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FIELDS WHOSE TORSION FREE PARTS DIVISIBLE WITH TRIVIAL BRAUER GROUP

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Abstract. Let F_0 be an absolutely algebraic field of characteristic $p > 0$ and κ an infinite cardinal. It is shown that there exists a field F such that $F^* \cong F_0^* \oplus (\oplus_\kappa \mathbb{Q})$ with $Br(F) = \{0\}$. Let L be an algebraic closure of F. Then for any finite subextension K of L/F, we have $K^* \cong T(K^*) \oplus (\oplus_\kappa \mathbb{Q})$, where $T(K^*)$ is the group of torsion elements of K^*. In addition, $Br(K) = \{0\}$ and $[K : F] = [T(K^*) \cup \{0\} : F_0]$.

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1. Introduction

In [5, p.299] L. Fuchs asks which abelian groups can be the multiplicative groups of fields. R. M. Dicker in [3] gives an answer to this question in terms of the existence of a certain function on the group with zero adjoined. This question is largely unsolved, though quite a few partial results have been obtained. We refer the reader to [1],[2],[3],[9],[10] and [13] for a sampling of what is known on this question. An abelian group G (written additively) is divisible if for every $g \in G$ and every positive integer n, there exists $h \in G$ with $nh = g$. An abelian group G is divisible modulo its torsion group if $G/T(G)$ is divisible, where $T(G)$ is the group of torsion elements of G. The famous example of a divisible abelian group is the additive group $\mathbb{Q}$ of rational numbers, a torsion free divisible abelian group. Also, $C(p^\infty)$ is a torsion divisible abelian group, when $C(p^\infty)$ is the p-Sylow subgroup of $\mathbb{Q}/\mathbb{Z}$. The structure of divisible abelian groups is well-understood by Theorem 4.1.5 of [15] as following:

Theorem A. *Let G be an abelian group. Then G is divisible if and only if G is a direct sum of copies of $\mathbb{Q}$ and $C(p^\infty)$ for various primes p.*

Given a field F, denote by F^* the multiplicative group of F. For any prime p, let $\mathbb{F}_p$ be its prime subfield, when $Char(F) = p$. An absolutely algebraic field, denoted by aaf, is an algebraic extension of $\mathbb{F}_p$. One may easily check that for any aaf F we have $F = \bigcup_{n\in S} \mathbb{F}_{p^n}$, where S is a nonempty subset of the positive integers such that for any $n, m \in S$ we have $\mathbb{F}_{p^{lcm(n,m)}} \subseteq F$. Also, if $n \in S$ and $x|n$, then $\mathbb{F}_{p^x} \subseteq F$. These conditions are necessary and sufficient conditions for when F is an absolutely algebraic field (aaf). It is also clear that any aaf is perfect.

Here we investigate the question of when a multiplicative group of a field is divisible modulo its torsion group. In this direction we have the following results from [2].

Theorem B. *Let G be an abelian group with finite, nonzero torsion free rank. Then G is not isomorphic to the multiplicative group of any field.*

Theorem C. *A nontrivial torsion-free divisible abelian group G can be realized as the multiplicative group of a field if and only if G has infinite rank.*

Theorem D. *Let G be a torsion-free divisible group of infinite rank and let p be an arbitrary prime integer. Then there is a field F of characteristic p such that F^* is isomorphic to $\mathbb{F}_p^* \oplus G$.*

Finally, the latest result in this direction is the following from [13]:

Theorem E. *Let G be a divisible abelian group. Then G is the multiplicative group of a field of positive characteristic if and only if $G = H \oplus (\oplus_\kappa \mathbb{Q})$ for some divisible abelian group H realizable as the multiplicative group of an absolutely algebraic field of positive characteristic, and either $\kappa = 0$ or κ is infinite.*

On the other hand, we know that for a field F of characteristic zero such that F^* is a divisible group, $Br(F) = \{0\}$. But we don't know what happen for the case $Char(F) > 0$. Also, we have the following question:

Question. Let F be a field such that $F^*/T(F^*)$ is a divisible group. If F is not Euclidean or F is Euclidean and $\sqrt{-1} \in F$, do we have $Br(F) = \{0\}$? If F is Euclidean and $\sqrt{-1} \notin F$, is it true that $Br(F) = \mathbb{Z}_2$?

In this paper by combining Oman's result with having trivial Brauer group, it is proved that given an absolutely algebraic field F_0 of characteristic $p > 0$ and κ an infinite cardinal, then there exists a field F such that $F^* \cong F_0^* \oplus (\oplus_\kappa \mathbb{Q})$ with $Br(F) = \{0\}$. Let L be an algebraic closure of F. Then for any finite subextension K of L/F, we have $K^* \cong T(K^*) \oplus (\oplus_\kappa \mathbb{Q})$. In addition, $Br(K) = \{0\}$ and $[K : F] = [T(K^*) \cup \{0\} : F_0]$. More precisely, it is shown:

Theorem F. *Let $F_0 \subseteq K_0$ be two absolutely algebraic fields of characteristic $p > 0$, and κ an arbitrary infinite cardinal. Then there exist two fields, namely, F and K such that $F \subseteq K$, $Br(F) = Br(K) = \{0\}$, $F^* \cong F_0^* \oplus (\oplus_\kappa \mathbb{Q})$ and $K^* \cong K_0^* \oplus (\oplus_\kappa \mathbb{Q})$.*

2. Preparatory results

Let $F = \bigcup_{n \in S} \mathbb{F}_{p^n}$ be an aaf. For any $m \notin S$ we denote by $F\mathbb{F}_{p^m} = F(\mathbb{F}_{p^m})$, the composition of F and $\mathbb{F}_{p^m}$. It is easily checked that $F(\mathbb{F}_{p^m}) = \bigcup_{t \in Q} \mathbb{F}_{p^t}$, where $Q = \{t \mid \forall n \in S, t = lcm(m,n)\}$. From the standard finite field argument, we obtain the following results.

Lemma 2.1. *Assume that q is a prime and F an aaf of characteristic p.*

(1) *If for any natural number n, $\mathbb{F}_{p^{q^n}} \subseteq F$, there is no extension of degree q over F.*

(2) *If n is the largest natural number such that $\mathbb{F}_{p^{q^n}} \subseteq F$, then $F\mathbb{F}_{p^{q^{n+1}}}$ is the unique extension of degree q over F.*

Corollary 2.2. *Let F be an absolutely algebraic field. If there exists some field extension K_n with $[K_n : F] = n$ for some natural number n, then K_n is unique.*

Let F be a perfect field and L be an algebraic closure of F. We shall call F a *quasi absolutely algebraic field* or simply qaaf if for each n there is at most one subextension K of L/F with $[K : F] = n$. In addition, given a perfect field F, we say that F is a quasi Galois field or qGf if for each n there is one subextension K of L/F with $[K : F] = n$. For example any finite field or any infinite algebraic extensions of them whose 'degree" has no infinite part is a qGf. For some results concerning qGf, we refer the reader to [17]. Further, a field F is called *pseudo absolutely algebraic field* or paaf if for any finite field extension K of F with $F \subsetneqq K$ one has $T(F^*) \subsetneqq T(K^*)$. It is clear that any absolutely algebraic field is a paaf. To prove our next result we shall need the following:

Abel's Theorem. [8, p.297] *Let K be a field, $n > 2$ an integer, and $a \in K$ with $a \neq 0$. Assume that for all prime numbers p such that $p|n$, we have $a \notin K^p$, and if $4|n$ then $a \notin -4K^4$. Then $X^n - a$ is irreducible in $K[X]$.*

Lemma 2.3. *Let F be an aaf of characteristic $p > 0$ containing a primitive $q-$root of unity ω_q for some prime $q \neq p$. Then, for any natural number i, $\mathbb{F}_{p^{q^i}} \subseteq F$ if and only if the polynomial $x^{q^i} - \omega_q$ has a root in F(or splits over F) for any natural number i.*

Proof. First, let $\mathbb{F}_{p^{q^i}} \subseteq F$ for any natural number i. So, by Lemma 2.1, there is no field extension of degree q over F. Now, assume on the contrary that there exists a natural number n such that $x^{q^{n-1}} - \omega_q$ has a root in F but $x^{q^n} - \omega_q$ has no root in F. Since every finite subgroup of the multiplicative group of a field is

cyclic, we conclude that the maximal q-group in F^* is a finite cyclic group with q^n elements. Take ω_{q^n}, a generator of this group, which is a primitive q^n-root of unity in F. This implies that there is no root in F for the polynomial $x^q - \omega_{q^n}$. But, by Abel's Theorem, F has an extension of degree q, which is a contradiction.
Conversely, assume that $x^{q^i} - \omega_q$ has a root in F for any natural number i. Let n be the least natural number such that $\omega_q \in \mathbb{F}_{p^n}$. By Fermat's Theorem, $q \nmid n$. Consider the maximal q-subgroup of $\mathbb{F}_{p^n}^*$ which is cyclic and take ω_{q^m}, a generator of this group, for some natural number m. Now, if $q \neq 2$, by Abel's Theorem, $x^{q^i} - \omega_{q^m}$ is irreducible over $\mathbb{F}_{p^n}$ for any natural number i. Let $a_i \in F$ be a root of this polynomial in some extension. Then, $[\mathbb{F}_{p^n}(a_i) : \mathbb{F}_{p^n}] = q^i$ and $\mathbb{F}_{p^n}(a_i) = \mathbb{F}_{p^{nq^i}}$ for any natural number i. Thus, $\mathbb{F}_{p^{q^i}} \subseteq \mathbb{F}_{p^n}(a_i) \subseteq F$, as desired. For the case $q = 2$ and $\sqrt{-1} \in \mathbb{F}_p$, it is easily checked that $a \notin -4\mathbb{F}_p^4$ and hence Abel's Theorem may be applied to obtain the result. Finally, if $q = 2$ and $\sqrt{-1} \notin \mathbb{F}_p$, then $\mathbb{F}_p(\sqrt{-1}) = \mathbb{F}_{p^2} \subseteq F$. The maximal 2-subgroup of $\mathbb{F}_{p^2}^*$ is then cyclic. Now, use the same argument as above to end the proof. □

Recall from the theory of ordered fields (cf. [16, Ch. 3]) that a field F is said to be formally real if F admits an ordering, if and only if -1 is not a sum of squares in F. F is said to be real Pythagorean, if every sum of squares is a square in F and $-1 \notin F^{*2}$. F is said to be Euclidean, if F has an ordering with respect to which every positive element is a square. Clearly, if F is Euclidean, then F is real Pythagorean and $F^* = F^{*2} \cup -F^{*2}$. Notice that an ordered field necessarily has characteristic 0. Therefore, any Euclidean field has characteristic 0. In the following proposition, we obtain some properties of quasi absolutely algebraic fields.

Proposition 2.4. *Let F be a qaaf and q a prime number. Then we have:*

(1) *Every finite extension of F is a qaaf.*
(2) *Every finite extension K/F is cyclic Galois.*
(3) *If the maximal q-subgroup of F^* is trivial, then $F^* = F^{*q}$.*
(4) *If the maximal q-subgroup of F^* is a nontrivial finite group of order q^n for some natural number n and ω_{q^n} is the primitive q^n-root of unity in F, then for any natural number i with $1 \leq i \leq n$ we have $|F^*/F^{*q^i}| = q^i$ and $F^* = F^{*q^n}\langle\omega_{q^n}\rangle$, the subgroup of F generated by F^{*q^n} and $\langle\omega_{q^n}\rangle$.*
(5) *If the maximal q-subgroup of F^* is an infinite group, then either $F^* = F^{*q}$ or $|F^*/F^{*q^i}| = q^i$ for any natural number i.*
(6) *If F is not Euclidean or F is Euclidean and $\sqrt{-1} \in F$, then $Br(F) = \{0\}$. Moreover, for any finite extension K of F we have $N_{K/F}(K) = F$.*
(7) *If F is Euclidean and $\sqrt{-1} \notin F$, then $Br(F) = \mathbb{Z}_2$.*

Proof. (1) This is clear by the definition of qaaf.

(2) Take the normal closure M of K/F. By our assumption, for each n there is at most one subextension L of M/F with $[L : F] = n$. So, by the Fundamental Theorem of Galois Theory, any subgroup of $G = Gal(M/F)$ is normal. Thus, by Dedekind-Baer Theorem (cf. [15, p.143]) and the uniqueness of the subgroups involved, we conclude that G is a cyclic group, i.e., K/F is a finite cyclic Galois extension.

(3) We know that the maximal q-subgroup of F^* is trivial. If $Char F = q > 0$, then $F^* = F^{*q}$ since F is perfect. Now, assume that $Char F \neq q$. On the contrary, choose $a \in F^* \backslash F^{*q}$. By Abel's Theorem and the previous case, the splitting field of $x^q - a$ is a cyclic Galois extension of degree q over F. Hence $[F(\omega_q) : F] = q$, which is a contradiction and therefore $F^* = F^{*q}$.

(4) We first take $q \neq 2$. By Abel's Theorem, $x^{q^i} - \omega_{q^n}$ is irreducible over F for $1 \leq i \leq n$ and the splitting field of $x^{q^i} - \omega_{q^n}$ for any natural number i is the unique extension of degree q^i over F. Now, using Kummer Theory, we conclude that we have $|F^*/F^{*q^i}| = q^i$, where $1 \leq i \leq n$. Finally, since $F^{*q^n} \cap \langle \omega_{q^n} \rangle$ is trivial we obtain $F^* = F^{*q^n} \langle \omega_{q^n} \rangle$. Now, consider the case $q = 2$ and $\sqrt{-1} \in F$. It is easily checked that $\omega_{2^n} \notin -4F^4$. Hence using Abel's Theorem and the same argument as above we obtain the result. If $q = 2$ and $\sqrt{-1} \notin F$, then the splitting field of $x^2 + 1$ is the unique extension of degree 2 over F. Thus, $F^* = F^{*2} \langle -1 \rangle$.

(5) Assume that $F^* \neq F^{*q}$ and take $a \in F^* \backslash F^{*q}$. If $q \neq 2$, then the Abel's Theorem shows that the splitting field of $x^{q^i} - a$ for any natural number i is the unique extension of degree q^i over F. Using Kummer Theory, one concludes that $|F^*/F^{*q^i}| = q^i$ for any natural number i. Now, if $q = 2$, assume on the contrary that $a = -4k^4$ for some $k \in F$. Since $\sqrt{-1} \in F$, we conclude that $a \in F^{*2}$, which is a contradiction. Therefore, we may apply the same argument as in the previous case to obtain the conclusion.

(6) Let, on the contrary, q be a prime and $0 \neq [A] \in Br(F)$ such that $q[A] = 0$. We may assume that A is an F-central division algebra. From (2) we conclude that A is a noncommutative cyclic division algebra. By Lemma 15.1 of (cf. [14, p.278]), for any maximal subfield L of A, we have $N_{L/F}(L^*) \neq F^*$.

If $\omega_q \notin F$, we obtain $F^* = F^{*q}$, which in turn implies that $F^* = F^{*q^n}$ for any natural number n. On the other hand, $o([A]) = q$, the order of $[A]$ in $Br(F)$. Thus, by Theorem 11 of (cf. [4, p.66]), $i(A) = q^i$ for some natural number i, where $i(A)^2 = [A : F]$. For any maximal subfield L of A we have $[L : F] = q^i$ and hence $F^{*q^i} \subseteq N_{L/F}(L^*)$. This means that $F^* = N_{L/F}(L^*)$ which contradicts our assumption.

Now, if $\omega_q \in F$, by Merkurjev-Suslin Theorem, (cf. [11, p.2616]), there exists a noncommutative F-central division algebra D such that $[D : F] = q^2$. Take a maximal subfield L of D so that $[L : F] = q$. Since L/F is cyclic and $\omega_q \in F$, by Theorem 9.5 of (cf. [12, p.89]), $L = F(\sqrt[q]{b})$ for some $b \in F$. By the case (4) we have $|F^*/F^{*q}| = q$. Take $a \in L$ with $a^q = b$ so that bF^{*q} is a generator for F^*/F^{*q}. Now, let q be odd. Since the minimal polynomial of a over F is $x^q - b$, we obtain $N_{L/F}(a) = (-1)^{q+1}b = b$. On the other hand, $F^{*q} \subseteq N_{L/F}(L^*)$ and hence $F^* = \langle b\rangle F^{*q} \subseteq N_{L/F}(L^*)$. Thus $N_{L/F}(L) = F$ which is a contradiction. We may therefore assume that q = 2. If $\sqrt{-1} \in F$, then $N_{L/F}(\sqrt{-1}a) = b$ and hence $N_{L/F}(L) = F$, which is also a contradiction. Thus, we must have $\sqrt{-1} \notin F$, $F = \langle -1\rangle F^{*2}$ and $L = F(\sqrt{-1})$. But since $F^{*2} \subseteq N_{L/F}(L^*)$ and $N_{L/F}(L^*) \neq F^*$, we obtain $N_{L/F}(L^*) = F^{*2}$. Therefore, given $c, d \in F^*$, from $L = F(\sqrt{-1})$ we obtain $c^2 + d^2 = N_{L/F}(c + d\sqrt{-1}) \in F^{*2}$, which means that F is Euclidean, a contradiction. Finally, since every finite extension K of F is cyclic and $Br(F) = \{0\}$, by Lemma 15.1 of (cf. [14, p.278]), we have $N_{K/F}(K) = F$.

(7) Assume that F is Euclidean and $\sqrt{-1} \notin F$. By the same argument as used in the previous case, for any prime $q \neq 2$ we have $qBr(F) = \{0\}$. Using cases (1) and (6), we obtain $Br(F(\sqrt{-1})) = \{0\}$ and $N_{F(\sqrt{-1})/F}(F(\sqrt{-1})^*) = F^{*2}$. Therefore, $Br(F) \subseteq Br(F(\sqrt{-1})/F)$. Now, by Theorem 7 of (cf. [4, p.64]), for every F-central division algebra D there exists an F-central simple algebra A such that $[D] = [A]$ and $F(\sqrt{-1})$ is a maximal subfield of A. Since $N_{F(\sqrt{-1})/F}(F(\sqrt{-1})^*) = F^{*2}$, A is a division algebra and hence a quaternion division algebra. Because the only quaternion division algebra over a Euclidean field is the ordinary quaternion division algebra, we obtain $Br(F) = \mathbb{Z}_2$. □

In the next proposition we present some properties of pseudo absolutely algebraic fields (paaf) with positive characteristic.

Proposition 2.5. *Let F be a pseudo absolutely algebraic field (paaf) of characteristic $p > 0$ and f the algebraic closure of $\mathbb{F}_p$ in F. Thus, f is absolutely algebraic with $T(F^*) = f^*$ and we have the following:*

(1) *F is a qaaf.*

(2) *F has a unique extension of degree n if and only if f has a unique extension of degree n. If K/F and k/f are the unique extensions of degree n, then $T(K^*) = k^*$.*

(3) *Any finite extension of* F *is a paaf.*

(4) $F^*/T(F^*)$ *is a torsion-free divisible group.*

(5) *If* $T(F^*)$ *is a direct product of a divisible and bounded group, then* $F^* \cong f^* \oplus (\oplus_\kappa \mathbb{Q})$. *If* F *is an aaf, then* $\kappa = 0$. *Otherwise,* $\kappa = |F|$.

Proof. (1) Let $\overline{F}$ and $\overline{f}$ be the algebraic closure of F and f, respectively, where $\overline{f} \subseteq \overline{F}$. First, assume that q is a prime and L/F a finite extension of degree q. We show that L is unique. Set $L \cap \overline{f} = N$. Since F is paaf we have $N \neq f$. Now, N/f is Galois as $\overline{f}/f$ is abelian Galois and hence by the Natural Irrationalities Theorem, $Gal(N/f) \cong Gal(NF/F)$ and so $F \neq NF \subseteq L$. This yields $[N : f] = [NF : F] = [L : F] = q$. On the other hand, by Lemma 2.1, if n is the largest natural number such that $\mathbb{F}_{p^{q^n}} \subseteq f$, then $f\mathbb{F}_{p^{q^{n+1}}}$ is the unique extension of degree q over f. This means that $L = F(\mathbb{F}_{p^{q^{n+1}}})$, as desired.

Next step is to prove that any finite extension L/F is cyclic. Since F is a paaf, it is easily checked that F is perfect. Thus, we can view L as a finite Galois extension of F. Set $G = Gal(L/F)$ and take a maximal subgroup H of G. By the Fundamental Theorem of Galois Theory, there exists an extension M/F such that $[M : F] = [G : H]$. Since F is a qaaf and H is maximal, we conclude that $[G : H]$ is a prime number. Using the first step, we conclude that H is normal in G. Hence, any maximal subgroup of G is normal and by Theorem 5.2.4 of [15], we obtain that G is nilpotent and therefore is a direct product of its Sylow subgroups. By the uniqueness of the maximal subgroups corresponding to prime indices dividing $|G|$, we conclude that G is a cyclic group. So, any finite extension of F is cyclic Galois. Now, if L_1/F and L_2/F are two finite extensions of degree n, then L_1L_2/F is also cyclic. Thus, $L_1 = L_2$ which means that F is a qaaf.

(2) For the natural number n, let, by Lemma 2.1, k/f be the unique extension of degree n. By the Natural Irrationalities Theorem, $Gal(k/f) \cong Gal(kF/F)$ and hence $[F(k) : F] = n$. Since F is a qaaf $F(k)/F$ is unique of degree n. On the other hand, we have $k \subseteq F(k) \cap \overline{f} = T(F(k)^*) \cup \{0\}$. By the Natural Irrationalities Theorem and the uniqueness of the extension, we conclude that $T(F(k)^*) = k^*$. Now, assume that K/F is finite of degree n for some natural number n. Since K/F is cyclic for any prime $q|n$ there exists a unique field K_q such that $F \subseteq K_q \subseteq K$ and $[K_q : F] = q$. By the same argument as used in the case (1), we conclude that there exists some natural number n_q such that $\mathbb{F}_{p^{q^{n_q}}} \nsubseteq f$. Now, by Lemma 2.1, we obtain that f has a unique extension of degree n. Hence, by the same argument as above the result follows.

(3) This is a consequence of the previous case.

(4) Assume first that F contains a primitive q-root of unity for some prime q and the polynomial $x^{q^i} - \omega_q$ has a root in F for any natural number i. Thus, by Lemma 2.3, $\mathbb{F}_{p^{q^i}} \subseteq f$ for any natural number i. Therefore, by Lemma 2.1, f as well as F has no extension of degree q. This implies that $F^* = F^{*q}$. Since F is a qaaf, by Proposition 2.4 and the above argument we have $F^* = F^{*q}$ for any prime q unless the maximal q-subgroup of F^* is a nontrivial finite group of order q^n for some natural number n and ω_{q^n} is the primitive q^n-root of unity in F. In that case we have $F^* = F^{*q^n}\langle\omega_{q^n}\rangle$. Now, setting $G = F^*/T(F^*)$, for any prime number q there exists some natural number n_q such that $G^{q^{n_q}} = G$. Hence $G^q = G$ for any prime number q and therefore $F^*/T(F^*)$ is divisible.

(5) If $T(F^*)$ is a direct product of a divisible and bounded group, then F^* splits over $T(F^*)$ by Theorem 4.3.9 of [15]. Now, by the structure theorem of divisible abelian groups and the previous case we obtain $F^* \cong f^* \oplus (\oplus_\kappa \mathbb{Q})$, where κ is either zero or infinite by Theorem B. Therefore, by Lemma 4.1.3 of [6], we conclude that when F is an aaf, then $\kappa = 0$ and otherwise $\kappa = |F|$. □

Based on the above results, one might propose the following problem.

Question 1. Let F be a pseudo absolutely algebraic field of positive characteristic. Then F^* splits over $T(F^*)$.

3. Main results

Lemma 3.1. *Let p be a prime, f an absolutely algebraic field of characteristic p, and κ any infinite cardinal. Then, there exists a field F of characteristic p such that $T(F^*) = f^*$, $|F| = \kappa$, and F is a pseudo absolutely algebraic field.*

Proof. Assume that X is a family of variables of cardinality κ and consider the rational function field $L = f(X)$. Let $\overline{L}$ and $\overline{f}$ be the algebraic closures of L and f, respectively, so that $\overline{f} \subseteq \overline{L}$. Set

$$S = \{M | L \leqslant M \leqslant \overline{L}; M \cap \overline{f} = f\}.$$

By Zorn's Lemma, S has a maximal element F, say. By the maximality of F, it is clear that F is a paaf and $T(F^*) = f^*$. □

This lemma and Proposition 2.5 enable us to obtain the following:

Corollary 3.2. *Let p be a prime, f an arbitrary finite field of characteristic p, and κ an arbitrary infinite cardinal. Then, there exists a field F of characteristic p such that $F^* \cong f^* \oplus (\oplus_\kappa \mathbb{Q})$. Furthermore, F is a paaf.*

Theorem 3.3. *Let f be an absolutely algebraic field of characteristic $p > 0$, κ an arbitrary infinite cardinal. Then there exist a field F of characteristic $p > 0$ with $Br(F) = \{0\}$ and $F^* \cong f^* \oplus (\oplus_\kappa \mathbb{Q})$. Furthermore, F is a paaf. Let L be an algebraic closure of F. Then for any finite subextension K of L/F, we have $K^* \cong T(K^*) \oplus (\oplus_\kappa \mathbb{Q})$. In addition, $Br(K) = \{0\}$ and $[K : F] = [T(K^*) \cup \{0\} : f]$.*

Proof. By Corollary 3.2, there exists a field, namely, F_0 of characteristic p such that $F_0^* \cong \mathbb{F}_p^* \oplus (\oplus_\kappa \mathbb{Q})$ and further F_0 is a paaf. Since $Char(F_0) > 0$, so F_0 is not Euclidean. Thus, by Proposition 2.4, $Br(F_0) = 0$. Take the chain $\mathbb{F}_p = f_0 \subseteq f_1 \subseteq \cdots \subseteq f$ of finite fields such that $f = \bigcup f_i$ and assume that M is the algebraic closure of F_0. Using the case 2 of Proposition 2.5, we obtain the chain $F_0 \subseteq F_1 \subseteq \cdots \subseteq M$ such that $T({F_i}^*) = {f_i}^*$. Moreover, $Br(F_i) = \{0\}$ and $F_i^* \cong f_i^* \oplus (\oplus_\kappa \mathbb{Q})$ with F_i is a paaf. On the other hand, we have $F_i^* = T({F_i}^*) \oplus N_i$ where N_i is a torsion-free divisible group. If $a \in N_i$, then $a \in F_{i+1}$ and we obtain $a = bc$ for some $b \in f_{i+1}^*$ and $c \in N_{i+1}$. Now, since f_{i+1}^* is a finite group the divisibility of N_{i+1} and N_i gives us $b = 1$ and hence $N_i \subseteq N_{i+1}$. Setting $N = \cup N_i$, we clearly have $f^* = \cup T({F_i}^*)$ and $N \cap f^* = \{1\}$. Thus, N is a torsion free divisible group. It is then easily checked that $\cup F_i = (f^* \oplus N) \cup \{0\}$. Now, we put $F = \cup F_i$ so that F is a field and $F^* \cong f^* \oplus (\oplus_\kappa \mathbb{Q})$, where $T(F^*) = f^*$. If we show that F is a paaf, then by Proposition 2.4 we have $Br(F) = \{0\}$. To end this, assume that K/F is a finite extension and take $\alpha \in K \backslash F$ with the minimal polynomial $f(x) = x^q + a_0 x^{q-1} + \cdots + a_q$ over F with $q = [F(\alpha) : F]$. So, there exists some natural number i such that $a_0, \cdots, a_q \in F_i$ and hence $q = [F_i(\alpha) : F_i]$. By the case 2 of Proposition 2.5, $F_i(\alpha) = F_i(l)$, where l is the unique extension of degree q over f_i. This means that $F(\alpha) = F(l)$. Hence $l \nsubseteq F$ and therefore $T(F^*) \subsetneqq T(K^*)$, i.e., F is a paaf as desired, and the rest follows from Proposition 2.5. □

Using the uniqueness of extensions of absolutely algebraic fields, one may easily check that for any two absolutely algebraic fields $F_0 \subseteq K_0$ of characteristic $p > 0$, we can find the chain $F_0 = f_0 \subseteq f_1 \subseteq \cdots \subseteq K_0$ of finite dimensional absolutely algebraic fields over F_0 such that $K_0 = \bigcup f_i$. Thus, by Propositions 2.4 and 2.5 and using the method of the previous theorem, we obtain

Corollary 3.4. *Let $F_0 \subseteq K_0$ be two absolutely algebraic fields of characteristic $p > 0$, and κ an infinite cardinal. Then there exist two fields, namely, F and K of characteristic $p > 0$ such that $F \subseteq K$, $Br(F) = Br(K) = \{0\}$, $F^* \cong F_0^* \oplus (\oplus_\kappa \mathbb{Q})$, and $K^* \cong K_0^* \oplus (\oplus_\kappa \mathbb{Q})$.*

Remark 3.5. Assume that $F \subsetneqq K$ are two fields of characteristic $p > 0$. Furthermore, set $F^* = T(F^*) \oplus N$ and $K^* = T(K^*) \oplus M$, where M and N are two groups. If $M = N$, then K^*/F^* is a torsion group. Now, by Kaplansky's Theorem (cf. [7, p.245]), F and K are absolutely algebraic fields.

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