International Electronic Journal of Algebra
Volume 32 (2022) 241-245
DOI: 10.24330/ieja.1144159

ON SOLUBILITY OF GROUPS WITH FINITELY MANY CENTRALIZERS

Igor Lima and Caio Rodrigues

Received: 22 December 2021; Revised: 29 March 2022; Accepted: 25 May 2022

Communicated by Abdullah Harmancı

Abstract. In this paper we present a new sufficient condition for a solubility criterion in terms of centralizers of elements. This result is a corrigendum of one of Zarrin's results. Furthermore, we extend some of K. Khoramshahi and M. Zarrin's results in the primitive case.

Mathematics Subject Classification (2020): 20D10, 20D15
Keywords: Centralizer, n-centralizer group, simple group

1. Introduction

Let G be a group, given $g \in G$ we define $C_G(g) = \{x \in G|\ xg = gx\}$ the centralizer of g in G and $Cent(G) = \{C_G(g)|\ g \in G\}$ the set of all centralizers of elements in G. Denote by $|W|$ the cardinal of the set W. If $|Cent(G)| = n \in \mathbb{N}$ we say that G is a $\mathtt{C}_n$-group or that G is a n-centralizer group. If $G/Z(G)$ is an n-centralizer too, we say that G is a primitive n-centralizer group, or simply primitive n-centralizer.

The study of finite groups in terms of $|Cent(G)|$ was started by Belcastro and Sherman in [3]. It is easy to see that a group is 1-centralizer if and only if it is abelian and there is no n-centralizer group for $n = 2, 3$. An n-centralizer group was constructed for each $n \neq 2, 3$ in [2]. We collect a few results in the following theorem.

Theorem 1.1. *Suppose G is a finite n-centralizer group. Then*

(1) $n = 4 \iff G/Z(G) \cong C_2 \times C_2$ *(see [3]).*
(2) $n = 5 \iff G/Z(G) \cong C_3 \times C_3$ *or* S_3 *(see [3]).*
(3) $n = 6 \Rightarrow G/Z(G) \cong D_8,\ A_4,\ C_2 \times C_2 \times C_2$ *or* $C_2 \times C_2 \times C_2 \times C_2$ *(see [2]).*
(4) $n = 7 \iff G/Z(G) \cong C_5 \times C_5,\ D_{10}$ *or* $\langle x, y|\ x^5 = y^4 = 1, x^y = x^3\rangle$ *(see [1]).*
(5) $n = 8 \Rightarrow G/Z(G) \cong C_2 \times C_2 \times C_2,\ A_4$ *or* D_{12} *(see [1]).*

The first author was supported by DPI/UnB and FAPDF, Brazil. The second author was supported by CNPq.

(6) $n = 9 \iff G/Z(G) \cong D_{14},\ C_7 \times C_7,\ Hol(C_7)$ *or a non-abelian group of order* 21 *(see* [6]*).*

(7) $n = 10 \Rightarrow G/Z(G) \cong D_{16},\ C_2^4,\ C_4 \times C_4,\ (C_4 \times C_2) \rtimes C_2,\ C_2 \times D_8,\ C_2^5,\ C_2^6$ *or* $C_2^3 \rtimes C_7$ *(see* [7]*).*

(8) *If* G *is a primitive* 11*-centralizer group of odd order, then* $G/Z(G) \cong (C_9 \times C_3) \rtimes C_3$ *(see* [10]*).*

The concept of isoclinic groups was introduced by P. Hall in [5]. Two groups G_1 and G_2 (not necessarily finite) are said to be *isoclinic* if there are isomorphisms $\varphi : G_1/Z(G_1) \to G_2/Z(G_2)$ and $\phi : G_1' \to G_2'$ such that if $\varphi(a_1Z(G_1)) = a_2Z(G_2)$ and $\phi(b_1Z(G_1)) = b_2(G_2)$, then $\phi([a_1, b_1]) = [a_2, b_2]$ for each $a_1, b_1 \in G_1$ and $a_2, b_2 \in G_2$. It is easy to see that isoclinism is an equivalence relation.

As noted by P. Hall [5], every group G_2 which is isoclinic with G_1 also is isoclinic with the product $G_1 \times A$, where A is an abelian group. Indeed, if G_1 is isoclinic with G_2 and G_3 is isoclinic with G_4, then the direct product $G_1 \times G_3$ is isoclinic with $G_2 \times G_4$. In particular if A is an abelian group, then A is isoclinic with the trivial group, say 1, and therefore G is isoclinic with $G \times A$, for all group G.

In [13] M. Zarrin establishes a relation between isoclinism and the number centralizers of elements of G. He proves that if G_1 and G_2 are isoclinic, then $|Cent(G_1)| = |Cent(G_2)|$. He also proves that if G is an arbitrary group with $|Cent(G)| = n$, then there are only finitely many groups J, up to isoclinism, with $|Cent(J)| = n$, moreover, there exists a finite group K that is isoclinic with G and $|Cent(G)| = |Cent(K)|$. Theorem 3.5 of the same article is an extension of Theorem 1.1 for arbitrary groups. Note that Zarrin proves in [13] the case $|Cent(G)| \leq 8$.

In this short paper we prove that if G is a finite n-centralizer group such that $n \geq 4$ and $|G| < \frac{30n}{19}$, then G is a non-nilpotent solvable group. This fact is a correction of the proof of Theorem B (2) in [12]. Moreover, we extend the Theorem 3.5 in [8] in the primitive case.

Let $I(G)$ be the set of all involutions of a group G, that is, $I(G) = \{a \in G|\ a = a^{-1}\}$. The problem with the proof of Theorem B (2) in [12] is that $|I(G)| \geq \frac{4|G|}{15}$ instead of $|I(G)| > \frac{4|G|}{15}$ and we cannot apply Potter's result, but this problem can be refined if we change the condition in the statement Theorem B (2) to $|G| < 30n/19$.

2. Preliminaries

We shall need the following results in [9] and [12] for the correction of Theorem B in [12]. For the convenience of the reader, we repeat the statements of the followings results.

Lemma 2.1. *Let G be a finite C_n-group. Then*

$$n \leq \frac{|G| + |I(G)|}{2}.$$

Theorem 2.2 (Potter, 1988)**.** *Suppose G admits an automorphism which inverts more than $\frac{4|G|}{15}$ elements. Then G is solvable.*

3. Correction

Now we are ready to prove the following theorem, which is similar to Theorem B in [12], using the same proof outline.

Theorem 3.1. *If G is a finite n-centralizer group with $n \geq 4$, then the following holds:*

(1) *$|G| < 2n$, then G is non-nilpotent.*
(2) *$|G| < \frac{30n}{19}$, then G is a non-nilpotent solvable group.*

Proof. We will just prove part (2). From part (1), which is proved in Theorem B (1) in [12], we have that G is non-nilpotent, since $|G| < \frac{30n}{19} < 2n$. Moreover, since $2n > \frac{19|G|}{15}$, Lemma 2.1 implies that

$$|I(G)| \geq 2n - |G| > \frac{4|G|}{15}.$$

Since $I(G)$ is the set of all elements of G that is inverted by the identity automorphism, Theorem 2.2 completes the proof. □

The condition (2) above is better than the part (2) of Theorem B in [12]. However using a GAP check [11] we don't know an example of a group G such that $|G| < \frac{30n+15}{19}$ and G is not a solvable group. It is immediate from Theorem 1.1 examples of groups where both conditions of Theorem 3.1 holds exist, for instance $G = S_3$ and $n = 5$.

4. A condition for isoclinism

We will need of a Lemma (see Lemma 3.3 in [8]).

Lemma 4.1. *Let H a subgroup of an arbitrary group G such that $|Cent(H)| = |Cent(G)|$. Then $H \cap Z(G) = Z(H)$ and $\frac{H}{Z(H)} \cong \frac{HZ(G)}{Z(G)}$. In particular, H is isoclinic with $HZ(G)$.*

Using similar arguments we extend Theorem 3.5 in [8] and for the case $n = 11$, we add the hypothesis that H is a primitive 11-centralizer group.

Theorem 4.2. *Let G be a non-abelian arbitrary group. If $H \leq G$, $|Cent(G)| = |Cent(H)| = n = 8$, then H is isoclinic with G. This result still holds if $n = 11$, H is primitive and G is a primitive 11-centralizer of odd order.*

Proof. From Lemma 4.1, $1 \neq \frac{H}{Z(H)} \cong \frac{HZ(G)}{Z(G)} \leq \frac{G}{Z(G)}$. By Zarrin's Theorem 3.3 (2) [13] there is a finite group K which is isoclinic with G and $|Cent(G)| = |Cent(K)|$, so $G/Z(G) \cong K/Z(K)$. Let $|Cent(G)| = |Cent(K)| = n = 8$. From Theorem 1.1 we have that $K/Z(K) \cong G/Z(G) \cong C_2 \times C_2 \times C_2$, A_4 or D_{12}. If $\frac{HZ(G)}{Z(G)} < \frac{G}{Z(G)}$, we have that $\frac{H}{Z(H)} \cong C_2$, $C_2 \times C_2$, C_3, C_6, or S_3. If $\frac{H}{Z(H)}$ is cyclic, then H is abelian, which is a contradiction. If $\frac{H}{Z(H)} \cong S_3$ or $C_2 \times C_2$, from Theorem 1.1, $|Cent(H)| = 5$ or 4, which is a contradiction. Therefore from Lemma 4.1 it follows that $H/Z(H) \cong \frac{HZ(G)}{Z(G)} = \frac{G}{Z(G)}$. Let $n = 11$ and suppose that G is a primitive 11-centralizer group of odd order. From Theorem 1.1 we have that $G/Z(G) \cong (C_9 \times C_3) \rtimes C_3$. If $\frac{HZ(G)}{Z(G)} < \frac{G}{Z(G)}$, we have that $\frac{H}{Z(H)} \cong C_3$, $C_3 \times C_3$, C_9, $C_9 \times C_3$, or $(C_3 \times C_3) \rtimes C_3$. Again, $\frac{H}{Z(H)}$ can't be cyclic. Using the GAP (see [10]), and the fact that H is primitive, we can verify that if $\frac{H}{Z(H)} \cong C_3 \times C_3$, $C_9 \times C_3$, or $(C_3 \times C_3) \rtimes C_3$ then $11 = |Cent(H)| = |Cent(\frac{H}{Z(H)})| = 1$ or 5, which is a contradiction. Therefore from Lemma 4.1 it follows that $H/Z(H) \cong \frac{HZ(G)}{Z(G)} = \frac{G}{Z(G)}$. In either case we obtain $\frac{H}{Z(H)} \cong \frac{HZ(G)}{Z(G)} = \frac{G}{Z(G)}$, so $HZ(G) = G$. Again by Lemma 4.1, H is isoclinic with $HZ(G) = G$. □

Acknowledgement. The authors would like to thank the referees for the valuable suggestions and comments.

References

[1] A. Abdollahi, S. M. Jafarian Amiri and A. M. Hassanabadi, *Groups with specific number of centralizers*, Houston J. Math., 33(1) (2007), 43-57.

[2] A. R. Ashrafi, *On finite groups with a given number of centralizers*, Algebra Colloq., 7 (2000), 139-146.

[3] S. M. Belcastro and G. J. Sherman, *Counting centralizers in finite groups*, Math. Mag., 67(5) (1994), 366-374.

[4] Z. Foruzanfar and Z. Mostaghim, *On 10-centralizer groups of odd order*, ISRN Algebra 2014, 607984 (4pp).

[5] P. Hall, *The classification of prime power groups*, J. Reine Agnew. Math., 182 (1940), 130-141.

[6] S. M. Jafarian Amiri, H. Madadi and H. Rostami, *On 9-centralizer groups*, J. Algebra Appl., 14 (1) (2015), 1550003 (13 pp).

[7] S. M. Jafarian Amiri, H. Madadi and H. Rostami, *Groups with exactly ten centralizers*, Bull. Iranian Math. Soc., 44 (2018), 1163-1170.

[8] K. Khoramshahi and M. Zarrin, *Groups with the same number of centralizers*, J. Algebra Appl. 20(2) (2021), 2150012 (6 pp).

[9] W. M. Potter, *Nonsolvable groups with an automorphism inverting many elements*, Arch. Math. (Basel), 50 (1998), 292-299.

[10] M. Rezaei and Z. Foruzanfar, *On primitive 11-centralizer groups of odd order*, Malays. J. Math. Sci., 10(3) (2016), 361-368.

[11] The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.10.2; 2019. (https://www.gap-system.org)

[12] M. Zarrin, *On solubility of groups with finitely many centralizers*, Bull. Iranian Math. Soc., 39 (2013), 517-521.

[13] M. Zarrin, *On noncommuting sets and centralisers in infinite groups*, Bull. Aust. Math. Soc., 93 (2016), 42-46.

Igor Lima (Corresponding Author)
Departamento de Matemática
Universidade de Brasília
Brasilia-DF, 70910-900 Brazil
e-mail: igor.matematico@gmail.com

Caio Rodrigues
IMPA
Estrada Dona Castorina, 110
22460-320 Rio de Janeiro, Brazil
e-mail: oiac10@gmail.com