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M-LAURICELLA HYPERGEOMETRIC FUNCTIONS: INTEGRAL REPRESENTATIONS AND SOLUTION OF FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper, using the modified beta function involving the generalized M-series in its kernel, we describe new extensions for the Lauricella hypergeometric functions $F_A^{(r)}$, $F_B^{(r)}$, $F_C^{(r)}$ and $F_D^{(r)}$. Furthermore, we find various integral representations for the newly defined extended Lauricella hypergeometric functions. Then, we obtain solution of fractional differential equations involving new extensions of Lauricella hypergeometric functions, as examples.

1. Introduction and Preliminaries

Scientists have conducted a lot of research in recent years on various generalizations of special functions (see for example [1, 5–13, 15, 19–22, 24, 26, 27, 29, 33]). Particularly, the modified gamma and beta functions for $\Re(\alpha) > 0$, $\Re(\rho) > 0$, $\Re(x) > 0$, $\Re(y) > 0$ and $\xi_1, \ldots, \xi_p, \eta_1, \ldots, \eta_q \neq 0, -1, -2, \ldots$ was introduced by Ata in [8], respectively, as follows:

$$\begin{aligned} {}^M\Gamma_{p,q}^{(\alpha,\beta)}(x;\rho) &= {}^M\Gamma_{p,q}^{(\alpha,\beta)}(\xi_1,\ldots,\xi_p;\eta_1,\ldots,\eta_q;x;\rho) \\ &= \int_0^1 \Delta^{x-1}\ {}_p^{\alpha}M_q^{\beta}\left(\xi_1,\ldots,\xi_p;\eta_1,\ldots,\eta_q;-\Delta-\frac{\rho}{\Delta}\right)d\Delta, \end{aligned}$$

and

$$\begin{aligned} {}^MB_{p,q}^{(\alpha,\beta)}(x,y;\rho) &= {}^MB_{p,q}^{(\alpha,\beta)}(\xi_1,\ldots,\xi_p;\eta_1,\ldots,\eta_q;x,y;\rho) \\ &= \int_0^1 \Delta^{x-1}(1-\Delta)^{y-1}\ {}_p^{\alpha}M_q^{\beta}\left(\xi_1,\ldots,\xi_p;\eta_1,\ldots,\eta_q;\frac{-\rho}{\Delta(1-\Delta)}\right)d\Delta. \qquad (1) \end{aligned}$$

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If we take $\Delta = (\sin\phi)^2$ in Eq. (1), then

$$
{}^{M}B_{p,q}^{(\alpha,\beta)}(x,y;\rho) = 2\int_0^{\frac{\pi}{2}} (\sin\phi)^{2x-1}(\cos\phi)^{2y-1} \times {}_{p}^{\alpha}M_{q}^{\beta}\left(\xi_1,\ldots,\xi_p;\eta_1,\ldots,\eta_q;-\rho(\sec\phi)^2(\csc\phi)^2\right)d\phi. \quad (2)
$$

If we take $\Delta = \frac{u}{1+u}$ in Eq. (1), then

$$
{}^{M}B_{p,q}^{(\alpha,\beta)}(x,y;\rho) = \int_0^{\infty} \frac{u^{x-1}}{(1+u)^{x+y}} \times {}_{p}^{\alpha}M_{q}^{\beta}\left(\xi_1,\ldots,\xi_p;\eta_1,\ldots,\eta_q;-2\rho-\rho\left(u+\frac{1}{u}\right)\right)du. \quad (3)
$$

If we take $\Delta = \frac{u-a}{b-a}$ in Eq. (1), then

$$
{}^{M}B_{p,q}^{(\alpha,\beta)}(x,y;\rho) = (b-a)^{1-x-y}\int_a^{b} (u-a)^{x-1}(b-u)^{y-1} \times {}_{p}^{\alpha}M_{q}^{\beta}\left(\xi_1,\ldots,\xi_p;\eta_1,\ldots,\eta_q;\frac{-\rho(b-a)^2}{(u-a)(b-u)}\right)du. \quad (4)
$$

Then, the modified confluent hypergeometric function for $\Re(\alpha) > 0$, $\Re(\rho) > 0$, $\Re(\chi_3) > \Re(\chi_2) > 0$ and $\xi_1,\ldots,\xi_p,\eta_1,\ldots,\eta_q \neq 0,-1,-2,\ldots$ was introduced by Ata in [8], as follows:

$$
\begin{aligned}
{}^{M}\Phi_{p,q}^{(\alpha,\beta)}(\chi_2;\chi_3;z;\rho) &= {}^{M}\Phi_{p,q}^{(\alpha,\beta)}(\xi_1,\ldots,\xi_p;\eta_1,\ldots,\eta_q;\chi_2;\chi_3;z;\rho) \\
&= \sum_{n=0}^{\infty} \frac{{}^{M}B_{p,q}^{(\alpha,\beta)}(\chi_2+n,\chi_3-\chi_2;\rho)}{B(\chi_2,\chi_3-\chi_2)}\frac{z^n}{n!}. \quad (5)
\end{aligned}
$$

Also, the following formula holds true [8]:

$$
{}^{M}\Phi_{p,q}^{(\alpha,\beta)}(\chi_2;\chi_3;z;\rho) = \exp(z)\ {}^{M}\Phi_{p,q}^{(\alpha,\beta)}(\chi_3-\chi_2;\chi_3;-z;\rho). \quad (6)
$$

Respectively, Ata called them as M-gamma, M-beta and M-confluent hypergeometric functions. If we put $\rho = 0$ and $p = q = \xi_1 = \eta_1 = \alpha = \beta = 1$ to the M-gamma, M-beta and M-confluent hypergeometric functions, we get the following classical special functions [3, 4], respectively:

- The gamma function for $\Re(x) > 0$

$$
\Gamma(x) = \int_0^{\infty} \Delta^{x-1}\exp(-\Delta)d\Delta.
$$

- The beta function for $\Re(x) > 0$ and $\Re(y) > 0$

$$
B(x,y) = \int_0^{1} \Delta^{x-1}(1-\Delta)^{y-1}d\Delta.
$$

- The confluent hypergeometric function for $\Re(\chi_3) > \Re(\chi_2) > 0$

$$\Phi(\chi_2;\chi_3;z) = \sum_{n=0}^{\infty} \frac{B(\chi_2+n,\chi_3-\chi_2)}{B(\chi_2,\chi_3-\chi_2)} \frac{z^n}{n!}.$$

The function ${}_p^{\alpha}M_q^{\beta}$ used above is known as the generalized M-series [28] for $\Re(\alpha) > 0$ and $\xi_1,\ldots,\xi_p,\eta_1,\ldots,\eta_q \neq 0,-1,-2,\ldots$ which defined as:

$${}_p^{\alpha}M_q^{\beta}(z) = {}_p^{\alpha}M_q^{\beta}(\xi_1,\ldots,\xi_p;\eta_1,\ldots,\eta_q;z) = \sum_{n=0}^{\infty} \frac{(\xi_1)_n\ldots(\xi_p)_n}{(\eta_1)_n\ldots(\eta_q)_n} \frac{z^n}{\Gamma(\alpha n+\beta)}.$$

$(\cdot)_n$ used above denotes the Pochhammer symbol [4] is defined by

$$(\zeta)_n = \frac{\Gamma(\zeta+n)}{\Gamma(\zeta)} = \begin{cases} \zeta(\zeta+1)\ldots(\zeta+n-1), & n=1,2,\ldots, \\ 1, & n=0. \end{cases} \tag{7}$$

The binomial theorem [4] is as follows:

$$(1-\Delta)^{-\zeta} = \sum_{n=0}^{\infty} (\zeta)_n \frac{\Delta^n}{n!}, \quad (|\Delta| < 1). \tag{8}$$

The Caputo fractional derivative [18] for $m-1 < \Re(\epsilon) < m$, $m \in \mathbb{N}$ is given by

$${}^cD_\rho^\epsilon\{f(\rho)\} = \frac{1}{\Gamma(m-\epsilon)} \int_0^\rho (\rho-\omega)^{m-\epsilon-1} f^{(m)}(\omega)d\omega, \quad (\Re(\epsilon) > 0; \rho > 0).$$

The Laplace and inverse Laplace transforms [14] are defined by

$$\mathfrak{L}\{f(\rho);s\} = F(s) = \int_0^\infty \exp(-s\rho)f(\rho)d\rho, \quad (\Re(s) > 0),$$

and

$$\mathfrak{L}^{-1}\{F(s)\} = f(\rho) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \exp(s\rho)F(s)ds, \quad (c > 0).$$

Also, the Laplace transform of the Caputo fractional derivative is as follows [25]:

$$\mathfrak{L}\left\{{}^cD_\rho^\epsilon\{f(\rho)\};s\right\} = s^\epsilon F(s) - \sum_{k=0}^{m-1} s^{\epsilon-k-1} f^{(k)}(0), \quad (m-1 < \Re(\epsilon) \leq m). \tag{9}$$

Respectively, the Lauricella hypergeometric functions $F_A^{(r)}$, $F_B^{(r)}$, $F_C^{(r)}$ and $F_D^{(r)}$ [30,31] are as follows:

$$\begin{aligned} &F_A^{(r)}(\kappa,\mu_1,\ldots,\mu_r;\nu_1,\ldots,\nu_r;x_1,\ldots,x_r) \\ &\quad = \sum_{n_1,\ldots,n_r=0}^{\infty} \frac{(\kappa)_{n_1+\ldots+n_r}(\mu_1)_{n_1}\ldots(\mu_r)_{n_r}}{(\nu_1)_{n_1}\ldots(\nu_r)_{n_r}} \frac{x_1^{n_1}}{n_1!}\ldots\frac{x_r^{n_r}}{n_r!}, \\ &\qquad (|x_1|+\ldots+|x_r| < 1), \end{aligned} \tag{10}$$

$$F_B^{(r)}(\kappa_1,\ldots,\kappa_r,\mu_1,\ldots,\mu_r;\nu;x_1,\ldots,x_r)$$

$$= \sum_{n_1,\ldots,n_r=0}^{\infty} \frac{(\kappa_1)_{n_1}\ldots(\kappa_r)_{n_r}(\mu_1)_{n_1}\ldots(\mu_r)_{n_r}}{(\nu)_{n_1+\ldots+n_r}} \frac{x_1^{n_1}}{n_1!}\cdots\frac{x_r^{n_r}}{n_r!}, \tag{11}$$

$$(\max\{|x_1|,\ldots,|x_r|\}<1),$$

$$F_C^{(r)}(\kappa,\mu;\nu_1,\ldots,\nu_r;x_1,\ldots,x_r)$$

$$= \sum_{n_1,\ldots,n_r=0}^{\infty} \frac{(\kappa)_{n_1+\ldots+n_r}(\mu)_{n_1+\ldots+n_r}}{(\nu_1)_{n_1}\ldots(\nu_r)_{n_r}} \frac{x_1^{n_1}}{n_1!}\cdots\frac{x_r^{n_r}}{n_r!}, \tag{12}$$

$$(\sqrt{x_1}+\ldots+\sqrt{x_r}<1),$$

$$F_D^{(r)}(\kappa,\mu_1,\ldots,\mu_r;\nu;x_1,\ldots,x_r)$$

$$= \sum_{n_1,\ldots,n_r=0}^{\infty} \frac{(\kappa)_{n_1+\ldots+n_r}(\mu_1)_{n_1}\ldots(\mu_r)_{n_r}}{(\nu)_{n_1+\ldots+n_r}} \frac{x_1^{n_1}}{n_1!}\cdots\frac{x_r^{n_r}}{n_r!}, \tag{13}$$

$$(\max\{|x_1|,\ldots,|x_r|\}<1).$$

2. New Extended Lauricella Hypergeometric Functions

Scientists have studied on various extended of Lauricella hypergeometric functions (see for example [2, 16, 17, 23, 32]). Motivated by these studies, we introduce the newly extended Lauricella hypergeometric functions $F_A^{(r)}$, $F_B^{(r)}$, $F_C^{(r)}$ and $F_D^{(r)}$ using the modified beta function involving the generalized M-series in its kernel, in this section.

Definition 1. *Let $\Re(\alpha)>0$, $\Re(\rho)>0$, $\Re(\nu_i)>\Re(\mu_i)>0$ for $i=1,\ldots,r$ and $|x_1|+\ldots+|x_r|<1$. Then, new extended Lauricella hypergeometric function $F_A^{(r)}$ is defined as:*

$$\begin{aligned}
&{}^{M}F_{A,p,q}^{(\alpha,\beta;r)}(\kappa,\mu_1,\ldots,\mu_r;\nu_1,\ldots,\nu_r;x_1,\ldots,x_r;\rho)\\
&= {}^{M}F_{A,p,q}^{(\alpha,\beta;r)}(\xi_1,\ldots,\xi_p;\eta_1,\ldots,\eta_q;\kappa,\mu_1,\ldots,\mu_r;\nu_1,\ldots,\nu_r;x_1,\ldots,x_r;\rho)\\
&:= \sum_{n_1,\ldots,n_r=0}^{\infty}(\kappa)_{n_1+\ldots+n_r}\frac{{}^{M}B_{p,q}^{(\alpha,\beta)}(\mu_1+n_1,\nu_1-\mu_1;\rho)}{B(\mu_1,\nu_1-\mu_1)}\cdots\\
&\quad\times\frac{{}^{M}B_{p,q}^{(\alpha,\beta)}(\mu_r+n_r,\nu_r-\mu_r;\rho)}{B(\mu_r,\nu_r-\mu_r)}\frac{x_1^{n_1}}{n_1!}\cdots\frac{x_r^{n_r}}{n_r!}.
\end{aligned}$$

Definition 2. *Let $\Re(\alpha)>0$, $\Re(\rho)>0$, $\Re(\lambda)>\Re(\mu_1)>0$ and $\max\{|x_1|,\ldots,|x_r|\}<1$. Then, new extended Lauricella hypergeometric function $F_B^{(r)}$ is defined as:*

$$\begin{aligned}
&{}^{M}F_{B,p,q}^{(\alpha,\beta;r)}(\kappa_1,\ldots,\kappa_r,\mu_1,\ldots,\mu_r;\nu;x_1,\ldots,x_r;\rho;\lambda)\\
&= {}^{M}F_{B,p,q}^{(\alpha,\beta;r)}(\xi_1,\ldots,\xi_p;\eta_1,\ldots,\eta_q;\kappa_1,\ldots,\kappa_r,\mu_1,\ldots,\mu_r;\nu;x_1,\ldots,x_r;\rho;\lambda)
\end{aligned}$$

$$:= \sum_{n_1,\ldots,n_r=0}^{\infty} \frac{(\kappa_1)_{n_1}\ldots(\kappa_r)_{n_r}(\lambda)_{n_1}(\mu_2)_{n_2}\ldots(\mu_r)_{n_r}}{(\nu)_{n_1+\ldots+n_r}}$$
$$\times \frac{{}^{M}B_{p,q}^{(\alpha,\beta)}(\mu_1+n_1,\lambda-\mu_1;\rho)}{B(\mu_1,\lambda-\mu_1)} \frac{x_1^{n_1}}{n_1!}\cdots\frac{x_r^{n_r}}{n_r!}.$$

Definition 3. *Let* $\Re(\alpha)>0, \Re(\rho)>0, \Re(\nu_1)>\Re(\lambda)>0$ *and* $\sqrt{|x_1|}+\ldots+\sqrt{|x_r|}<1$. *Then, new extended Lauricella hypergeometric function* $F_C^{(r)}$ *is defined as:*

$$\begin{aligned}&{}^{M}F_{C,p,q}^{(\alpha,\beta;r)}(\kappa,\mu;\nu_1,\ldots,\nu_r;x_1,\ldots,x_r;\rho;\lambda)\\ &= {}^{M}F_{C,p,q}^{(\alpha,\beta;r)}(\xi_1,\ldots,\xi_p;\eta_1,\ldots,\eta_q;\kappa,\mu;\nu_1,\ldots,\nu_r;x_1,\ldots,x_r;\rho;\lambda)\\ &:= \sum_{n_1,\ldots,n_r=0}^{\infty} \frac{(\kappa)_{n_1+\ldots+n_r}(\mu)_{n_1+\ldots+n_r}}{(\lambda)_{n_1}(\nu_2)_{n_2}\ldots(\nu_r)_{n_r}} \frac{{}^{M}B_{p,q}^{(\alpha,\beta)}(\lambda+n_1,\nu_1-\lambda;\rho)}{B(\lambda,\nu_1-\lambda)} \frac{x_1^{n_1}}{n_1!}\cdots\frac{x_r^{n_r}}{n_r!}.\end{aligned}$$

Definition 4. *Let* $\Re(\alpha)>0, \Re(\rho)>0, \Re(\nu)>\Re(\kappa)>0$ *and* $\max\{|x_1|,\ldots,|x_r|\}<1$. *Then, new extended Lauricella hypergeometric function* $F_D^{(r)}$ *is defined as:*

$$\begin{aligned}&{}^{M}F_{D,p,q}^{(\alpha,\beta;r)}(\kappa,\mu_1,\ldots,\mu_r;\nu;x_1,\ldots,x_r;\rho)\\ &= {}^{M}F_{D,p,q}^{(\alpha,\beta;r)}(\xi_1,\ldots,\xi_p;\eta_1,\ldots,\eta_q;\kappa,\mu_1,\ldots,\mu_r;\nu;x_1,\ldots,x_r;\rho)\\ &:= \sum_{n_1,\ldots,n_r=0}^{\infty} (\mu_1)_{n_1}\ldots(\mu_r)_{n_r} \frac{{}^{M}B_{p,q}^{(\alpha,\beta)}(\kappa+n_1+\ldots+n_r,\nu-\kappa;\rho)}{B(\kappa,\nu-\kappa)} \frac{x_1^{n_1}}{n_1!}\cdots\frac{x_r^{n_r}}{n_r!}.\end{aligned}$$

Respectively, we call them as M-Lauricella hypergeometric function $F_A^{(r)}$, M-Lauricella hypergeometric function $F_B^{(r)}$, M-Lauricella hypergeometric function $F_C^{(r)}$ and M-Lauricella hypergeometric function $F_D^{(r)}$.

Remark 1. *If we take* $\rho=0$ *and* $p=q=\xi_1=\eta_1=\alpha=\beta=1$ *in these functions, we get Eqs.* (10), (11), (12) *and* (13).

3. Integral Representations for M-Lauricella Hypergeometric Function $F_A^{(r)}$

Theorem 1. *Let* $\Re(\alpha)>0$, $\Re(\rho)>0$, $\Re(\nu_i)>\Re(\mu_i)>0$ *for* $i=1,\ldots,r$. *Then,*

$$\begin{aligned}&{}^{M}F_{A,p,q}^{(\alpha,\beta;r)}(\kappa,\mu_1,\ldots,\mu_r;\nu_1,\ldots,\nu_r;x_1,\ldots,x_r;\rho)\\ &\qquad= \frac{1}{B(\mu_1,\nu_1-\mu_1)\ldots B(\mu_r,\nu_r-\mu_r)} \int_0^1\cdots\int_0^1 \Delta_1^{\mu_1-1}\cdots\ \Delta_r^{\mu_r-1}\\ &\qquad\times(1-\Delta_1)^{\nu_1-\mu_1-1}\cdots\ (1-\Delta_r)^{\nu_r-\mu_r-1}\\ &\qquad\times {}_{p}^{\alpha}M_q^{\beta}\left(\frac{-\rho}{\Delta_1(1-\Delta_1)}\right)\cdots\ {}_{p}^{\alpha}M_q^{\beta}\left(\frac{-\rho}{\Delta_r(1-\Delta_r)}\right)\\ &\qquad\times F_A^{(r)}(\kappa,\mu_1,\ldots,\mu_r;\mu_1,\ldots,\mu_r;\Delta_1x_1,\ldots,\Delta_rx_r)d\Delta_1\ldots d\Delta_r.\end{aligned}$$

Proof. Using the integral representation (1) of M-beta function in the definition of M-Lauricella hypergeometric function $F_A^{(r)}$, we have

$$\begin{aligned}
&{}^{M}F_{A,p,q}^{(\alpha,\beta;r)}(\kappa,\mu_1,\ldots,\mu_r;\nu_1,\ldots,\nu_r;x_1,\ldots,x_r;\rho)\\
&= \sum_{n_1,\ldots,n_r=0}^{\infty} (\kappa)_{n_1+\ldots+n_r} \frac{{}^{M}B_{p,q}^{(\alpha,\beta)}(\mu_1+n_1,\nu_1-\mu_1;\rho)}{B(\mu_1,\nu_1-\mu_1)}\ldots\\
&\times \frac{{}^{M}B_{p,q}^{(\alpha,\beta)}(\mu_r+n_r,\nu_r-\mu_r;\rho)}{B(\mu_r,\nu_r-\mu_r)}\frac{x_1^{n_1}}{n_1!}\ldots\frac{x_r^{n_r}}{n_r!}\\
&= \frac{1}{B(\mu_1,\nu_1-\mu_1)\ldots B(\mu_r,\nu_r-\mu_r)}\sum_{n_1,\ldots,n_r=0}^{\infty}(\kappa)_{n_1+\ldots+n_r}\\
&\times \int_0^1 \Delta_1^{\mu_1+n_1-1}(1-\Delta_1)^{\nu_1-\mu_1-1}\ {}_{p}^{\alpha}M_q^{\beta}\left(\frac{-\rho}{\Delta_1(1-\Delta_1)}\right)\ldots\\
&\times \int_0^1 \Delta_r^{\mu_r+n_r-1}(1-\Delta_r)^{\nu_r-\mu_r-1}\ {}_{p}^{\alpha}M_q^{\beta}\left(\frac{-\rho}{\Delta_r(1-\Delta_r)}\right)\\
&\times \frac{x_1^{n_1}}{n_1!}\ldots\frac{x_r^{n_r}}{n_r!}d\Delta_1\ldots d\Delta_r\\
&= \frac{1}{B(\mu_1,\nu_1-\mu_1)\ldots B(\mu_r,\nu_r-\mu_r)}\sum_{n_1,\ldots,n_r=0}^{\infty}(\kappa)_{n_1+\ldots+n_r}\\
&\times \int_0^1 \Delta_1^{\mu_1-1}(1-\Delta_1)^{\nu_1-\mu_1-1}\ {}_{p}^{\alpha}M_q^{\beta}\left(\frac{-\rho}{\Delta_1(1-\Delta_1)}\right)\ldots\\
&\times \int_0^1 \Delta_r^{\mu_r-1}(1-\Delta_r)^{\nu_r-\mu_r-1}\ {}_{p}^{\alpha}M_q^{\beta}\left(\frac{-\rho}{\Delta_r(1-\Delta_r)}\right)\\
&\times \frac{(x_1\Delta_1)^{n_1}}{n_1!}\ldots\frac{(x_r\Delta_r)^{n_r}}{n_r!}d\Delta_1\ldots d\Delta_r.
\end{aligned}$$

Multiplied by $\frac{(\mu_1)_{n_1}\ldots(\mu_r)_{n_r}}{(\mu_1)_{n_1}\ldots(\mu_r)_{n_r}}$ and considering Eq. (10), we get

$$\begin{aligned}
&{}^{M}F_{A,p,q}^{(\alpha,\beta;r)}(\kappa,\mu_1,\ldots,\mu_r;\nu_1,\ldots,\nu_r;x_1,\ldots,x_r;\rho)\\
&= \frac{1}{B(\mu_1,\nu_1-\mu_1)\ldots B(\mu_r,\nu_r-\mu_r)}\int_0^1\ldots\int_0^1 \Delta_1^{\mu_1-1}\ldots\ \Delta_r^{\mu_r-1}\\
&\times (1-\Delta_1)^{\nu_1-\mu_1-1}\ldots\ (1-\Delta_r)^{\nu_r-\mu_r-1}\\
&\times {}_{p}^{\alpha}M_q^{\beta}\left(\frac{-\rho}{\Delta_1(1-\Delta_1)}\right)\ldots\ {}_{p}^{\alpha}M_q^{\beta}\left(\frac{-\rho}{\Delta_r(1-\Delta_r)}\right)\\
&\times F_A^{(r)}(\kappa,\mu_1,\ldots,\mu_r;\mu_1,\ldots,\mu_r;\Delta_1x_1,\ldots,\Delta_rx_r)d\Delta_1\ldots d\Delta_r. \qquad \square
\end{aligned}$$

Theorem 2. *Let $\Re(\alpha) > 0$, $\Re(\rho) > 0$, $\Re(\nu_i) > \Re(\mu_i) > 0$ for $i = 1, \ldots, r$. Then,*

$$\begin{aligned} &{}^{M}F_{A,p,q}^{(\alpha,\beta;r)}(\kappa, \mu_1, \ldots, \mu_r; \nu_1, \ldots, \nu_r; x_1, \ldots, x_r; \rho) \\ &\quad = \frac{1}{\Gamma(\kappa)} \int_0^\infty \Delta^{\kappa-1} \exp(-\Delta) \\ &\quad \times {}^{M}\Phi_{p,q}^{(\alpha,\beta)}(\mu_1; \nu_1; \Delta x_1; \rho) \ldots {}^{M}\Phi_{p,q}^{(\alpha,\beta)}(\mu_r; \nu_r; \Delta x_r; \rho) d\Delta. \end{aligned} \tag{14}$$

Proof. Using Eq. (7) in the definition of M-Lauricella hypergeometric function $F_A^{(r)}$, we have

$$\begin{aligned} &{}^{M}F_{A,p,q}^{(\alpha,\beta;r)}(\kappa, \mu_1, \ldots, \mu_r; \nu_1, \ldots, \nu_r; x_1, \ldots, x_r; \rho) \\ &= \sum_{n_1,\ldots,n_r=0}^{\infty} (\kappa)_{n_1+\ldots+n_r} \frac{{}^{M}B_{p,q}^{(\alpha,\beta)}(\mu_1 + n_1, \nu_1 - \mu_1; \rho)}{B(\mu_1, \nu_1 - \mu_1)} \ldots \\ &\quad \times \frac{{}^{M}B_{p,q}^{(\alpha,\beta)}(\mu_r + n_r, \nu_r - \mu_r; \rho)}{B(\mu_r, \nu_r - \mu_r)} \frac{x_1^{n_1}}{n_1!} \ldots \frac{x_r^{n_r}}{n_r!} \\ &= \sum_{n_1,\ldots,n_r=0}^{\infty} \frac{\Gamma(\kappa + n_1 + \ldots + n_r)}{\Gamma(\kappa)} \frac{{}^{M}B_{p,q}^{(\alpha,\beta)}(\mu_1 + n_1, \nu_1 - \mu_1; \rho)}{B(\mu_1, \nu_1 - \mu_1)} \ldots \\ &\quad \times \frac{{}^{M}B_{p,q}^{(\alpha,\beta)}(\mu_r + n_r, \nu_r - \mu_r; \rho)}{B(\mu_r, \nu_r - \mu_r)} \frac{x_1^{n_1}}{n_1!} \ldots \frac{x_r^{n_r}}{n_r!}. \end{aligned}$$

Using the integral representation of gamma function and considering Eq. (5), we get

$$\begin{aligned} &{}^{M}F_{A,p,q}^{(\alpha,\beta;r)}(\kappa, \mu_1, \ldots, \mu_r; \nu_1, \ldots, \nu_r; x_1, \ldots, x_r; \rho) \\ &\quad = \frac{1}{\Gamma(\kappa)} \int_0^\infty \Delta^{\kappa-1} \exp(-\Delta) \\ &\quad \times {}^{M}\Phi_{p,q}^{(\alpha,\beta)}(\mu_1; \nu_1; \Delta x_1; \rho) \ldots {}^{M}\Phi_{p,q}^{(\alpha,\beta)}(\mu_r; \nu_r; \Delta x_r; \rho) d\Delta. \end{aligned}$$

□

Theorem 3. *Let $\Re(\alpha) > 0$, $\Re(\rho) > 0$, $\Re(\nu_i) > \Re(\mu_i) > 0$ for $i = 1, \ldots, r$. Then,*

$$\begin{aligned} &{}^{M}F_{A,p,q}^{(\alpha,\beta;r)}(\kappa, \mu_1, \ldots, \mu_r; \nu_1, \ldots, \nu_r; x_1, \ldots, x_r; \rho) \\ &\quad = \frac{1}{\Gamma(\kappa)} \int_0^\infty \Delta^{\kappa-1} \exp\big(-\Delta(1 - x_1 - \ldots - x_r)\big) \\ &\quad \times {}^{M}\Phi_{p,q}^{(\alpha,\beta)}(\nu_1 - \mu_1; \nu_1; -\Delta x_1; \rho) \ldots {}^{M}\Phi_{p,q}^{(\alpha,\beta)}(\nu_r - \mu_r; \nu_r; -\Delta x_r; \rho) d\Delta. \end{aligned}$$

Proof. Using Eq. (6) in Eq. (14), proof is complete. □

Theorem 4. *Let $\Re(\alpha) > 0$, $\Re(\rho) > 0$, $\Re(\nu_i) > \Re(\mu_i) > 0$ for $i = 1, \ldots, r$. Then,*

$$\begin{aligned} &{}^{M}F_{A,p,q}^{(\alpha,\beta;r)}(\kappa, \mu_1, \ldots, \mu_r; \nu_1, \ldots, \nu_r; x_1, \ldots, x_r; \rho) \\ &= \frac{2^r}{B(\mu_1, \nu_1 - \mu_1) \ldots B(\mu_r, \nu_r - \mu_r)} \int_0^{\frac{\pi}{2}} \ldots \int_0^{\frac{\pi}{2}} \end{aligned}$$

$$
\begin{aligned}
&\times (\sin\phi_1)^{2\mu_1-1}\dots\ (\sin\phi_r)^{2\mu_r-1}(\cos\phi_1)^{2\nu_1-2\mu_1-1}\dots\ (\cos\phi_r)^{2\nu_r-2\mu_r-1}\\
&\times {}_p^{\alpha}M_q^{\beta}\left(-\rho(\sec\phi_1)^2(\csc\phi_1)^2\right)\dots\ {}_p^{\alpha}M_q^{\beta}\left(-\rho(\sec\phi_r)^2(\csc\phi_r)^2\right)\\
&\times F_A^{(r)}\left(\kappa,\mu_1,\dots,\mu_r;\mu_1,\dots,\mu_r;x_1(\sin\phi_1)^2,\dots,x_r(\sin\phi_r)^2\right)d\phi_1\dots d\phi_r.
\end{aligned}
$$

Proof. Using the integral representation (2) of M-beta function in the definition of M-Lauricella hypergeometric function $F_A^{(r)}$ and making similar calculations in the proof of Theorem 1, proof is complete. □

Theorem 5. *Let $\Re(\alpha)>0$, $\Re(\rho)>0$, $\Re(\nu_i)>\Re(\mu_i)>0$ for $i=1,\dots,r$. Then,*

$$
\begin{aligned}
&{}^{M}F_{A,p,q}^{(\alpha,\beta;r)}(\kappa,\mu_1,\dots,\mu_r;\nu_1,\dots,\nu_r;x_1,\dots,x_r;\rho)\\
&=\frac{1}{B(\mu_1,\nu_1-\mu_1)\dots B(\mu_r,\nu_r-\mu_r)}\int_0^\infty\dots\int_0^\infty\frac{u_1^{\mu_1-1}}{(1+u_1)^{\nu_1}}\dots\frac{u_r^{\mu_r-1}}{(1+u_r)^{\nu_r}}\\
&\times {}_p^{\alpha}M_q^{\beta}\left(-2\rho-\rho\left(u_1+\frac{1}{u_1}\right)\right)\dots\ {}_p^{\alpha}M_q^{\beta}\left(-2\rho-\rho\left(u_r+\frac{1}{u_r}\right)\right)\\
&\times F_A^{(r)}\left(\kappa,\mu_1,\dots,\mu_r;\mu_1,\dots,\mu_r;\frac{x_1u_1}{1+u_1},\dots,\frac{x_ru_r}{1+u_r}\right)du_1\dots du_r.
\end{aligned}
$$

Proof. Using the integral representation (3) of M-beta function in the definition of M-Lauricella hypergeometric function $F_A^{(r)}$ and making similar calculations in the proof of Theorem 1, proof is complete. □

Theorem 6. *Let $\Re(\alpha)>0$, $\Re(\rho)>0$, $\Re(\nu_i)>\Re(\mu_i)>0$ for $i=1,\dots,r$. Then,*

$$
\begin{aligned}
&{}^{M}F_{A,p,q}^{(\alpha,\beta;r)}(\kappa,\mu_1,\dots,\mu_r;\nu_1,\dots,\nu_r;x_1,\dots,x_r;\rho)\\
&=\frac{(b-a)^{r-(\nu_1+\dots+\nu_r)}}{B(\mu_1,\nu_1-\mu_1)\dots B(\mu_r,\nu_r-\mu_r)}\int_a^b\dots\int_a^b(u_1-a)^{\mu_1-1}\dots\ (u_r-a)^{\mu_r-1}\\
&\times (b-u_1)^{\nu_1-\mu_1-1}\dots\ (b-u_r)^{\nu_r-\mu_r-1}\\
&\times {}_p^{\alpha}M_q^{\beta}\left(\frac{-\rho(b-a)^2}{(u_1-a)(b-u_1)}\right)\dots\ {}_p^{\alpha}M_q^{\beta}\left(\frac{-\rho(b-a)^2}{(u_r-a)(b-u_r)}\right)\\
&\times F_A^{(r)}\left(\kappa,\mu_1,\dots,\mu_r;\mu_1,\dots,\mu_r;\frac{x_1(u_1-a)}{b-a},\dots,\frac{x_r(u_r-a)}{b-a}\right)du_1\dots du_r.
\end{aligned}
$$

Proof. Using the integral representation (4) of M-beta function in the definition of M-Lauricella hypergeometric function $F_A^{(r)}$ and making similar calculations in the proof of Theorem 1, proof is complete. □

4. Integral Representations for M-Lauricella Hypergeometric Function $F_B^{(r)}$

Theorem 7. *Let $\Re(\alpha)>0$, $\Re(\rho)>0$, $\Re(\lambda)>\Re(\mu_1)>0$. Then,*

$$
{}^{M}F_{B,p,q}^{(\alpha,\beta;r)}(\kappa_1,\dots,\kappa_r,\mu_1,\dots,\mu_r;\nu;x_1,\dots,x_r;\rho;\lambda)
$$

$$= \frac{1}{B(\mu_1, \lambda-\mu_1)} \int_0^1 \Delta^{\mu_1-1}(1-\Delta)^{\lambda-\mu_1-1} \, {}_p^{\alpha}M_q^{\beta}\left(\frac{-\rho}{\Delta(1-\Delta)}\right) \times F_B^{(r)}(\kappa_1, \ldots, \kappa_r, \lambda, \mu_2, \ldots, \mu_r; \nu; \Delta x_1, x_2, \ldots, x_r) d\Delta.$$

Proof. Using the integral representation (1) of M-beta function in the definition of M-Lauricella hypergeometric function $F_B^{(r)}$, we have

$$\begin{aligned} &{}^{M}F_{B,p,q}^{(\alpha,\beta;r)}(\kappa_1, \ldots, \kappa_r, \mu_1, \ldots, \mu_r; \nu; x_1, \ldots, x_r; \rho; \lambda) \\ &= \sum_{n_1,\ldots,n_r=0}^{\infty} \frac{(\kappa_1)_{n_1} \ldots (\kappa_r)_{n_r} (\lambda)_{n_1} (\mu_2)_{n_2} \ldots (\mu_r)_{n_r}}{(\nu)_{n_1+\ldots+n_r}} \times \frac{{}^{M}B_{p,q}^{(\alpha,\beta)}(\mu_1+n_1, \lambda-\mu_1; \rho)}{B(\mu_1, \lambda-\mu_1)} \frac{x_1^{n_1}}{n_1!} \cdots \frac{x_r^{n_r}}{n_r!} \\ &= \frac{1}{B(\mu_1, \lambda-\mu_1)} \sum_{n_1,\ldots,n_r=0}^{\infty} \frac{(\kappa_1)_{n_1} \ldots (\kappa_r)_{n_r} (\lambda)_{n_1} (\mu_2)_{n_2} \ldots (\mu_r)_{n_r}}{(\nu)_{n_1+\ldots+n_r}} \\ &\quad \times \int_0^1 \Delta^{\mu_1+n_1-1}(1-\Delta)^{\lambda-\mu_1-1} \, {}_p^{\alpha}M_q^{\beta}\left(\frac{-\rho}{\Delta(1-\Delta)}\right) \frac{x_1^{n_1}}{n_1!} \cdots \frac{x_r^{n_r}}{n_r!} d\Delta \\ &= \frac{1}{B(\mu_1, \lambda-\mu_1)} \int_0^1 \Delta^{\mu_1-1}(1-\Delta)^{\lambda-\mu_1-1} \, {}_p^{\alpha}M_q^{\beta}\left(\frac{-\rho}{\Delta(1-\Delta)}\right) \\ &\quad \times \sum_{n_1,\ldots,n_r=0}^{\infty} \frac{(\kappa_1)_{n_1} \ldots (\kappa_r)_{n_r} (\lambda)_{n_1} (\mu_2)_{n_2} \ldots (\mu_r)_{n_r}}{(\nu)_{n_1+\ldots+n_r}} \frac{(\Delta x_1)^{n_1}}{n_1!} \frac{x_2^{n_2}}{n_2!} \cdots \frac{x_r^{n_r}}{n_r!} d\Delta. \end{aligned}$$

Considering Eq. (11), we get

$$\begin{aligned} &{}^{M}F_{B,p,q}^{(\alpha,\beta;r)}(\kappa_1, \ldots, \kappa_r, \mu_1, \ldots, \mu_r; \nu; x_1, \ldots, x_r; \rho; \lambda) \\ &= \frac{1}{B(\mu_1, \lambda-\mu_1)} \int_0^1 \Delta^{\mu_1-1}(1-\Delta)^{\lambda-\mu_1-1} \, {}_p^{\alpha}M_q^{\beta}\left(\frac{-\rho}{\Delta(1-\Delta)}\right) \\ &\quad \times F_B^{(r)}(\kappa_1, \ldots, \kappa_r, \lambda, \mu_2, \ldots, \mu_r; \nu; \Delta x_1, x_2, \ldots, x_r) d\Delta. \end{aligned}$$

□

Theorem 8. *Let* $\Re(\alpha) > 0$, $\Re(\rho) > 0$, $\Re(\lambda) > \Re(\mu_1) > 0$. *Then,*

$$\begin{aligned} &{}^{M}F_{B,p,q}^{(\alpha,\beta;r)}(\kappa_1, \ldots, \kappa_r, \mu_1, \ldots, \mu_r; \nu; x_1, \ldots, x_r; \rho; \lambda) \\ &= \frac{2}{B(\mu_1, \lambda-\mu_1)} \int_0^{\frac{\pi}{2}} (\sin\phi)^{2\mu_1-1}(\cos\phi)^{2\lambda-2\mu_1-1} \, {}_p^{\alpha}M_q^{\beta}\left(-\rho(\sec\phi)^2(\csc\phi)^2\right) \\ &\quad \times F_B^{(r)}(\kappa_1, \ldots, \kappa_r, \lambda, \mu_2, \ldots, \mu_r; \nu; x_1(\sin\phi)^2, x_2, \ldots, x_r) d\phi. \end{aligned}$$

Proof. Using the integral representation (2) of M-beta function in the definition of M-Lauricella hypergeometric function $F_B^{(r)}$ and making similar calculations in the proof of Theorem 7, proof is complete. □

Theorem 9. *Let $\Re(\alpha) > 0$, $\Re(\rho) > 0$, $\Re(\lambda) > \Re(\mu_1) > 0$. Then,*

$$
\begin{aligned}
&{}^{M}F_{B,p,q}^{(\alpha,\beta;r)}(\kappa_1,\ldots,\kappa_r,\mu_1,\ldots,\mu_r;\nu;x_1,\ldots,x_r;\rho;\lambda)\\
&= \frac{1}{B(\mu_1,\lambda-\mu_1)}\int_0^\infty \frac{u^{\mu_1-1}}{(1+u)^\lambda}\,{}_{p}^{\alpha}M_q^{\beta}\left(-2\rho-\rho\left(u+\frac{1}{u}\right)\right)\\
&\quad\times F_B^{(r)}\left(\kappa_1,\ldots,\kappa_r,\lambda,\mu_2,\ldots,\mu_r;\nu;\frac{x_1u}{1+u},x_2,\ldots,x_r\right)du.
\end{aligned}
$$

Proof. Using the integral representation (3) of M-beta function in the definition of M-Lauricella hypergeometric function $F_B^{(r)}$ and making similar calculations in the proof of Theorem 7, proof is complete. □

Theorem 10. *Let $\Re(\alpha) > 0$, $\Re(\rho) > 0$, $\Re(\lambda) > \Re(\mu_1) > 0$. Then,*

$$
\begin{aligned}
&{}^{M}F_{B,p,q}^{(\alpha,\beta;r)}(\kappa_1,\ldots,\kappa_r,\mu_1,\ldots,\mu_r;\nu;x_1,\ldots,x_r;\rho;\lambda)\\
&= \frac{(b-a)^{1-\lambda}}{B(\mu_1,\lambda-\mu_1)}\int_a^b (u-a)^{\mu_1-1}(b-u)^{\lambda-\mu_1-1}\,{}_{p}^{\alpha}M_q^{\beta}\left(\frac{-\rho(b-a)^2}{(u-a)(b-u)}\right)\\
&\quad\times F_B^{(r)}\left(\kappa_1,\ldots,\kappa_r,\lambda,\mu_2,\ldots,\mu_r;\nu;\frac{x_1(u-a)}{b-a},x_2,\ldots,x_r\right)du.
\end{aligned}
$$

Proof. Using the integral representation (4) of M-beta function in the definition of M-Lauricella hypergeometric function $F_B^{(r)}$ and making similar calculations in the proof of Theorem 7, proof is complete. □

5. Integral Representations for M-Lauricella Hypergeometric Function $F_C^{(r)}$

Theorem 11. *Let $\Re(\alpha) > 0$, $\Re(\rho) > 0$, $\Re(\nu_1) > \Re(\lambda) > 0$. Then,*

$$
\begin{aligned}
&{}^{M}F_{C,p,q}^{(\alpha,\beta;r)}(\kappa,\mu;\nu_1,\ldots,\nu_r;x_1,\ldots,x_r;\rho;\lambda)\\
&= \frac{1}{B(\lambda,\nu_1-\lambda)}\int_0^1 \Delta^{\lambda-1}(1-\Delta)^{\nu_1-\lambda-1}\,{}_{p}^{\alpha}M_q^{\beta}\left(\frac{-\rho}{\Delta(1-\Delta)}\right)\\
&\quad\times F_C^{(r)}(\kappa,\mu;\lambda,\nu_2,\ldots,\nu_r;\Delta x_1,x_2,\ldots,x_r)d\Delta.
\end{aligned}
$$

Proof. Using the integral representation (1) of M-beta function in the definition of M-Lauricella hypergeometric function $F_C^{(r)}$, we have

$$
\begin{aligned}
&{}^{M}F_{C,p,q}^{(\alpha,\beta;r)}(\kappa,\mu;\nu_1,\ldots,\nu_r;x_1,\ldots,x_r;\rho;\lambda)\\
&= \sum_{n_1,\ldots,n_r=0}^{\infty}\frac{(\kappa)_{n_1+\ldots+n_r}(\mu)_{n_1+\ldots+n_r}}{(\lambda)_{n_1}(\nu_2)_{n_2}\ldots(\nu_r)_{n_r}}\frac{{}^{M}B_{p,q}^{(\alpha,\beta)}(\lambda+n_1,\nu_1-\lambda;\rho)}{B(\lambda,\nu_1-\lambda)}\frac{x_1^{n_1}}{n_1!}\cdots\frac{x_r^{n_r}}{n_r!}\\
&= \frac{1}{B(\lambda,\nu_1-\lambda)}\sum_{n_1,\ldots,n_r=0}^{\infty}\frac{(\kappa)_{n_1+\ldots+n_r}(\mu)_{n_1+\ldots+n_r}}{(\lambda)_{n_1}(\nu_2)_{n_2}\ldots(\nu_r)_{n_r}}
\end{aligned}
$$

$$\times \int_0^1 \Delta^{\lambda+n_1-1}(1-\Delta)^{\nu_1-\lambda-1} \, {}_p^{\alpha}M_q^{\beta}\left(\frac{-\rho}{\Delta(1-\Delta)}\right)\frac{x_1^{n_1}}{n_1!}\cdots\frac{x_r^{n_r}}{n_r!}d\Delta$$

$$= \frac{1}{B(\lambda,\nu_1-\lambda)}\int_0^1 \Delta^{\lambda-1}(1-\Delta)^{\nu_1-\lambda-1} \, {}_p^{\alpha}M_q^{\beta}\left(\frac{-\rho}{\Delta(1-\Delta)}\right)$$

$$\times \sum_{n_1,\ldots,n_r=0}^{\infty}\frac{(\kappa)_{n_1+\ldots+n_r}(\mu)_{n_1+\ldots+n_r}}{(\lambda)_{n_1}(\nu_2)_{n_2}\ldots(\nu_r)_{n_r}}\frac{(\Delta x_1)^{n_1}}{n_1!}\frac{x_2^{n_2}}{n_2!}\cdots\frac{x_r^{n_r}}{n_r!}d\Delta.$$

Considering Eq. (12), we get

$$ {}^{M}F_{C,p,q}^{(\alpha,\beta;r)}(\kappa,\mu;\nu_1,\ldots,\nu_r;x_1,\ldots,x_r;\rho;\lambda)$$

$$= \frac{1}{B(\lambda,\nu_1-\lambda)}\int_0^1 \Delta^{\lambda-1}(1-\Delta)^{\nu_1-\lambda-1} \, {}_p^{\alpha}M_q^{\beta}\left(\frac{-\rho}{\Delta(1-\Delta)}\right)$$

$$\times F_C^{(r)}(\kappa,\mu;\lambda,\nu_2,\ldots,\nu_r;\Delta x_1,x_2,\ldots,x_r)d\Delta.$$ □

Theorem 12. *Let* $\Re(\alpha)>0$, $\Re(\rho)>0$, $\Re(\nu_1)>\Re(\lambda)>0$. *Then,*

$$ {}^{M}F_{C,p,q}^{(\alpha,\beta;r)}(\kappa,\mu;\nu_1,\ldots,\nu_r;x_1,\ldots,x_r;\rho;\lambda)$$

$$= \frac{2}{B(\lambda,\nu_1-\lambda)}\int_0^{\frac{\pi}{2}}(\sin\phi)^{2\lambda-1}(\cos\phi)^{2\nu_1-2\lambda-1} \, {}_p^{\alpha}M_q^{\beta}\left(-\rho(\sec\phi)^2(\csc\phi)^2\right)$$

$$\times F_C^{(r)}(\kappa,\mu;\lambda,\nu_2,\ldots,\nu_r;x_1(\sin\phi)^2,x_2,\ldots,x_r)d\phi.$$

Proof. Using the integral representation (2) of M-beta function in the definition of M-Lauricella hypergeometric function $F_C^{(r)}$ and making similar calculations in the proof of Theorem 11, proof is complete. □

Theorem 13. *Let* $\Re(\alpha)>0$, $\Re(\rho)>0$, $\Re(\nu_1)>\Re(\lambda)>0$. *Then,*

$$ {}^{M}F_{C,p,q}^{(\alpha,\beta;r)}(\kappa,\mu;\nu_1,\ldots,\nu_r;x_1,\ldots,x_r;\rho;\lambda)$$

$$= \frac{1}{B(\lambda,\nu_1-\lambda)}\int_0^{\infty}\frac{u^{\lambda-1}}{(1+u)^{\nu_1}} \, {}_p^{\alpha}M_q^{\beta}\left(-2\rho-\rho\left(u+\frac{1}{u}\right)\right)$$

$$\times F_C^{(r)}\left(\kappa,\mu;\lambda,\nu_2,\ldots,\nu_r;\frac{x_1u}{1+u},x_2,\ldots,x_r\right)du.$$

Proof. Using the integral representation (3) of M-beta function in the definition of M-Lauricella hypergeometric function $F_C^{(r)}$ and making similar calculations in the proof of Theorem 11, proof is complete. □

Theorem 14. *Let* $\Re(\alpha)>0$, $\Re(\rho)>0$, $\Re(\nu_1)>\Re(\lambda)>0$. *Then,*

$$ {}^{M}F_{C,p,q}^{(\alpha,\beta;r)}(\kappa,\mu;\nu_1,\ldots,\nu_r;x_1,\ldots,x_r;\rho;\lambda)$$

$$= \frac{(b-a)^{1-\nu_1}}{B(\lambda,\nu_1-\lambda)}\int_a^b(u-a)^{\lambda-1}(b-u)^{\nu_1-\lambda-1} \, {}_p^{\alpha}M_q^{\beta}\left(\frac{-\rho(b-a)^2}{(u-a)(b-u)}\right)$$

$$\times F_C^{(r)}\left(\kappa,\mu;\lambda,\nu_2,\ldots,\nu_r;\frac{x_1(u-a)}{b-a},x_2,\ldots,x_r\right)du.$$

Proof. Using the integral representation (4) of M-beta function in the definition of M-Lauricella hypergeometric function $F_C^{(r)}$ and making similar calculations in the proof of Theorem 11, proof is complete. □

6. Integral Representations for M-Lauricella Hypergeometric Function $F_D^{(r)}$

Theorem 15. *Let* $\Re(\alpha)>0$, $\Re(\rho)>0$, $\Re(\nu)>\Re(\kappa)>0$. *Then,*

$$\begin{aligned}&{}^{M}F_{D,p,q}^{(\alpha,\beta;r)}(\kappa,\mu_1,\ldots,\mu_r;\nu;x_1,\ldots,x_r;\rho)\\&\quad=\frac{1}{B(\kappa,\nu-\kappa)}\int_0^1\Delta^{\kappa-1}(1-\Delta)^{\nu-\kappa-1}\,{}_p^{\alpha}M_q^{\beta}\left(\frac{-\rho}{\Delta(1-\Delta)}\right)\\&\quad\times F_D^{(r)}(\kappa,\mu_1,\ldots,\mu_r;\kappa;\Delta x_1,\ldots,\Delta x_r)d\Delta.\end{aligned}$$

Proof. Using the integral representation (1) of M-beta function in the definition of M-Lauricella hypergeometric function $F_D^{(r)}$, we have

$$\begin{aligned}&{}^{M}F_{D,p,q}^{(\alpha,\beta;r)}(\kappa,\mu_1,\ldots,\mu_r;\nu;x_1,\ldots,x_r;\rho)\\&=\sum_{n_1,\ldots,n_r=0}^{\infty}(\mu_1)_{n_1}\ldots(\mu_r)_{n_r}\frac{{}^{M}B_{p,q}^{(\alpha,\beta)}(\kappa+n_1+\ldots+n_r,\nu-\kappa;\rho)}{B(\kappa,\nu-\kappa)}\frac{x_1^{n_1}}{n_1!}\ldots\frac{x_r^{n_r}}{n_r!}\\&=\frac{1}{B(\kappa,\nu-\kappa)}\sum_{n_1,\ldots,n_r=0}^{\infty}(\mu_1)_{n_1}\ldots(\mu_r)_{n_r}\\&\times\int_0^1\Delta^{\kappa+n_1+\ldots+n_r-1}(1-\Delta)^{\nu-\kappa-1}\,{}_p^{\alpha}M_q^{\beta}\left(\frac{-\rho}{\Delta(1-\Delta)}\right)\frac{x_1^{n_1}}{n_1!}\ldots\frac{x_r^{n_r}}{n_r!}d\Delta\\&=\frac{1}{B(\kappa,\nu-\kappa)}\int_0^1\Delta^{\kappa-1}(1-\Delta)^{\nu-\kappa-1}\,{}_p^{\alpha}M_q^{\beta}\left(\frac{-\rho}{\Delta(1-\Delta)}\right)\\&\times\sum_{n_1,\ldots,n_r=0}^{\infty}(\mu_1)_{n_1}\ldots(\mu_r)_{n_r}\frac{(\Delta x_1)^{n_1}}{n_1!}\ldots\frac{(\Delta x_r)^{n_r}}{n_r!}d\Delta.\end{aligned}$$

Multiplied by $\frac{(\kappa)_{n_1+\ldots+n_r}}{(\kappa)_{n_1+\ldots+n_r}}$ and considering Eq. (13), we get

$$\begin{aligned}&{}^{M}F_{D,p,q}^{(\alpha,\beta;r)}(\kappa,\mu_1,\ldots,\mu_r;\nu;x_1,\ldots,x_r;\rho)\\&\quad=\frac{1}{B(\kappa,\nu-\kappa)}\int_0^1\Delta^{\kappa-1}(1-\Delta)^{\nu-\kappa-1}\,{}_p^{\alpha}M_q^{\beta}\left(\frac{-\rho}{\Delta(1-\Delta)}\right)\\&\quad\times\sum_{n_1,\ldots,n_r=0}^{\infty}\frac{(\kappa)_{n_1+\ldots+n_r}(\mu_1)_{n_1}\ldots(\mu_r)_{n_r}}{(\kappa)_{n_1+\ldots+n_r}}\frac{(\Delta x_1)^{n_1}}{n_1!}\ldots\frac{(\Delta x_r)^{n_r}}{n_r!}d\Delta\end{aligned}$$

$$= \frac{1}{B(\kappa,\nu-\kappa)} \int_0^1 \Delta^{\kappa-1}(1-\Delta)^{\nu-\kappa-1} \, {}_p^{\alpha}M_q^{\beta}\left(\frac{-\rho}{\Delta(1-\Delta)}\right) \times F_D^{(r)}(\kappa,\mu_1,\ldots,\mu_r;\kappa;\Delta x_1,\ldots,\Delta x_r)d\Delta.$$

□

Theorem 16. *Let* $\Re(\alpha)>0$, $\Re(\rho)>0$, $\Re(\nu)>\Re(\kappa)>0$. *Then,*

$$\begin{aligned} &{}^{M}F_{D,p,q}^{(\alpha,\beta;r)}(\kappa,\mu_1,\ldots,\mu_r;\nu;x_1,\ldots,x_r;\rho) \\ &= \frac{1}{B(\kappa,\nu-\kappa)} \int_0^1 \Delta^{\kappa-1}(1-\Delta)^{\nu-\kappa-1} \, {}_p^{\alpha}M_q^{\beta}\left(\frac{-\rho}{\Delta(1-\Delta)}\right) \\ &\times (1-\Delta x_1)^{-\mu_1}\ldots(1-\Delta x_r)^{-\mu_r}d\Delta. \end{aligned}$$

Proof. Using the integral representation (1) of M-beta function in the definition of M-Lauricella hypergeometric function $F_D^{(r)}$, we have

$$\begin{aligned} &{}^{M}F_{D,p,q}^{(\alpha,\beta;r)}(\kappa,\mu_1,\ldots,\mu_r;\nu;x_1,\ldots,x_r;\rho) \\ &= \sum_{n_1,\ldots,n_r=0}^{\infty} (\mu_1)_{n_1}\ldots(\mu_r)_{n_r} \frac{{}^{M}B_{p,q}^{(\alpha,\beta)}(\kappa+n_1+\ldots+n_r,\nu-\kappa;\rho)}{B(\kappa,\nu-\kappa)} \frac{x_1^{n_1}}{n_1!}\cdots\frac{x_r^{n_r}}{n_r!} \\ &= \frac{1}{B(\kappa,\nu-\kappa)} \int_0^1 \Delta^{\kappa-1}(1-\Delta)^{\nu-\kappa-1} \, {}_p^{\alpha}M_q^{\beta}\left(\frac{-\rho}{\Delta(1-\Delta)}\right) \\ &\times \sum_{n_1=0}^{\infty}(\mu_1)_{n_1}\frac{(\Delta x_1)^{n_1}}{n_1!}\cdots\sum_{n_r=0}^{\infty}(\mu_r)_{n_r}\frac{(\Delta x_r)^{n_r}}{n_r!}d\Delta. \end{aligned}$$

Considering Eq. (8), we get

$$\begin{aligned} &{}^{M}F_{D,p,q}^{(\alpha,\beta;r)}(\kappa,\mu_1,\ldots,\mu_r;\nu;x_1,\ldots,x_r;\rho) \\ &= \frac{1}{B(\kappa,\nu-\kappa)} \int_0^1 \Delta^{\kappa-1}(1-\Delta)^{\nu-\kappa-1} \, {}_p^{\alpha}M_q^{\beta}\left(\frac{-\rho}{\Delta(1-\Delta)}\right) \\ &\times (1-\Delta x_1)^{-\mu_1}\ldots\ (1-\Delta x_r)^{-\mu_r}d\Delta. \end{aligned}$$

□

Theorem 17. *Let* $\Re(\alpha)>0$, $\Re(\rho)>0$, $\Re(\nu)>\Re(\kappa)>0$. *Then,*

$$\begin{aligned} &{}^{M}F_{D,p,q}^{(\alpha,\beta;r)}(\kappa,\mu_1,\ldots,\mu_r;\nu;x_1,\ldots,x_r;\rho) \\ &= \frac{2}{B(\kappa,\nu-\kappa)} \int_0^{\frac{\pi}{2}} (\sin\phi)^{2\kappa-1}(\cos\phi)^{2\nu-2\kappa-1} \, {}_p^{\alpha}M_q^{\beta}\left(-\rho(\sec\phi)^2(\csc\phi)^2\right) \\ &\times F_D^{(r)}\left(\kappa,\mu_1,\ldots,\mu_r;\kappa;x_1(\sin\phi)^2,\ldots,x_r(\sin\phi)^2\right)d\phi. \end{aligned}$$

Proof. Using the integral representation (2) of M-beta function in the definition of M-Lauricella hypergeometric function $F_D^{(r)}$ and making similar calculations in the proof of Theorem 15, proof is complete. □

Theorem 18. *Let* $\Re(\alpha) > 0$, $\Re(\rho) > 0$, $\Re(\nu) > \Re(\kappa) > 0$. *Then,*

$$\begin{aligned} &{}^{M}F_{D,p,q}^{(\alpha,\beta;r)}(\kappa,\mu_1,\ldots,\mu_r;\nu;x_1,\ldots,x_r;\rho) \\ &= \frac{1}{B(\kappa,\nu-\kappa)}\int_0^\infty \frac{u^{\kappa-1}}{(1+u)^\nu}\ {}_p^\alpha M_q^\beta\left(-2\rho-\rho\left(u+\frac{1}{u}\right)\right) \\ &\times F_D^{(r)}\left(\kappa,\mu_1,\ldots,\mu_r;\kappa;\frac{x_1u}{1+u},\ldots,\frac{x_ru}{1+u}\right)du. \end{aligned}$$

Proof. Using the integral representation (3) of M-beta function in the definition of M-Lauricella hypergeometric function $F_D^{(r)}$ and making similar calculations in the proof of Theorem 15, proof is complete. □

Theorem 19. *Let* $\Re(\alpha) > 0$, $\Re(\rho) > 0$, $\Re(\nu) > \Re(\kappa) > 0$. *Then,*

$$\begin{aligned} &{}^{M}F_{D,p,q}^{(\alpha,\beta;r)}(\kappa,\mu_1,\ldots,\mu_r;\nu;x_1,\ldots,x_r;\rho) \\ &= \frac{(b-a)^{1-\nu}}{B(\kappa,\nu-\kappa)}\int_a^b (u-a)^{\kappa-1}(b-u)^{\nu-\kappa-1}\ {}_p^\alpha M_q^\beta\left(\frac{-\rho(b-a)^2}{(u-a)(b-u)}\right) \\ &\times F_D^{(r)}\left(\kappa,\mu_1,\ldots,\mu_r;\kappa;\frac{x_1(u-a)}{b-a},\ldots,\frac{x_r(u-a)}{b-a}\right)du. \end{aligned}$$

Proof. Using the integral representation (4) of M-beta function in the definition of M-Lauricella hypergeometric function $F_D^{(r)}$ and making similar calculations in the proof of Theorem 15, proof is complete. □

7. Applications of M-Lauricella Hypergeometric Functions

In this section, we obtain the solution of fractional differential equations involving the M-Lauricella hypergeometric functions.

Example 1. *Let* $1 < \Re(\epsilon) \le 2$, $\Re(\alpha) > 0$, $\Re(\lambda) > \Re(\mu_1) > 0$. *We consider the fractional differential equation*

$${}^{c}D_\rho^\epsilon\{f(\rho)\} = {}^{M}F_{B,p,q}^{(\alpha,\beta;r)}(\xi_1,\ldots,\xi_p;\eta_1,\ldots,\eta_q;\kappa_1,\ldots,\kappa_r,\mu_1,\ldots,\mu_r;\nu;x_1,\ldots,x_r;\epsilon\rho;\lambda),$$

with the initial conditions

$$f(0) = f'(0) = 0.$$

Applying the Laplace transform to the fractional differential equation and using Eq. (9)*, we have*

$$\begin{aligned} &\mathfrak{L}\left\{{}^{c}D_\rho^\epsilon\{f(\rho)\};s\right\} \\ &= \mathfrak{L}\left\{{}^{M}F_{B,p,q}^{(\alpha,\beta;r)}(\xi_1,\ldots,\xi_p;\eta_1,\ldots,\eta_q;\kappa_1,\ldots,\kappa_r,\mu_1,\ldots,\mu_r;\nu;x_1,\ldots,x_r;\epsilon\rho;\lambda);s\right\}, \end{aligned}$$

then

$$s^{\epsilon}F(s)-s^{\epsilon-1}f(0)-s^{\epsilon-2}f'(0)$$
$$=\frac{{}^{M}F_{B,p+1,q}^{(\alpha,\beta;r)}\left(\xi_1,\ldots,\xi_p,1;\eta_1,\ldots,\eta_q;\kappa_1,\ldots,\kappa_r,\mu_1,\ldots,\mu_r;\nu;x_1,\ldots,x_r;\frac{\epsilon}{s};\lambda\right)}{s}.$$

Using the initial conditions, we get

$$F(s)=\frac{{}^{M}F_{B,p+1,q}^{(\alpha,\beta;r)}\left(\xi_1,\ldots,\xi_p,1;\eta_1,\ldots,\eta_q;\kappa_1,\ldots,\kappa_r,\mu_1,\ldots,\mu_r;\nu;x_1,\ldots,x_r;\frac{\epsilon}{s};\lambda\right)}{s^{\epsilon+1}}.$$

Applying the inverse Laplace transform, we obtain

$$f(\rho)=\frac{{}^{M}F_{B,p+1,q+1}^{(\alpha,\beta;r)}(\xi_1,\ldots,\xi_p,1;\eta_1,\ldots,\eta_q,1+\epsilon;\kappa_1,\ldots,\kappa_r,\mu_1,\ldots,\mu_r;\nu;x_1,\ldots,x_r;\epsilon\rho;\lambda)}{\Gamma(1+\epsilon)\rho^{-\epsilon}}.$$

Example 2. *Let $1<\Re(\epsilon)\leq 2$, $\Re(\alpha)>0$, $\Re(\nu_1)>\Re(\lambda)>0$. We consider the fractional differential equation*

$${}^{c}D_{\rho}^{\epsilon}\{f(\rho)\}={}^{M}F_{C,p,q}^{(\alpha,\beta;r)}(\xi_1,\ldots,\xi_p;\eta_1,\ldots,\eta_q;\kappa,\mu;\nu_1,\ldots,\nu_r;x_1,\ldots,x_r;\epsilon\rho;\lambda),$$

with the initial conditions

$$f(0)=f'(0)=0.$$

Applying the Laplace transform to the fractional differential equation and using Eq. (9), we have

$$\mathfrak{L}\left\{{}^{c}D_{\rho}^{\epsilon}\{f(\rho)\};s\right\}$$
$$=\mathfrak{L}\left\{{}^{M}F_{C,p,q}^{(\alpha,\beta;r)}(\xi_1,\ldots,\xi_p;\eta_1,\ldots,\eta_q;\kappa,\mu;\nu_1,\ldots,\nu_r;x_1,\ldots,x_r;\epsilon\rho;\lambda);s\right\},$$

then

$$s^{\epsilon}F(s)-s^{\epsilon-1}f(0)-s^{\epsilon-2}f'(0)$$
$$=\frac{{}^{M}F_{C,p+1,q}^{(\alpha,\beta;r)}\left(\xi_1,\ldots,\xi_p,1;\eta_1,\ldots,\eta_q;\kappa,\mu;\nu_1,\ldots,\nu_r;x_1,\ldots,x_r;\frac{\epsilon}{s};\lambda\right)}{s}.$$

Using the initial conditions, we get

$$F(s)=\frac{{}^{M}F_{C,p+1,q}^{(\alpha,\beta;r)}\left(\xi_1,\ldots,\xi_p,1;\eta_1,\ldots,\eta_q;\kappa,\mu;\nu_1,\ldots,\nu_r;x_1,\ldots,x_r;\frac{\epsilon}{s};\lambda\right)}{s^{\epsilon+1}}.$$

Applying the inverse Laplace transform, we obtain

$$f(\rho)=\frac{{}^{M}F_{C,p+1,q+1}^{(\alpha,\beta;r)}(\xi_1,\ldots,\xi_p,1;\eta_1,\ldots,\eta_q,1+\epsilon;\kappa,\mu;\nu_1,\ldots,\nu_r;x_1,\ldots,x_r;\epsilon\rho;\lambda)}{\Gamma(1+\epsilon)\rho^{-\epsilon}}.$$

Example 3. *Let $1<\Re(\epsilon)\leq 2$, $\Re(\alpha)>0$, $\Re(\nu)>\Re(\kappa)>0$. We consider the fractional differential equation*

$${}^{c}D_{\rho}^{\epsilon}\{f(\rho)\}={}^{M}F_{D,p,q}^{(\alpha,\beta;r)}(\xi_1,\ldots,\xi_p;\eta_1,\ldots,\eta_q;\kappa,\mu_1,\ldots,\mu_r;\nu;x_1,\ldots,x_r;\epsilon\rho),$$

with the initial conditions

$$f(0) = f'(0) = 0.$$

Applying the Laplace transform to the fractional differential equation and using Eq. (9)*, we have*

$$\mathfrak{L}\left\{{}^{c}D_{\rho}^{\epsilon}\left\{f(\rho)\right\};s\right\} = \mathfrak{L}\left\{{}^{M}F_{D,p,q}^{(\alpha,\beta;r)}(\xi_1,\ldots,\xi_p;\eta_1,\ldots,\eta_q;\kappa,\mu_1,\ldots,\mu_r;\nu;x_1,\ldots,x_r;\epsilon\rho);s\right\},$$

then

$$s^{\epsilon}F(s) - s^{\epsilon-1}f(0) - s^{\epsilon-2}f'(0) = \frac{{}^{M}F_{D,p+1,q}^{(\alpha,\beta;r)}\left(\xi_1,\ldots,\xi_p,1;\eta_1,\ldots,\eta_q;\kappa,\mu_1,\ldots,\mu_r;\nu;x_1,\ldots,x_r;\frac{\epsilon}{s}\right)}{s}.$$

Using the initial conditions, we get

$$F(s) = \frac{{}^{M}F_{D,p+1,q}^{(\alpha,\beta;r)}\left(\xi_1,\ldots,\xi_p,1;\eta_1,\ldots,\eta_q;\kappa,\mu_1,\ldots,\mu_r;\nu;x_1,\ldots,x_r;\frac{\epsilon}{s}\right)}{s^{\epsilon+1}}.$$

Applying the inverse Laplace transform, we obtain

$$f(\rho) = \frac{{}^{M}F_{D,p+1,q+1}^{(\alpha,\beta;r)}(\xi_1,\ldots,\xi_p,1;\eta_1,\ldots,\eta_q,1+\epsilon;\kappa,\mu_1,\ldots,\mu_r;\nu;x_1,\ldots,x_r;\epsilon\rho)}{\Gamma(1+\epsilon)\rho^{-\epsilon}}.$$

8. Conclusion

In this paper, we introduced the M-Lauricella hypergeometric functions using the modified beta function involving the generalized M-series in its kernel. Then we presented various integral representations of M-Lauricella hypergeometric functions. As examples, we obtained the solution of fractional differential equations involving the M-Lauricella hypergeometric functions.

M-Lauricella hypergeometric functions can be used not only in various fractional differential equations, but also in various ordinary and partial differential equations. Therefore, we conclude this paper by believing that M-Lauricella hypergeometric functions can be used in various research areas of interest to many fields of science such as mathematics, statistics, physics, chemistry, biology, medicine, engineering, astronomy and space sciences and will contribute to the scientific world.

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