{n}{m}\cos(nt)\Big(\cos t\cos(nt) + n\sin t\sin(nt), \cos(nt)\sin t - n\cos t\sin(nt), \frac{n}{m}\sin(nt)\Big)\\
&\quad +\frac{n}{m}\sin(nt)\Big(\cos t\sin(nt) - n\cos(nt)\sin t, \sin t\sin(nt) + n\cos t\cos(nt), -\frac{n}{m}\cos(nt)\Big)\\
&= -\frac{n}{m}\left(\cos(nt)\,T(t) + \sin(nt)\,B(t)\right) \qquad (21)
\end{aligned}$$

$$\begin{aligned}
B'(t) &= \frac{n^2}{m^2}\sin(nt)(\sin t, -\cos t, -m)\\
&= \frac{n}{m}\sin(nt)\,N(t) \qquad (22)
\end{aligned}$$

We could have obtained these equations using the Equation 3, but we preferred to arrive at these equations by operations.

Corollary 3.2. The Frenet derivative formulas for Salkowski curves are as following:

$$\begin{bmatrix} T'(t)\\ N'(t)\\ B'(t)\end{bmatrix} = \begin{bmatrix} 0 & \frac{n}{m}\cos(nt) & 0\\ -\frac{n}{m}\cos(nt) & 0 & -\frac{n}{m}\sin(nt)\\ 0 & \frac{n}{m}\sin(nt) & 0\end{bmatrix}\begin{bmatrix} T(t)\\ N(t)\\ B(t)\end{bmatrix} \qquad (23)$$

3.4. The Darboux Vector of Salkowski Curves in $\mathbb{E}^3$

Now, let us make sense of the angle (nt) for Salkowski curves. We know that the Darboux vector of any regular curve is found by

$$W(t) = N(t) \wedge N'(t). \qquad (24)$$

Then, for the Darboux vector of Salkowski curves, we get

$$\begin{aligned}
W(t) &= \frac{n}{m}(\sin t, -\cos t, -m) \wedge \frac{n}{m}(\cos t, \sin t, 0)\\
&= \frac{n^2}{m}\left(\sin t, -\cos t, \frac{1}{m}\right) \qquad (25)
\end{aligned}$$

Or, we can get it another way. If we use $\tau(t) = -\tan(nt)$, from the Equations 3 and 12, since Salkowski curves are not a unit speed curve, from the Equation

$$W(t) = v(t)\left(\tau(t)T(t) + \kappa(t)B(t)\right),$$

the Darboux vector of Salkowski curve is obtained as follows:

$$\begin{aligned}
W(t) &= \frac{n}{m}\cos(nt)\big(\cos t\sin(nt) + n\sin t\sin(nt)\tan(nt) - \cos t\sin(nt) + n\cos(nt)\sin t,\\
&\quad \sin t\sin(nt) - n\cos t\sin(nt)\tan(nt) - \sin t\sin(nt) - n\cos t\cos(nt),\\
&\quad \frac{n}{m}\sin(nt)\tan(nt) + \frac{n}{m}\cos(nt)\Big)\\
&= \frac{n^2}{m}\left(\sin t, -\cos t, \frac{1}{m}\right) \qquad (26)
\end{aligned}$$

Here we see the equality of Equations 25 and 26. Also, the norm of the Darboux vector is

$$\|W(t)\| = \frac{n}{m} \tag{27}$$

3.5. The Angle (nt) of Salkowski Curves in $\mathbb{E}^3$

Let the angle between the vectors $B(t)$ and $W(t)$ of Salkowski curves be $\varphi(t)$.

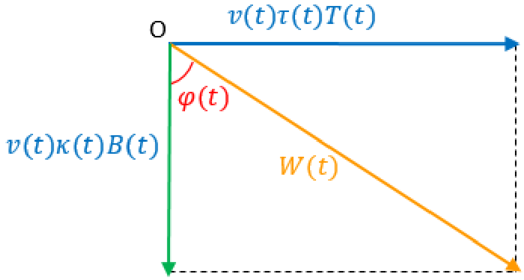


Fig. 2. The Darboux vector of Salkowski curves

In this case, from Figure 2, by using the Equation 27, we can write

$$\begin{aligned} \langle B(t), W(t)\rangle &= \|B(t)\|\,\|W(t)\|\cos\varphi \\ &= \frac{n}{m}\cos\varphi \end{aligned} \tag{28}$$

Also, if the inner product operation is applied to the vectors $B(t)$ and $W(t)$, we get

$$\begin{aligned} \langle B(t), W(t)\rangle &= -\frac{n^2}{m}\Big(\cos t\sin t\sin(nt) - n\sin^2 t\cos(nt) - \cos t\sin t\sin(nt) \\ &\quad -n\cos^2 t\cos(nt) - \frac{n}{m^2}\cos(nt)\Big) \\ &= -\frac{n^2}{m}\left(-n\cos(nt) - \frac{n}{m^2}\cos(nt)\right) \\ &= \frac{n}{m}\cos(nt) \end{aligned} \tag{29}$$

From the equality of the Equations 28 and 29, we have

$$\cos\varphi = \cos(nt) \tag{30}$$

On the other hand, from Figure 2, by using the Equation 27, we can write

$$\begin{aligned} \langle T(t), W(t)\rangle &= \|T(t)\|\,\|W(t)\|\sin\varphi, \\ &= \frac{n}{m}\sin\varphi \end{aligned} \tag{31}$$

Also, if the inner product operation is applied to the vectors $T(t)$ and $W(t)$, we get

$$\begin{aligned} \langle T(t), W(t)\rangle &= -\frac{n^2}{m}\left(n\sin^2 t\sin(nt) + n\cos^2 t\sin(nt) + \frac{n}{m^2}\sin(nt)\right) \\ &= -\frac{n^2}{m}\left(n\sin(nt) + \frac{n}{m^2}\sin(nt)\right), \\ &= -\frac{n}{m}\sin(nt) \end{aligned} \tag{32}$$

From the equality of the Equations 31 and 32, we have

$$\sin\varphi = -\sin(nt) \tag{33}$$

Hence, from the Equations 30 and 33, we write

$$\begin{aligned} \cos\varphi - \cos(nt) &= 0 \Longrightarrow \cos^2\varphi - 2\cos\varphi\cos(nt) + \cos^2(nt) = 0 \\ \sin\varphi + \sin(nt) &= 0 \Longrightarrow \sin^2\varphi + 2\sin\varphi\sin(nt) + \sin^2(nt) = 0 \end{aligned}$$

If these two equations are added side by side, we get

$$\begin{aligned} 2 - 2(\cos\varphi\cos(nt) - \sin\varphi\sin(nt)) &= 0 \\ \cos\varphi\cos(nt) - \sin\varphi\sin(nt) &= \cos(\varphi + nt) = 1 \end{aligned}$$

And thus, we obtain

$$nt = -\varphi \tag{34}$$

Corollary 3.3. The angle (nt) is the angle opposite sign to the angle between of the binormal vector $B(t)$ and the Darboux vector $W(t)$ of Salkowski curves.

3.6. The Unit Vector in the Direction of Darboux Vector of Salkowski Curves in $\mathbb{E}^3$

Let the unit vector in the direction of Darboux vector $W(t)$ of Salkowski curves be $C(t)$. Then, from the Equations 25 and 27, we have

$$C(t) = \frac{W(t)}{\|W(t)\|} = \left(n\sin t, -n\cos t, \frac{n}{m}\right) \tag{35}$$

On the other hand, from the Figure 2, we write

$$\begin{aligned} C(t) &= \frac{v(t)}{\|W(t)\|}(\tau(t)T(t) + \kappa(t)B(t)) \\ &= \sin\varphi T(t) + \cos\varphi B(t) \end{aligned} \tag{36}$$

[27]. If the Equation 34 is substituted in the Equation 36, we get

$$\begin{aligned} C(t) &= -\sin(nt)\,T(t) + \cos(nt)\,B(t) \\ &= \left(\cos t\cos(nt)\sin(nt) + n\sin t\sin^2(nt), \cos(nt)\sin t\sin(nt) - n\cos t\sin^2(nt), \frac{n}{m}\sin^2(nt)\right) \\ &- \left(\cos t\cos(nt)\sin(nt) - n\cos^2(nt)\sin t, \cos(nt)\sin t\sin(nt) + n\cos t\cos^2(nt), -\frac{n}{m}\cos^2(nt)\right) \end{aligned}$$

Then, we indeed obtain

$$C(t) = \left(n\sin t, -\cos t, \frac{n}{m}\right) \tag{37}$$

Here, we see that the Equations 35 and 37 are equal.

4. The Modified Orthogonal Frames of Salkowski Curves in $\mathbb{E}^3$

In this section, we calculate the modified orthogonal frames with the both curvature and torsion of Salkowski curves. And we give the relations between the derivatives of the vectors of these frames and the Frenet vectors or the vectors of the modified orthogonal frames. Let the Frenet frame, the curvature and the torsion of Salkowski curves $\gamma_m(t)$ be $\{T(t), N(t), B(t)\}$, $\kappa(t)$ and $\tau(t)$, respectively.

4.1. The Modified Orthogonal Frame with the Curvature of Salkowski Curves in $\mathbb{E}^3$

Considering the Gram-Schmidt orthogonalization procedure, let us define the orthogonal frame $\{E_1(t), E_2(t), E_3(t)\}$ below for Salkowski curves $\gamma_m(t)$, [26]:

$$E_1(t) = \gamma_m'(t), \quad E_2(t) = E_1'(t) - \frac{\langle E_1'(t), E_1(t)\rangle}{\langle E_1(t), E_1(t)\rangle} E_1(t), \quad E_3(t) = E_1(t) \wedge E_2(t) \tag{38}$$

Here, since Salkowski curves are not a unit speed curve, we can not use the Equation 8. From the Equations 11 and 38, the vector $E_1(t)$ is obtained as follows:

$$E_1(t) = \frac{n}{m}\cos(nt)\, T(t) \tag{39}$$

From the Equation 39, the following equation is gotten:

$$\begin{aligned}
\frac{\langle E_1'(t), E_1(t)\rangle}{\langle E_1(t), E_1(t)\rangle} &= \frac{\left\langle \left(\frac{n}{m}\cos(nt)\, T(t)\right)', \frac{n}{m}\cos(nt)\, T(t)\right\rangle}{\left\langle \frac{n}{m}\cos(nt)\, T(t), \frac{n}{m}\cos(nt)\, T(t)\right\rangle} \\
&= \frac{\left\langle -n\sin(nt)\, T(t) + \frac{n}{m}\cos^2(nt)\, N(t), \cos(nt)\, T(t)\right\rangle}{\cos^2(nt)} \\
&= -n\tan(nt)
\end{aligned} \tag{40}$$

From the Equations 3, 38 and 40, the vectors $E_2(t)$ and $E_3(t)$ are obtained as follows:

$$\begin{aligned}
E_2(t) &= -\frac{n^2}{m}\sin(nt)\, T(t) + \frac{n^2}{m^2}\cos^2(nt)\, N(t) + \frac{n^2}{m}\sin(nt)\, T(t) \\
&= \frac{n^2}{m^2}\cos^2(nt)\, N(t)
\end{aligned} \tag{41}$$

$$E_3(t) = \frac{n^3}{m^3}\cos^3(nt)\, B(t) \tag{42}$$

Since the curvature of Salkowski curves is $\kappa(t) = 1$, using the Equation 12, from the Equations 39, 41 and 42, the modified orthogonal frame $\{E_1(t), E_2(t), E_3(t)\}$ with the curvature $\kappa(t)$ of Salkowski curves $\gamma_m(t)$ is written as follows:

$$E_1(t) = v(t)T(t), \quad E_2(t) = v^2(t)\kappa(t)N(t), \quad E_3(t) = v^3(t)\kappa(t)B(t) \tag{43}$$

where

$$\langle E_1(t), E_2(t)\rangle = \langle E_2(t), E_3(t)\rangle = \langle E_1(t), E_3(t)\rangle = 0 \tag{44}$$

and

$$\begin{aligned}
\langle E_1(t), E_1(t)\rangle &= v^2(t) = \frac{n^2}{m^2}\cos^2(nt) \\
\langle E_2(t), E_2(t)\rangle &= v^4(t)\kappa^2(t) = \frac{n^4}{m^4}\cos^4(nt) \\
\langle E_3(t), E_3(t)\rangle &= v^6(t)\kappa^2(t) = \frac{n^6}{m^6}\cos^6(nt)
\end{aligned}$$

The frame $\{E_1(t), E_2(t), E_3(t)\}$ is indeed orthogonal (from the Equation 44), but is not orthonormal, because the vectors $E_1(t), E_2(t), E_3(t)$ are not unit vectors. Now, let us find the derivative vectors of the frame, respectively. First, by using the Equation 3, from the Equations 39, 41 and 42, we obtain the derivative vectors $E_1'(t), E_2'(t), E_3'(t)$ in terms of the Frenet vectors $T(t), N(t), B(t)$ as follows:

$$E_1'(t) = -\frac{n^2}{m}\sin(nt)\,T(t) + \frac{n^2}{m^2}\cos^2(nt)\,N(t) \tag{45}$$

$$\begin{aligned} E_2'(t) &= -\frac{2n^3}{m^2}\cos(nt)\sin(nt)\,N(t) + \frac{n^2}{m^2}\cos^2(nt)\,N'(t) \\ &= -\frac{2n^3}{m^2}\cos(nt)\sin(nt)\,N(t) + \frac{n^2}{m^2}\cos^2(nt)\left(-\frac{n}{m}\cos(nt)\,T(t) - \frac{n}{m}\sin(nt)\,B(t)\right) \\ &= -\frac{n^3}{m^3}\cos^3(nt)\,T(t) - \frac{2n^3}{m^2}\cos(nt)\sin(nt)\,N(t) - \frac{n^3}{m^3}\cos^2(nt)\sin(nt)\,B(t), \end{aligned} \tag{46}$$

$$\begin{aligned} E_3'(t) &= -\frac{3n^4}{m^3}\cos^2(nt)\sin(nt)\,B(t) + \frac{n^3}{m^3}\cos^3(nt)\,B'(t) \\ &= \frac{n^4}{m^4}\cos^3(nt)\sin(nt)\,N(t) - \frac{3n^4}{m^3}\cos^2(nt)\sin(nt)\,B(t) \end{aligned} \tag{47}$$

Corollary 4.1. From the Equations 45, 46 and 47, we obtain the following equalities between the Frenet vectors and the derivative vectors of the modified orthogonal frame with $\kappa(t)$ of Salkowski curves:

$$\begin{bmatrix} E_1'(t) \\ E_2'(t) \\ E_3'(t) \end{bmatrix} = \begin{bmatrix} -\frac{n^2}{m}\sin(nt) & \frac{n^2}{m^2}\cos^2(nt) & 0 \\ -\frac{n^3}{m^3}\cos(nt) & -\frac{2n^3}{m^2}\cos(nt)\sin(nt) & -\frac{n^3}{m^3}\cos^2(nt)\sin(nt) \\ 0 & \frac{n^4}{m^4}\cos^3(nt)\sin(nt) & -\frac{3n^4}{m^3}\cos^2(nt)\sin(nt) \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix}$$

Second, let's get the derivative vectors $E_1'(t), E_2'(t), E_3'(t)$ in terms of the vectors $E_1(t), E_2(t), E_3(t)$. From the Equations 39, 41 and 42, the Frenet vectors are written in terms of the vectors $E_1(t)$, $E_2(t)$, $E_3(t)$ as follows:

$$\begin{bmatrix} T(t) \\ N(t) \\ B(t) \end{bmatrix} = \begin{bmatrix} \frac{m}{n\cos(nt)} & 0 & 0 \\ 0 & \frac{m^2}{n^2\cos^2(nt)} & 0 \\ 0 & 0 & \frac{m^3}{n^3\cos^3(nt)} \end{bmatrix} \begin{bmatrix} E_1(s) \\ E_2(s) \\ E_3(s) \end{bmatrix} \tag{48}$$

If the Equation 48 are substituted in the Equations 45, 46 and 47, respectively, we get

$$\begin{aligned} E_1'(t) &= -\frac{n^2}{m}\sin(nt)\left(\frac{m}{n\cos(nt)}E_1(t)\right) + \frac{n^2}{m^2}\cos^2(nt)\left(\frac{m^2}{n^2\cos^2(nt)}E_2(t)\right), \\ E_1'(t) &= -n\tan(nt)\,E_1(t) + E_2(t) \end{aligned} \tag{49}$$

$$
\begin{aligned}
E_2'(t) &= -\frac{n^3}{m^3}\cos^3(nt)\left(\frac{m}{n\cos(nt)}E_1(t)\right) - \frac{2n^3}{m^2}\cos(nt)\sin(nt)\left(\frac{m^2}{n^2\cos^2(nt)}E_2(t)\right) \\
&\quad -\frac{n^3}{m^3}\cos^2(nt)\sin(nt)\left(\frac{m^3}{n^3\cos^3(nt)}E_3(t)\right), \\
&= -\frac{n^2}{m^2}\cos^2(nt)\,E_1(t) - 2n\tan(nt)\,E_2(t) - \tan(nt)\,E_3(t)
\end{aligned} \tag{50}
$$

$$
\begin{aligned}
E_3'(t) &= \frac{n^4}{m^4}\cos^3(nt)\sin(nt)\left(\frac{m^2}{n^2\cos^2(nt)}E_2(t)\right) - \frac{3n^4}{m^3}\cos^2(nt)\sin(nt)\left(\frac{m^3}{n^3\cos^3(nt)}E_3(t)\right) \\
&= \frac{n^2}{m^2}\cos(nt)\sin(nt)\,E_2(t) - 3n\tan(nt)\,E_3(t)
\end{aligned} \tag{51}
$$

Corollary 4.2. From the Equations 49, 50 and 51, we obtain the following equalities between the vectors of the modified orthogonal frame with $\kappa(t)$ of Salkowski curves and its derivative vectors:

$$
\begin{bmatrix} E_1'(t) \\ E_2'(t) \\ E_3'(t) \end{bmatrix} = \begin{bmatrix} -n\tan(nt) & 1 & 0 \\ -\frac{n^2}{m^2}\cos^2(nt) & -2n\tan(nt) & -\tan(nt) \\ 0 & \frac{n^2}{m^2}\cos(nt)\sin(nt) & -3n\tan(nt) \end{bmatrix} \begin{bmatrix} E_1(s) \\ E_2(s) \\ E_3(s) \end{bmatrix}
$$

4.2. The Modified Orthogonal Frame with the Torsion of Salkowski curves in $\mathbb{E}^3$

Considering the Gram-Schmidt orthogonalization procedure, let us define the orthogonal frame $\{A_1(t), A_2(t), A_3(t)\}$ below for Salkowski curves $\gamma_m(t)$, [26]:

$$
A_1(t) = \gamma_m'(t), \quad A_2(t) = \frac{\tau(t)}{\kappa(t)}\left(A_1'(t) - \frac{\langle A_1'(t), A_1(t)\rangle}{\langle A_1(t), A_1(t)\rangle}A_1(t)\right), \quad A_3(t) = A_1(t)\wedge A_2(t) \tag{52}
$$

From the Equations 11 and 52, the vector $A_1(t)$ is obtained as follows:

$$
A_1(t) = \frac{n}{m}\cos(nt)\,T(t) \tag{53}
$$

from the Equations 39 and 53, we see that the vectors $E_1(t)$ and $A_1(t)$ are equal. Therefore, from the Equation 40, we have

$$
\frac{\langle A_1'(t), A_1(t)\rangle}{\langle A_1(t), A_1(t)\rangle} = -n\tan(nt) \tag{54}
$$

From the Equations 3, 52 and 54, the vectors $A_2(t)$ and $A_3(t)$ are obtained as follows:

$$
\begin{aligned}
A_2(t) &= -\tan(nt)\left(-\frac{n^2}{m}\sin(nt)\,T(t) + \frac{n^2}{m^2}\cos^2(nt)\,N(t) + \frac{n^2}{m}\sin(nt)\,T(t)\right) \\
&= -\frac{n^2}{m^2}\cos(nt)\sin(nt)\,N(t)
\end{aligned} \tag{55}
$$

$$
A_3(t) = -\frac{n^3}{m^3}\cos^2(nt)\sin(nt)\,B(t) \tag{56}
$$

Since the torsion of Salkowski curves is $\tau(t) = -\tan(nt)$, using the Equation 12, from the Equations 53, 55 and 56, the modified orthogonal frame $\{A_1(t), A_2(t), A_3(t)\}$ with the torsion $\tau(t)$ of Salkowski curves $\gamma_m(t)$ is written as follows:

$$
A_1(t) = v(t)T(t), \quad A_2(t) = v^2(t)\tau(t)N(t), \quad A_3(t) = v^3(t)\tau(t)B(t) \tag{57}
$$

where

$$\langle A_1(t), A_2(t)\rangle = \langle A_2(t), A_3(t)\rangle = \langle A_1(t), A_3(t)\rangle = 0 \tag{58}$$

and

$$\begin{aligned}
\langle A_1(t), A_1(t)\rangle &= v^2(t) = \frac{n^2}{m^2}\cos^2(nt) \\
\langle A_2(t), A_2(t)\rangle &= v^4(t)\tau^2(t) = \frac{n^4}{m^4}\cos^2(nt)\sin^2(nt) \\
\langle A_3(t), A_3(t)\rangle &= v^6(t)\tau^2(t) = \frac{n^6}{m^6}\cos^4(nt)\sin^2(nt)
\end{aligned}$$

The frame $\{A_1(t), A_2(t), A_3(t)\}$ is indeed orthogonal (from the Equation 58), but is not orthonormal, because the vectors $A_1(t), A_2(t), A_3(t)$ are not unit vectors. Now, let us find the derivative vectors of the frame, respectively. First, by using the Equation 3, from the Equations 53, 55 and 56, we obtain the derivative vectors $A_1'(t), A_2'(t), A_3'(t)$ in terms of the Frenet vectors as follows:

$$A_1'(t) = -\frac{n^2}{m}\sin(nt)\,T(t) + \frac{n^2}{m^2}\cos^2(nt)\,N(t) \tag{59}$$

$$\begin{aligned}
A_2'(t) &= -\frac{n^2}{m^2}\left(-n\sin^2(nt) + n\cos^2(nt)\right)N(t) - \frac{n^2}{m^2}\cos(nt)\sin(nt)\,N'(t) \\
&= -\frac{n^3}{m^2}\left(-\sin^2(nt) + \cos^2(nt)\right)N(t) \\
&\quad -\frac{n^2}{m^2}\cos(nt)\sin(nt)\left(-\frac{n}{m}\cos(nt)\,T(t) - \frac{n}{m}\sin(nt)\,B(t)\right) \\
&= \frac{n^3}{m^3}\cos^2(nt)\sin(nt)\,T(t) + \frac{n^3}{m^2}\left(2\sin^2(nt) - 1\right)N(t) + \frac{n^3}{m^3}\cos(nt)\sin^2(nt)\,B(t)
\end{aligned} \tag{60}$$

$$\begin{aligned}
A_3'(t) &= -\frac{n^3}{m^3}\left(\left(-2n\cos(nt)\sin^2(nt) + n\cos^3(nt)\right)B(t) + \cos^2(nt)\sin(nt)\,B'(t)\right) \\
&= -\frac{n^4}{m^4}\cos^2(nt)\sin^2(nt)\,N(t) + \frac{n^4}{m^3}\cos(nt)\left(3\sin^2(nt) - 1\right)B(t)
\end{aligned} \tag{61}$$

Corollary 4.3. From the Equations 59, 60 and 61, we obtain the following equalities between the Frenet vectors and the derivative vectors of the modified orthogonal frame with $\tau(t)$ of Salkowski curves:

$$\begin{bmatrix} A_1'(t) \\ A_2'(t) \\ A_3'(t) \end{bmatrix} = \begin{bmatrix} -\frac{n^2}{m}\sin(nt) & \frac{n^2}{m^2}\cos^2(nt) & 0 \\ \frac{n^3}{m^3}\cos^2(nt)\sin(nt) & \frac{n^3}{m^2}\left(2\sin^2(nt)-1\right) & \frac{n^3}{m^3}\cos(nt)\sin^2(nt) \\ 0 & -\frac{n^4}{m^4}\cos^2(nt)\sin^2(nt) & \frac{n^4}{m^3}\cos(nt)\left(3\sin^2(nt)-1\right) \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix}$$

Second, let's get the derivative vectors $A_1'(t), A_2'(t), A_3'(t)$ in terms of the vectors $A_1(t), A_2(t), A_3(t)$. From the Equations 53, 55, and 56, the Frenet vectors are written in terms of the vectors $A_1(t)$, $A_2(t)$, and $A_3(t)$ as follows:

$$\begin{bmatrix} T(t) \\ N(t) \\ B(t) \end{bmatrix} = \begin{bmatrix} \frac{m}{n\cos(nt)} & 0 & 0 \\ 0 & -\frac{m^2}{n^2\cos(nt)\sin(nt)} & 0 \\ 0 & 0 & -\frac{m^3}{n^3\cos^2(nt)\sin(nt)} \end{bmatrix} \begin{bmatrix} A_1(s) \\ A_2(s) \\ A_3(s) \end{bmatrix} \tag{62}$$

If the Equation 62 are substituted in the Equations 59, 60 and 61, respectively, we get

$$\begin{aligned} A_1'(t) &= -\frac{n^2}{m}\sin(nt)\left(\frac{m}{n\cos(nt)}A_1(t)\right)+\frac{n^2}{m^2}\cos^2(nt)\left(-\frac{m^2}{n^2\cos(nt)\sin(nt)}A_2(t)\right) \\ &= -n\tan(nt)\,A_1(t)-\frac{1}{\tan(nt)}A_2(t) \end{aligned} \tag{63}$$

$$\begin{aligned} A_2'(t) = & \frac{n^3}{m^3}\cos^2(nt)\sin(nt)\left(\frac{m}{n\cos(nt)}A_1(t)\right)+\frac{n^3}{m^2}\left(2\sin^2(nt)-1\right)\left(-\frac{m^2}{n^2\cos(nt)\sin(nt)}A_2(t)\right) \\ & + \frac{n^3}{m^3}\cos(nt)\sin^2(nt)\left(-\frac{m^3}{n^3\cos^2(nt)\sin(nt)}A_3(t)\right) \\ & = \frac{n^2}{m^2}\cos(nt)\sin(nt)\,A_1(t)-n\left(\frac{2\sin^2(nt)-1}{\cos(nt)\sin(nt)}\right)A_2(t)-\tan(nt)\,A_3(t) \end{aligned} \tag{64}$$

$$\begin{aligned} A_3'(t) &= -\frac{n^4}{m^4}\cos^2(nt)\sin^2(nt)\left(-\frac{m^2}{n^2\cos(nt)\sin(nt)}A_2(t)\right) \\ &\quad +\frac{n^4}{m^3}\cos(nt)\left(3\sin^2(nt)-1\right)\left(-\frac{m^3}{n^3\cos^2(nt)\sin(nt)}A_3(t)\right) \\ &= \frac{n^2}{m^2}\cos(nt)\sin(nt)\,A_2(t)-n\left(\frac{3\sin^2(nt)-1}{\cos(nt)\sin(nt)}\right)A_3(t) \end{aligned} \tag{65}$$

Corollary 4.4. From the Equations 63, 64 and 65, we obtain the following equalities between the modified orthogonal frame with $\tau(t)$ of Salkowski curves and its derivative vectors:

$$\begin{bmatrix} A_1'(t) \\ A_2'(t) \\ A_3'(t) \end{bmatrix} = \begin{bmatrix} -n\tan(nt) & -\frac{1}{\tan(nt)} & 0 \\ \frac{n^2}{m^2}\cos(nt)\sin(nt) & -\frac{n\left(2\sin^2(nt)-1\right)}{\cos(nt)\sin(nt)} & -\tan(nt) \\ 0 & \frac{n^2}{m^2}\cos(nt)\sin(nt) & -\frac{n\left(3\sin^2(nt)-1\right)}{\cos(nt)\sin(nt)} \end{bmatrix} \begin{bmatrix} A_1(t) \\ A_2(t) \\ A_3(t) \end{bmatrix}$$

5. Conclusion

The modified orthogonal frame is a tool that can be used to solve the problem at singular points where the Frenet frame of the analytical or discontinuous curves cannot be calculated. But there is no harm in constructing a modified orthogonal frame of any regular curve. For example, in this study, the modified orthogonal frames of Salkowski curves are calculated and the relationships between the Frenet frame and the modified orthogonal frames are given. The characteristic properties of Salkowski curves can be studied with its modified orthogonal frames, as well as with its Frenet frame.

Author Contributions

All the authors contributed equally to this work. They all read and approved the last version of the paper.

Conflicts of Interest

The authors declare no conflict of interest.

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