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# ON THE ALGEBRA OF CONSTANTS OF FREE METABELIAN LIE ALGEBRAS

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Abstract. Let $K$ be a field of characteristic zero, $X_n = \{x_1, \dots, x_n\}$ be a set of variables, $K[X_n]$ be the polynomial algebra and $F_n$ be the free metabelian Lie algebra of rank $n$ generated by $X_n$ over the base field $K$. Well known result of Weitzenböck states that $K[X_n]^\delta = \{u \in K[X_n] \mid \delta(u) = 0\}$ is finitely generated as an algebra, where $\delta$ is a locally nilpotent linear derivation of $K[X_n]$. Extending this idea to the non-commutative algebras, recently the algebra $F_n^\delta$ of constants in free metabelian Lie algebras have been investigated. According to the findings, $F_n^\delta$ is not finitely generated as a Lie algebra; whereas, $F_n^\delta \cap F_n'$ is a finitely generated $K[X_n]^\delta$-module and a list of generators for $n \leq 4$ was given. In this work, in filling the gap in the list of small $n'$s we work in $F_5$ and give a list of generators of $F_5^\delta$ where $\delta(x_5) = x_4$, $\delta(x_4) = 0$, $\delta(x_3) = x_2$, $\delta(x_2) = x_1$ and $\delta(x_1) = 0$.

## 1. Introduction

Let $K[X_n]$ be the polynomial algebra over a field $K$ of characteristic 0 freely generated by the set of variables $X_n = \{x_1, \dots, x_n\}$ and let $KX_n$ be the vector space with basis $X_n$. A nonzero locally nilpotent linear derivation $\delta$ of the polynomial algebra $K[X_n]$ is called a Weitzenböck derivation. The canonical action of the general linear group $GL_n(K)$ on the polynomial algebra $K[X_n]$ defines the set

$$K[X_n]^G = \{p \in K[X_n] \mid g \cdot p = p, \quad \forall \quad g \in G\}$$

which is the algebra of invariants of the subgroup $G \leq GL_n(K)$. The $14^{th}$ of 23 Hilbert problems asked whether the algebra $K[X_n]^G$ is finitely generated as an algebra for any $G \leq GL_n(K)$. Different results have been found on the topic such as Nagata [7] who gave a counterexample which consists of 13 commuting locally nilpotent derivations of a polynomial ring in 32 variables for which the kernel is not finitely generated. Even if the concept was not correct in general some affirmative results have been found one can see the approach of Noether [8] who proved the case for finite groups $G$, and Weitzenböck [10] who confirmed that the algebra of constants of the derivation $\delta$ in the algebra $K[X_n]$

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$$K[X_n]^\delta = Ker\delta = \{u \in K[X_n] \mid \delta(u) = 0\}$$

is finitely generated.

For associative polynomial algebra $K\langle X_n\rangle$ generated by $X_n$ over a field $K$, the algebra $K\langle X_n\rangle^G$ of invariants is finitely generated if and only if $G$ is a cyclic group which acts by scalar multiplication via [4, 6]. Another approach here is when the group acts on relatively free algebras in a variety of associative algebras, Bryant and Drensky [2, 5] proved that the algebra $L_n^G$ and $F_n^G$ are not finitely generated for finite groups $G$ where $L_n$ and $F_n$ are the free Lie algebra and the free metabelian Lie algebra of rank $n$ over $K$ respectively. In the case of free metabelian Lie algebra, even if the algebra $F_n^G$ is not finitely generated, the commutator ideal $F_n'$ of $F_n$ is finitely generated $K[X_n]^G$-module under an appropriate action of $K[X_n]$ on $F_n'$.

If $\delta$ is a Weitzenböck derivation, then it is locally nilpotent and the linear map $exp(\delta)$ acting on the vector space $KX_n$ is unipotent. Hence, the algebra $K[X_n]^\delta$ of constants of $\delta$ is equal to the algebra of invariants

$$K[X_n]^{<exp(\delta)>} = \{u \in K[X_n] \mid exp(\delta)(u) = u\}.$$

The algebra of invariants of $exp(\delta)$ is equal to the algebra of invariants $K[X_n]^G$, where the group $G$ consists of all elements $exp(c\delta)$, $c \in K$.

Let $\delta_n$ be the Weitzenböck derivation of $K[X_n]$ which corresponds to a single $n \times n$ Jordan block of size $n$, then,

$$\delta_n : x_n \to x_{n-1} \to \cdots \to x_1 \to 0.$$

and such Weitzenböck derivations is called basic.

Dangovski et al.[3] proved that if $\delta$ is a Weitzenböck derivation of the free metabelian Lie algebra $L_n/L_n''$, then the algebra $(L_n'/L_n'')^\delta$ of constants of $\delta$ in the commutator ideal $L_n'/L_n''$ of $L_n/L_n''$ is a finitely generated $K[X_n]^\delta$-module. In this work, we give the explicit set of generators of the algebra of constants of the $K[X_n]^{\delta_n}$-module $(L_n'/L_n'')^{\delta_n}$ for $n = 5$ as a non-basic Weitzenböck derivation.

## 2. Preliminaries

It is well known that a Lie algebra $L$ is a vector space over $K$ endowed with a bilinear map: $[.,.] : L \times L \to L, (x_1, x_2) \to [x_1, x_2]$ satisfying the following conditions:

$$[x_1, x_1] = 0$$

$$[[x_1, x_2], x_3] + [[x_2, x_3], x_1] + [[x_3, x_1], x_2] = 0$$

for all $x_1, x_2, x_3 \in L$ and the second condition is known as Jacobi identity.

Let $L_n$ be the free Lie algebra over the field $K$ of characteristic zero freely generated by a finite set $X_n = \{x_1, \ldots, x_n\}$ with $n \geq 2$. The elements of $X_n$ are Lie monomials of length 1 and those Lie monomials has the form of

$$[x_1, x_2, x_3] = [x_1, x_2]ad\ x_3 = [[x_1, x_2], x_3]$$

for the sake of simplicity. Hence, we may extent this inductively as

$$[x_1, x_2, x_3, \cdots, x_n] = [\cdots[[x_1, x_2], x_3], \cdots, x_n], n \geq 3,$$

for all $x_1, x_2, \ldots, x_n \in L_n$. In this way, every element of $L_n$ can be written as a linear combination of the left normed monomials, which means that, the set of all left normed Lie monomials spans the whole free Lie algebra $L_n$ on $X_n$.

The quotient algebra

$$F_n = L_n/L_n'' = L_n/[[L_n, L_n], [L_n, L_n]]$$

is called the free metabelian Lie algebra of rank $n$ defined by metabelian identity $[[x_1, x_2], [x_3, x_4]] = 0$. $F_n$ is generated by free generators $y_1 = x_1 + L_n'', \cdots, y_n = x_n + L_n''$ hence, the commutator ideal $F_n'$ has a basis consisting of all

$$[y_{i_1}, y_{i_2}, \cdots, y_{i_n}] = [\cdots[[y_{i_1}, y_{i_2}], y_{i_3}], \cdots, y_{i_m}],\ i_1 > i_2 \leq i_3 \leq \cdots \leq i_n \leq n.$$

Therefore, $F_n'$ is furnished with a natural structure of module of $K[y_1, \cdots, y_n]$. For more information see [1].

A vector space $V$ over a field $K$ of characteristic zero is called graded $K$-vector space if it has a direct sum

$$V = \sum_{k \geq 0} V^{(k)}$$

where $V^{(k)}$ is a subspace $k \geq 0$. In particular, we define a multigrading on $V$ if

$$V = \sum_{i=1}^{m} \sum_{k_i \geq 0} V^{(k_1, \ldots, k_m)},$$

and $V^{(k_1, \ldots, k_m)}$ is called homogeneous component of degree $(k_1, \ldots, k_m)$.

Since $V = \bigoplus_{k \geq 0} V^{(k)}$ is finitely generated graded $K$-algebra, then we define the Hilbert function of $V$ as

$$H(V, k) = dim_K(V^{(k)})$$

where $dim_K(V^{(k)})$ is the dimension of the vector space $V^{(k)}$ over $K$.

Let $F_n$ be a relatively free algebra of the variety $\mathcal{B}$ of (not necessarily associative or Lie) algebra, then is graded vector space with finite dimension of homogenous component of degree $k$ of $F_n$. Hence, we can define formal power series also known as the Hilbert or Poincar*é* series

$$H(F_n, t) = \sum_{k \geq 0} dim_K(V^{(k)}) \cdot t^k.$$

For multigrated of algebra $F_n$ with a $\mathbb{Z}^n$-grading; the Hilbert series of $F_n$ is

$$H(F_n, t_1, \ldots, t_k) = \sum_{k_j \geq 0} dim F_n^{(m_1, \ldots, m_k)} \cdot t_1^{m_1} \cdots t_k^{m_k}$$

where $F_n^{(m_1, \ldots, m_k)}$ is the miltihomogeneous component of degree $(m_1, \ldots, m_k)$.

Equivalently, for a Weitzenböck dervations $\delta$ of $F_n$, the algebra of constants $F_n^\delta$ is also a graded and evantually its Hilbert series is

$$H(F_n^\delta, t) = \sum_{m \geq 0} dim_K(F_n^\delta)^{(m)} \cdot t^m$$

By using direct standards computations from Maple we give the Hilbert series of the the algebra $K[X_n]^{\delta_n}$ of constant of polynomial algebra and the algebra $L_n/L_n''$ of constant of free metabelian Lie algebra for small $n \leq 4$ as a basic and for $n = 5$ Weitzenböck derivation.

For $n = 2$, $\delta = \delta(1)$

The Hilbert series of algebra of constants $K[X_2]^{\delta_2}$ is

$$H_{GL_2}(K[X_2]^{\delta_2}, t_1, t_2, z) = \frac{1}{1 - t_1 z}$$

and the Hilbert series of algebra of constants $(L_2/L_2^{''})^{\delta_2}$ is

$$H_{GL_2}((L_2/L_2^{''})^{\delta_2}, t_1, t_2, z) = t_1 z + \frac{t_1 t_2 z^2}{1 - t_1 z}$$

For $n = 3$, $\delta = \delta(2)$

The Hilbert series of algebra of constants $K[X_3]^{\delta_3}$ becomes

$$H_{GL_2}(K[X_3]^{\delta_3}, t_1, t_2, z) = \frac{1}{(1 - t_1^2 z)(1 - t_1^2 t_2^2 z^2)}$$

and the Hilbert series of algebra of constants $(L_3/L_3^{''})^{\delta_3}$ is

$$H_{GL_2}((L_3/L_3^{''})^{\delta_3}, t_1, t_2, z) = \frac{t_1^3 t_2 z^2}{(1 - t_1^2 z)(1 - t_1 t_2 z)}$$

For $n = 4$, $\delta = \delta(3)$

The Hilbert series of algebra of constants $K[X_4]^{\delta_4}$ is

$$H_{GL_2}(K[X_4]^{\delta_4}, t_1, t_2, z) = \frac{1 - t_1^{12} t_2^6 z^6}{(1 - t_1^3 z)(1 - t_1^4 t_2^2 z^2)(1 - t_1^6 t_2^3 z^3)(1 - t_1^6 t_2^6 z^4)}$$

while the Hilbert series of algebra of constants $(L_4/L_4^{''})^{\delta_4}$ is

$$H_{GL_2}((L_4/L_4^{''})^{\delta_4}, t_1, t_2, z) = \frac{t_1^3 t_2 z^2 (t_1^2 + t_2^2 + t_1^4 t_2^4 z^2 + t_1^5 t_2^6 z^3 - t_1^8 t_2^6 z^4)}{(1 - t_1^3 z)(1 - t_1^2 t_2 z)(1 - t_1^6 t_2^6 z^4)}$$

The followings are well known generators of the algebra $K[X_n]^{\delta_n}$ of constants of polynomial algebras and the algebras $L_n'/L_n''$ of constant of free metabelian Lie algebra for $n = 3, 4$ such that $\delta_n : x_n \to x_{n-1} \to \cdots \to x_1 \to 0$ and this can found by using the Hilbert series of polynomial algebra and the Hilbert series of free metabelian Lie algebra respectively where the bidegree of denominator represent the generators and the bidegree of numerator represent the relations. For more information one can see [9, 3]. For $n = 3$, the algebra of constants $K[X_3]^{\delta_3}$ is generated by $x_1$ and $-2x_1x_3 + x_2^2$ while the $K[X_3]^{\delta_3}$-module $L_3'/L_3''$ is generated by $[x_2, x_1]$ and $[x_3, x_1, x_1] - [x_2, x_1, x_2]$. For $n = 4$, the algebra of constants $K[X_4]^{\delta_4}$ is generated by $f_1 = x_1$, $f_2 = -2x_1x_3 + x_2^2$, $f_3 = 3x_1^2x_4 - 3x_1x_2x_3 + x_2^3$ and $f_4 = 6x_1x_2x_3x_4 - 2x_2^3x_4 - 3x_1^2x_4^2 - 8/3x_1x_3^3 + x_2^2x_3^2$ with single relation $3f_1^2f_4 + f_3^2 - f_2^3 = 0$ while the $K[X_4]^{\delta_4}$-module $L_4'/L_4''$ is generated by

$$[x_2, x_1],$$

$$[x_4, x_1] - [x_3, x_2],$$

$$[x_3, x_1, x_1] - [x_2, x_1, x_2],$$

$$3[x_2, x_1, x_4] - 2[x_3, x_1, x_3] + [x_3, x_2, x_2],$$

$$\begin{aligned}&[x_3, x_2, x_2, x_2] - 3[x_3, x_1, x_1, x_4] + 4[x_4, x_1, x_1, x_3] \\ &+ 3[x_2, x_1, x_2, x_4] - 2[x_4, x_1, x_2, x_2] - [x_3, x_1, x_2, x_3]\end{aligned}$$

$$\begin{aligned}&3[x_4, x_2, x_2, x_2, x_2] - 3[x_3, x_2, x_2, x_2, x_3] + 6[x_3, x_1, x_2, x_3, x_3] \\ &- 4[x_2, x_1, x_3, x_3, x_3] + 9[x_2, x_1, x_1, x_4, x_4] + 12[x_4, x_1, x_1, x_3, x_3] \\ &- 18[x_3, x_1, x_1, x_3, x_4] - 12[x_4, x_1, x_2, x_2, x_3] + 9[x_3, x_1, x_2, x_2, x_4]\end{aligned}$$

and,

$$\begin{aligned} c_7 =& 3[x_4,x_2,x_2,x_2,x_2,x_2]-3[x_3,x_2,x_2,x_2,x_2,x_3]-8[x_3,x_1,x_1,x_3,x_3,x_3]\\ &-18[x_4,x_1,x_1,x_1,x_3,x_4]+18[x_3,x_1,x_1,x_1,x_4,x_4]-15[x_4,x_1,x_2,x_2,x_2,x_3]\\ &+3[x_3,x_1,x_2,x_2,x_2,x_4]+9[x_2,x_1,x_2,x_2,x_3,x_4]+12[x_3,x_1,x_2,x_2,x_3,x_3]\\ &-10[x_2,x_1,x_2,x_3,x_3,x_3]+9[x_4,x_1,x_1,x_2,x_2,x_4]-18[x_2,x_1,x_1,x_2,x_4,x_4]\\ &+18[x_4,x_1,x_1,x_2,x_3,x_3]-18[x_3,x_1,x_1,x_2,x_3,x_4]+18[x_2,x_1,x_1,x_3,x_3,x_4] \end{aligned}$$

## 3. MAIN RESULTS

**Theorem 3.1.** *Let $\delta_{3x2}$ be a nonbasic Weitzenböck derivation such that $\delta_{3x2}: x_3 \to x_2 \to x_1 \to 0$ and $\delta_{3x2}: x_5 \to x_4 \to 0$. Then $K[X_5]^{\delta_{3x2}}$-module $(L_5'/L_5'')^{\delta_{3x2}}$ is generated by $\{c_1,\dots,c_{11}\}$ where,*

$$c_1 = [x_2,x_1]$$
$$c_2 = [x_4,x_1]$$
$$c_3 = [x_5,x_1]-[x_4,x_2]$$
$$c_4 = [x_5,x_4]$$
$$c_5 = [x_2,x_1,x_5]+[x_4,x_2,x_2]-2[x_4,x_1,x_3]$$
$$c_6 = [x_2,x_1,x_5]-[x_3,x_1,x_4]$$
$$c_7 = [x_3,x_1,x_1]-[x_2,x_1,x_2]$$
$$c_8 = [x_5,x_1,x_5]-[x_5,x_2,x_4]-[x_4,x_2,x_5]+2[x_4,x_3,x_4]$$
$$c_9 = [x_2,x_1,x_5,x_5]-2[x_3,x_1,x_4,x_5]+2[x_3,x_2,x_4,x_4]$$
$$c_{10} = [x_3,x_1,x_1,x_5]-[x_2,x_1,x_2,x_5]-[x_3,x_1,x_2,x_4]+2[x_2,x_1,x_3,x_4]$$

$$\begin{aligned} c_{11} =& [x_5,x_2,x_2,x_2,x_4]+[x_4,x_2,x_2,x_2,x_5]+2[x_5,x_1,x_1,x_3,x_5]\\ &-[x_3,x_1,x_1,x_5,x_5]-[x_5,x_1,x_2,x_2,x_5]+[x_2,x_1,x_2,x_5,x_5]\\ &-2[x_4,x_2,x_2,x_3,x_4]+4[x_4,x_1,x_3,x_3,x_4]-2[x_3,x_1,x_3,x_4,x_4]\\ &-2[x_5,x_1,x_2,x_3,x_4]-2[x_4,x_1,x_2,x_3,x_5]+2[x_3,x_1,x_2,x_4,x_5]. \end{aligned}$$

*Proof.* Let $K[X_5]$ be the polynomial algebras generated by $X_5=\{x_1,x_2,x_3,x_4,x_5\}$. Then $K[X_5]^{\delta_{3x2}}$ is the algebras of constants of polynomial algebras for nonbasic Weitzenböck derivations such that $\delta_{3x2}: x_3 \to x_2 \to x_1 \to 0$ and $\delta_{3x2}: x_5 \to x_4 \to 0$. By using Maple computation, we have

$$h_0 = \frac{v_1^2v_2z^2+1}{(v_1z-1)(v_1v_2z+1)(v_1^2v_2^2z^3-1)(v_1v_2z-1)(v_1^2z-1)}$$

Let consider that $v_1=t_1$ and $v_2=t_2$, then we have the Hilbert series of algebras of constants of polynomial algebra which is

$$H_{GL_2}(K[X_5]^{\delta_{3x2}},t_1,t_2,z^2) = \frac{1+t_1^2t_2z^2}{(1-t_1^2z)(1-t_1^2t_2^2z^2)(1-t_1^2t_2^2z^3)(1-t_1z)}$$

Therefore, the algebras $(K[X_5]^{\delta_{3x2}}$ is generated by the algebraically independent polynomials

$f_1=x_1$, $f_2=x_2^2-2x_1x_3$, $f_3=x_1x_5^2-2x_2x_4x_5+2x_3x_4^2$, $f_4=x_4$ and $f_5=x_1x_5-x_2x_4$; with single relation $f_5^2=f_2f_4^2+f_1f_3$. Similarly, the Hilbert series of the algebra $(L_5'/L_5'')^{\delta_{3x2}}$ of constant of free metabelian Lie algebra is

$$\begin{aligned}
&H_{GL_2}((L'_5/L''_5)^{\delta_{3x2}}, t_1, t_2, z) = \\
&= \frac{N}{(1-t_1^2z)(1-t_1t_2z)(1-t_1^2t_2^2z^3)(1-t_1z)} \\
&= (\frac{N}{(1-t_1^2z)(1-t_1^2t_2^2z^2)(1-t_1^2t_2^2z^3)(1-t_1z)})(1+t_1t_2z)
\end{aligned}$$

where

$$\begin{aligned}
N =&(t_1^3t_2 + t_1^3 + t_1^2t_2 + t_1t_2)z^2 + (t_1^3t_2^2 - t_1^4t_2)z^3 + (t_1^3t_2^2 + t_1^3t_2^3)z^4 \\
&+ (-t_1^4t_2^3 - t_1^5t_2^2 - t_1^5t_2^3)z^5 + t_1^6t_2^3z^6
\end{aligned}$$

Therefore,

$$\begin{aligned}
H_{GL_2}((L'_5/L''_5)^{\delta_{3x2}}, t_1, t_2, z) = &(t_1^3t_2 + t_1^3 + t_1^2t_2 + t_1t_2)z^2 \\
&+ (t_1^3t_2^2 - t_1^4t_2 + t_1^4t_2^2 + t_1^4t_2 + t_1^3t_2^2 + t_1^2t_2^2)z^3 \\
&+ (t_1^3t_2^2 + t_1^3t_2^3 + t_1^4t_2^3 - t_1^5t_2^2)z^4 \\
&+ (-t_1^4t_2^3 - t_1^5t_2^2 - t_1^5t_2^3 + t_1^4t_2^3 + t_1^4t_2^4)z^5 \\
&+ (t_1^6t_2^3 - t_1^5t_2^4 - t_1^6t_2^3 - t_1^6t_2^4)z^6 + t_1^7t_2^4z^7
\end{aligned}$$

The power of positive sign of $t's$ in Hilbert series of algebras of constants of free metabelian Lie algebra represents the bidegree of generators and the power negative sign represents the relations while the power of $z's$ represents the length. This means that we have the bidegrees $(3,1)$; $(3,0)$; $(2,1)$ and $(2,2)$ of length 2 which corresponds to the generators $c_1$,$c_2$,$c_3$ and $c_4$ respectively. Therefore, $K[X_5]^{\delta_{3x2}}$-module $(L'_5/L''_5)^{\delta_{3x2}}$ has four generators $c_1, c_2, c_3$ and $c_4$ of length 2 as follows:

$$c_1 = [x_2, x_1]$$

$$c_2 = [x_4, x_1]$$

$$c_3 = [x_5, x_1] - [x_4, x_2]$$

$$c_4 = [x_5, x_4].$$

Next, we have 4 bidegrees $(3,2)$;$(3,2)$;$(4,2)$ and $(2,2)$ of length 3 which corresponds to the generators $c_5$,$c_6$,$c_7$ and $c_8$ respectively. Therefore, $K[X_5]^{\delta_{3x2}}$-module $(L'_5/L''_5)^{\delta_{3x2}}$ has four generators $c_5$,$c_6$,$c_7$ and $c_8$ of length 3 as follow:

$$c_5 = [x_2, x_1, x_5] + [x_4, x_2, x_2] - 2[x_4, x_1, x_3]$$

$$c_6 = [x_2, x_1, x_5] - [x_3, x_1, x_4]$$

$$c_7 = [x_3, x_1, x_1] - [x_2, x_1, x_2]$$

$$c_8 = [x_5, x_1, x_5] - [x_5, x_2, x_4] - [x_4, x_2, x_5] + 2[x_4, x_3, x_4].$$

Similarly, we expect two bidegrees $(3,3)$ and $(4,3)$ of length 4 which can produce two generators $c_9$ and $c_{10}$. Therefore, $K[X_5]^{\delta_{3x2}}$-module $(L'_5/L''_5)^{\delta_{3x2}}$ has two generators $c_9$ and $c_{10}$ of length 4 as follows:

$$c_9 = [x_2, x_1, x_5, x_5] - 2[x_3, x_1, x_4, x_5] + 2[x_3, x_2, x_4, x_4]$$

$$c_{10} = [x_3, x_1, x_1, x_5] - [x_2, x_1, x_2, x_5] - [x_3, x_1, x_2, x_4] + 2[x_2, x_1, x_3, x_4].$$

By continuing in the same way, we found another bidegree $(4,4)$ of length 5 that has new generator and is denoted as $c_{11}$ so, $K[X_5]^{\delta_{3x2}}$-module $(L_5'/L_5'')^{\delta_{3x2}}$ has single generators $c_{11}$ of length 5 such that

$$\begin{aligned} c_{11} =& [x_5,x_2,x_2,x_2,x_4]+[x_4,x_2,x_2,x_2,x_5]+2[x_5,x_1,x_1,x_3,x_5] \\ & -[x_3,x_1,x_1,x_5,x_5]-[x_5,x_1,x_2,x_2,x_5]+[x_2,x_1,x_2,x_5,x_5] \\ & -2[x_4,x_2,x_2,x_3,x_4]+4[x_4,x_1,x_3,x_3,x_4]-2[x_3,x_1,x_3,x_4,x_4] \\ & -2[x_5,x_1,x_2,x_3,x_4]-2[x_4,x_1,x_2,x_3,x_5]+2[x_3,x_1,x_2,x_4,x_5]. \end{aligned}$$

Note that the not-used bidegrees have correspondence with the other generators hence, we conclude that those bidegrees do not generate the new generators. □

## 4. Conclusion

In this study, we found the set of generators of the algebra $F_5^{\delta_5}$ of constants of free metabelian Lie algebra such that $\delta_5$ is a non-basic Weitzenböck derivation and as recommendation one can extend our work and give the explicit set of generators of the algebra $F_n^{\delta_n}$ of constants for $n \geq 5$ where $\delta_5$ is a basic Weitzenböck derivation.

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