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**Research Article**

# Generalization of an Integer Sequence Associated with Tribonacci Numbers

Barış Arslan[1*], Kemal Uslu[2]

[1*] Selçuk University, Faculty of Science, Departmant of Mathematics, Konya, Turkey, (ORCID: 0000-0002-6972-3317), barismath@gmail.com
[2] Selçuk University, Faculty of Science, Departmant of Mathematics, Konya, Turkey, (ORCID: 0000-0001-6265-3128), kuslu@selcuk.edu.tr

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**Abstract**

In this paper we first consider an integer sequence which enumerates the number of subsets of S of the set $[n] = \{1, 2, \ . \ . \ . \ , n\}$ containing no three consecutive odd integers. Then we generalize this sequence to a polynomial sequence which is associated with the Tribonacci polynomials. Next, we obtain some basic properties of the polynomial sequence.

**Keywords:** Polynomial sequence, Tribonacci polynomials, Consecutive odd integers, Generating function.

# Tribonacci Sayıları ile İlişkili Bir Tamsayı Dizisininin Genellemesi

**Öz**

Bu çalışmada önce $\{1, 2, \ . \ . \ . \ , n\}$ kümesinin ardışık üç tek sayı içermeyen S alt kümelerinin sayısına karşılık gelen tam sayı dizisini göz önüne aldık. Sonra bu diziyi, Tribonacci polinomları ile ilişkili bir polinom dizisine genelledik. Daha sonar polinom dizisinin bazı temel özelliklerini elde ettik.

**Anahtar Kelimeler:** Polinom dizisi, Tribonacci polinomları, Ardışık tek sayılar, Üreteç fonksiyon.

* Corresponding Author: barismath@gmail.com

# 1. Introduction

Tribonacci numbers can be generalized as Tribonacci polynomials. There are many studies about Tribonacci polynomials. You can see [4-9] for the studies. In 1973 Tribonacci polynomial sequence $(T_n(x))$was defined by Hoggatt and Bicknell in [3].

For $n \geq 3$,

$$T_n(x) = x^2 T_{n-1}(x) + xT_{n-2}(x) + T_{n-3}(x), \quad (1)$$

with initial conditions

$$T_0(x) = 0, \quad T_1(x) = 1, \quad T_2(x) = x^2.$$

When $x = 1$, we obtain the Tribonacci sequence $(T_n)_{n\geq 0}$. Generating function for Tribonacci polynomial sequence is given in [9] as follows

$$\sum_{n=0}^{\infty} T_n(x)\, t^n = \frac{t}{1 - x^2 t - x t^2 - t^3}. \quad (2)$$

Binet's formula of Tribonacci polynomial is given in [5] by

$$T_n(x) = \frac{\alpha(x)^{n+1}}{(\alpha(x) - \beta(x))(\alpha(x) - \gamma(x))}$$

$$+ \frac{\beta(x)^{n+1}}{(\beta(x) - \alpha(x))(\beta(x) - \gamma(x))}$$

$$+ \frac{\gamma(x)^{n+1}}{(\gamma(x) - \alpha(x)))(\gamma(x) - \beta(x))}, \quad (3)$$

Where $\alpha(x)$, $\beta(x)$ and $\gamma(x)$ are the distinct roots of $t^3 - x^2 t^2 - xt - 1 = 0$ which is the characteristic equation of (1).

Consider the sequence $(a_n)_{n\geq 0}$ counting the number of subsets S of the set [n] = $\{1, 2, \ . \ . \ . \ , n\}$ such that S contains no three consecutive odd integers. You can see [1] for a detailed study about the sequence $(a_n)_{n\geq 0}$ .

$$a_n = 2a_{n-2} + 4a_{n-4} + 8a_{n-6}, \quad n \geq 6, \quad (4)$$

$$a_0 = 1, \ a_1 = 2, \ a_2 = 4, \ a_3 = 8, \ a_4 = 16, \ a_5 = 28.$$

Here we first define the polynomial sequence $(a_n(\mathrm{x}))_{n\geq 0}$ using (4) and next, we obtain some basic properties of the polynomial sequence.

# 2. Main Results

## 2.1. Recursive Definition of the Polynomial Sequence

Let us define the polinomial sequence $(a_n(x))$ with the help of the recurrence relation (4) as follows:

$$a_n(x) = 2x^4 a_{n-2}(x) + 4x^2 a_{n-4}(x) + 8a_{n-6}(x). \quad (5)$$

The first few polynomials are:

$$a_0(x) = 1$$

$$a_1(x) = 2$$

$$a_2(x) = 4$$

$$a_3(x) = 8$$

$$a_4(x) = 16$$

$$a_5(x) = 28$$

$$a_6(x) = 32x^4 + 16x^2 + 8$$

$$a_7(x) = 56x^4 + 32x^2 + 16$$

$$a_8(x) = 64x^8 + 32x^6 + 16x^4 + 64x^2 + 32$$

Notice that $a_n(1) = a_n$.

## 2.2. Generating Function and the Closed Form Formula of the Polynomial Sequence

Let us try to find the generating function $G(x,t)$ of the polynomial sequence $(a_n(x))$ using the formal power series.

$$G(x,t) = \sum_{n=0}^{\infty} a_n(x) t^n$$

To find $G(x,t)$, multiply both sides of the recurrence relation (5) by $t^n$ and sum over the values of $n$ for which the recurrence is valid, namely, over $n \geq 6$. We get,

$$\sum_{n\geq 6} a_n(x) t^n = \sum_{n\geq 6} 2x^4 a_{n-2}(x) t^n$$

$$+ \sum_{n\geq 6} 4x^2 a_{n-4}(x) t^n + \sum_{n\geq 6} 8a_{n-6}(x) t^n \quad (6)$$

Then try to relate these sums to the unknown generating function $G(x,t)$. We have

$$\sum_{n\geq 6} a_n(x)\, t^n$$

$$= G(x,t) - a_0(x) - a_1(x)t - a_2(x)t^2 - a_3(x)t^3 - a_4(x)t^4 - a_5(x)t^5$$

$$= G(x,t) - 1 - 2t - 4t^2 - 8t^3 - 16t^4 - 28t^5,$$

$$\sum_{n\geq 6} 2x^4 a_{n-2}(x)t^n = 2x^4t^2 \sum_{n\geq 6} a_{n-2}(x)t^{n-2}$$

$$= 2x^4t^2\,(G(x,t) - a_0(x) - a_1(x)t - a_2(x)t^2 - a_3(x)t^3)$$

$$= 2x^4t^2\,(G(x,t) - 1 - 2t - 4t^2 - 8t^3),$$

$$\sum_{n\geq 6} 4x^2 a_{n-4}(x)t^n = 4x^2t^4 \sum_{n\geq 6} a_{n-4}(x)t^{n-4}$$

$$= 4x^2t^4\,(G(x,t) - 1 - 2t),$$

$$\sum_{n\geq 6} 8a_{n-6}(x)t^n = 8t^6 \sum_{n\geq 6} a_{n-6}(x)t^{n-6}$$

$$= 8t^6 G(x,t).$$

If we write these results on the two sides of (6), we find that

$$G(x,t) - 1 - 2t - 4t^2 - 8t^3 - 16t^4 - 28t^5$$

$$= 2x^4t^2\,(G(x,t) - 1 - 2t - 4t^2 - 8t^3)$$

$$+4x^2t^4\,(G(x,t) - 1 - 2t) + 8t^6G(x,t),$$

$$G(x,t) = \frac{1 + 2t + (4-2x^4)t^2 + (8-4x^4)t^3}{1 - 2x^4t^2 - 4x^2t^4 - 8t^6}$$

$$+\frac{(16 - 4x^2 - 8x^4)t^4 + (28 - 8x^2 - 16x^4)t^5}{1 - 2x^4t^2 - 4x^2t^4 - 8t^6}. \qquad (7)$$

Substituting $x = 1$, we get the generating function for the integer sequence $(a_n)_{n\geq 0}$.

**Theorem 1.** *Let $(a_n(x))$ is the polynomial sequence defined by (5). Then we have*

$$a_{2n}(x) = 2^n[T_{n+1}(x^2) + (2 - x^4)T_n(x^2)$$
$$+(4 - x^2 - 2x^4)T_{n-1}(x^2)],$$

$$a_{2n+1}(x) = 2^n[2T_{n+1}(x^2) + (4 - 2x^4)T_n(x^2)$$
$$+(7 - 2x^2 - 4x^4)T_{n-1}(x^2),$$

*where $T_n(x)$ is the nth Tribonacci polynomial.*

**Proof.** If $A(x,t)$ is the generating function for even terms of the polinomial sequence $(a_n(x))_{n\geq 0}$ then it is clear that $A(x,t) = \frac{1}{2}(G(x,t) + G(x,-t))$. From (7) we get,

$$A(x,t) = \frac{1 + (2 - x^4)2t^2 + (4 - x^2 - 2x^4)4t^4}{1 - 2x^4t^2 - 4x^2t^4 - 8t^6},$$

$$A(x,t) = \frac{1}{1 - 2x^4t^2 - 4x^2t^4 - 8t^6}$$

$$+(2 - x^4)\frac{2t^2}{1 - 2x^4t^2 - 4x^2t^4 - 8t^6}$$

$$+(4 - x^2 - 2x^4)\frac{4t^4}{1 - 2x^4t^2 - 4x^2t^4 - 8t^6}. \qquad (8)$$

Let's write the generating function of the Tribonacci polynomial sequence with initial conditions $T_0(x) = 0$, $T_1(x) = 1$, $T_2(x) = x^2$ which is given in (2):

$$t(x,z) = \frac{z}{1 - x^2z - xz^2 - z^3}.$$

Let us indicate the correspondence between a sequence and its generating function with a double-sided arrow as follows:

$$\langle 1, x^2, x^4 + x, \ldots \rangle \leftrightarrow \frac{1}{1 - x^2z - xz^2 - z^3} \qquad (9)$$

$$\langle 0, 1, x^2, x^4 + x, \ldots \rangle \leftrightarrow \frac{z}{1 - x^2z - xz^2 - z^3} \qquad (10)$$

$$\langle 0, 0, 1, x^2, x^4 + x, \ldots \rangle \leftrightarrow \frac{z^2}{1 - x^2z - xz^2 - z^3} \qquad (11)$$

If we right-shift the polynomial sequence in (9) by adding respectively one and two leading zeros, we obtain the polynomial sequences (10) and (11). Hence (9), (10) and (11) are respectively generating functions of the polynomial sequences $(T_{n+1}(x))$, $(T_n(x))$ and $(T_{n-1}(x))$.

Substituting $x^2$ for $x$ and writing $z = 2t^2$ into (9), (10) and (11). Together with these and using (8) we get the coefficients of $t^{2n}$ which gives the exact formula for the polynomial sequence $(a_{2n}(x))$,

$$a_{2n}(x) = 2^n[T_{n+1}(x^2) + (2 - x^4)T_n(x^2)$$
$$+(4 - x^2 - 2x^4)T_{n-1}(x^2)]$$

where $T_n(x)$ is the Tribonacci polynomial sequence defined by (1).

If $B(x,t)$ is the generating function for odd terms of the polinomial sequence then it is clear that $B(x,t) = \frac{1}{2}(G(x,t) - G(x,-t))$. Using (7) we get,

$$B(x,t) = \frac{2t + (8-4x^4)t^3 + (28-8x^2-16x^4)t^5}{1-2x^4t^2-4x^2t^4-8t^6}$$

$$B(x,t) = t[\frac{2}{1-2x^4t^2-4x^2t^4-8t^6}$$

$$+(4-2x^4)\frac{2t^2}{1-2x^4t^2-4x^2t^4-8t^6}$$

$$+(7-2x^2-4x^4)\ \frac{4t^4}{1-2x^4t^2-4x^2t^4-8t^6}] \quad (12)$$

Substituting (9), (10) and (11) into the equation (12) we get the coefficients of $t^{2n+1}$ which gives the general term for the polynomial subsequence $(a_{2n+1}(x))$

$$a_{2n+1}(x) = 2^n [2T_{n+1}(x^2) + (4-2x^4)T_n(x^2)$$
$$+(7-2x^2-4x^4)T_{n-1}(x^2)],$$

where $T_n(x)$ is the Tribonacci polynomial defined by (1).

The proof is completed.

Notice that,

$$a_{2n}(1) = a_{2n} = 2^n T_{n+2},$$
$$a_{2n+1}(1) = a_{2n+1} = 2^n T_{n+3}.$$

## 2.3. The Sum of the First n Terms of the Polynomial Sequence

In [4] the sum of the Tribonacci polynomials is obtained as

$$\sum_{k=0}^{n} T_k(x) = \frac{T_{n+2}(x) + (1-x^2)T_{n+1}(x) + T_n(x) - 1}{x^2+x}.$$

***Theorem 2.*** *Let $(a_n(x))$ is the polynomial sequence defined by (5) and $T_n(x)$ is the nth Tribonacci polynomial. Then for $n \geq 4$ we have*

$$\sum_{k=0}^{2n} a_k(x) = \frac{2^n A_n(x) + 30x^4 + 12x^2 - 59}{2x^4+4x^2+7}$$

$$\sum_{k=0}^{2n+1} a_k(x) = \frac{2^n B_n(x) + 30x^4 + 12x^2 - 59}{2x^4+4x^2+7}$$

*Where*

$$A_n(x) = [(2x^4+4x^2+8)T_{n+1}(x^2)$$
$$+(-2x^8-4x^6-2x^4+14x^2+28)T_n(x^2)$$
$$+(-6x^8-16x^6-20x^4+22x^2+62)T_{n-1}(x^2)$$
$$+(-4x^8-16x^6-29x^4+14x^2+58)T_{n-2}(x^2)$$
$$+(-4x^6-16x^4+x^2+26)T_{n-3}(x^2)$$
$$+(-4x^4-2x^2+7)T_{n-4}(x^2),$$

$$B_n(x) = [(6x^4+12x^2+22)T_{n+1}(x^2)$$
$$+(-6x^8-12x^6-8x^4+30x^2+56)T_n(x^2)$$
$$+(-14x^8-36x^6-42x^4+36x^2+111)T_{n-1}(x^2)]$$
$$+(-4x^8-16x^6-29x^4+14x^2+58)T_{n-2}(x^2)$$
$$+(-4x^6-16x^4+x^2+26)T_{n-3}(x^2)$$
$$+(-4x^4-2x^2+7)T_{n-4}(x^2).$$

***Proof.*** Let $(S_n(x))_{n\geq 0}$ be the sum of first n terms of the polynomial sequence $(a_n(x))$:

$$S_n(x) = \sum_{k=0}^{n} a_k(x)$$

Using recurrence relation (5) and its initial conditions we have

$$a_n(x) = 2x^4 a_{n-2}(x) + 4x^2 a_{n-4}(x) + 8a_{n-6}(x),$$

$$a_0(x) = 1,\ a_1(x) = 2,\ a_2(x) = 4,\ a_3(x) = 8,$$
$$a_4(x) = 16,\ a_5(x) = 28.$$

For $n > 5$, we can write the following equalities:

$$a_6(x) = 2x^4 a_4(x) + 4x^2 a_2(x) + 8a_0(x)$$
$$a_7(x) = 2x^4 a_5(x) + 4x^2 a_3(x) + 8a_1(x)$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$a_n(x) = 2x^4 a_{n-2}(x) + 4x^2 a_{n-4}(x) + 8a_{n-6}(x)$$

Adding all these equations term by term and substituting initial values we have

$$S_n(x) = \frac{(2x^4+4x^2+8)[a_n(x)+a_{n-1}(x)]}{2x^4+4x^2+7}$$
$$+\frac{(4x^2+8)\ [a_{n-2}(x)+a_{n-3}(x)]}{2x^4+4x^2+7}$$
$$+\frac{8\,[a_{n-4}(x)+a_{n-5}(x)] + 30x^4 + 12x^2 - 59}{2x^4+4x^2+7}.$$

Let's write $S_{2n}(x)$:

$$S_{2n}(x) = \frac{(2x^4+4x^2+8)[a_{2n}(x)+a_{2n-1}(x)]}{2x^4+4x^2+7}$$

$$+\frac{(4x^2+8)\,[a_{2n-2}(x)+a_{2n-3}(x)]}{2x^4+4x^2+7}$$

$$+\frac{8\,[a_{2n-4}(x)+a_{2n-5}(x)]}{2x^4+4x^2+7}+\frac{30x^4+12x^2-59}{2x^4+4x^2+7}. \quad (13)$$

Using (13) and Theorem 1 we have

$$S_{2n}(x)=\sum_{k=0}^{2n} a_k(x)=\frac{2^n A_n(x)+30x^4+12x^2-59}{2x^4+4x^2+7}, \quad (14)$$

where

$$A_n(x)=[(2x^4+4x^2+8)T_{n+1}(x^2)$$
$$+(-2x^8-4x^6-2x^4+14x^2+28)T_n(x^2)$$
$$+(-6x^8-16x^6-20x^4+22x^2+62)T_{n-1}(x^2)$$
$$+(-4x^8-16x^6-29x^4+14x^2+58)T_{n-2}(x^2)$$
$$+(-4x^6-16x^4+x^2+26)T_{n-3}(x^2)$$
$$+(-4x^4-2x^2+7)T_{n-4}(x^2).$$

Let us find the following summation formula for $n\geq 4$,

$$\sum_{k=0}^{2n+1} a_k(x).$$

$$\sum_{k=0}^{2n+1} a_k(x)=a_{2n+1}(x)+\sum_{k=0}^{2n} a_k(x)$$

From Theorem 1 and (14) we have

$$\sum_{k=0}^{2n+1} a_k(x)=\frac{2^n B_n(x)+30x^4+12x^2-59}{2x^4+4x^2+7}, \quad (15)$$

where

$$B_n(x)=[(6x^4+12x^2+22)T_{n+1}(x^2)$$
$$+(-6x^8-12x^6-8x^4+30x^2+56)T_n(x^2)$$
$$+(-14x^8-36x^6-42x^4+36x^2+111)T_{n-1}(x^2)]$$
$$+(-4x^8-16x^6-29x^4+14x^2+58)T_{n-2}(x^2)$$
$$+(-4x^6-16x^4+x^2+26)T_{n-3}(x^2)$$

The proof is completed.

***Example 1.*** *Let us compute the following sum.*

$$\sum_{k=0}^{8} a_k(x)$$

From Theorem 2 we have

$$\sum_{k=0}^{8} a_k(x)=\frac{2^4 A_4(x)+30x^4+12x^2-59}{2x^4+4x^2+7}, \quad (16)$$

where

$$A_4(x)=[(2x^4+4x^2+8)T_5(x^2)$$
$$+(-2x^8-4x^6-2x^4+14x^2+28)T_4(x^2)$$
$$+(-6x^8-16x^6-20x^4+22x^2+62)T_3(x^2)$$
$$+(-4x^8-16x^6-29x^4+14x^2+58)T_2(x^2)$$
$$+(-4x^6-16x^4+x^2+26)T_1(x^2)$$
$$+(-4x^4-2x^2+7)T_0(x^2).$$

Using the definition of Tribonacci polynomials we have

$$A_4(x)=[(2x^4+4x^2+8)(x^{16}+3x^{10}+3x^4)$$
$$+(-2x^8-4x^6-2x^4+14x^2+28)(x^{12}+2x^6+1$$
$$+(-6x^8-16x^6-20x^4+22x^2+62)(x^8+x^2)$$
$$+(-4x^8-16x^6-29x^4+14x^2+58)(x^4)$$
$$+(-4x^6-16x^4+x^2+26).1$$
$$+(-4x^4-2x^2+7).0$$

Multiplying the polynomials we have

$$A_4(x)=2x^{20}+4x^{18}+8x^{16}+6x^{14}+12x^{12}$$
$$+24x^{10}+6x^8+12x^6+24x^4-2x^{20}-4x^{18}$$
$$-2x^{16}+10x^{14}+20x^{12}-4x^{10}+26x^8+52x^6$$
$$-2x^4+14x^2+28-6x^{16}-16x^{14}-20x^{12}$$
$$+16x^{10}+46x^8-20x^6+22x^4+62x^2-4x^{12}$$
$$-16x^{10}-29x^8+14x^6+58x^4-4x^6-16x^4$$
$$+x^2+26,$$

$$A_4(x)=$$

$$8x^{12}+20x^{10}+49x^8+54x^6+86x^4+77x^2+54. \quad (17)$$

From (16) and (17) we get

$$\sum_{k=0}^{8} a_k(x)=64x^8+32x^6+104x^4+112x^2+115.$$

# 3. Conclusions

In this paper, we define a polynomial sequence $(a_n(x))$ which is a generalization of the integer sequence $(a_n)$ given in [1]. The polynomial sequence is associated with the Tribonacci polynomials and we get some properties of the polynomial sequence.

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