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Note on the convergence of fractional conformable diffusion equation with linear source term

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Abstract

In this paper, we study the diffusion equation with conformable derivative. The main goal is to prove the convergence of the mild solution to our problem when the order of fractional Laplacian tends to 1^-. The principal techniques of our paper is based on some useful evaluations for exponential kernels.

Keywords: Conformable derivative, diffusion equation, convergent estimate.
2010 MSC: 35R11, 35B65, 26A33.

1. Introduction

In this paper, we are interested to study the following problem

$$\begin{cases} \dfrac{{}^{C}\partial^{\alpha}}{\partial t^{\alpha}} v + (-\Delta)^{s} v(x,t) = F(x,t), \quad (x,t) \in \Omega \times (0,T), \\ v(x,t) = 0, \quad x \in \partial\Omega, \quad t \in (0,T), \end{cases} \tag{1.1}$$

with the initial condition

$$v(x,0) = v_0(x), \quad x \in \Omega, \tag{1.2}$$

where v_0 and F are the input data. The symbol $(-\Delta)^s$, $s > 0$ says that the fractional Laplacian which is defined later. Here $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is a bounded domain with the smooth boundary $\partial\Omega$, and $T > 0$ is a given positive number. The above equation has various applications in areas such as the harmonic oscillator, the damped oscillator and the forced

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oscillator (see [3]), electrical circuits (see [5]), chaotic systems in dynamics (see [6]), projectile motion (see [7]). Our paper is one of the braches of directions about fractional PDEs, see [15, 16, 17, 18, 20].

The symbol $\frac{{}^{C}\partial^{\alpha}}{\partial t^{\alpha}}$ is understood as the conformable derivative. Let us now give a clear definition of conformable derivative on the Banach space. Let us given B is a Banach space, and $f : [0, \infty) \to B$. Let $\frac{{}^{C}\partial^{\alpha}}{\partial t^{\alpha}}$ be the conformable derivative of order $0 < \alpha \le 1$ which is given by

$$\frac{{}^{C}\partial^{\alpha} f(t)}{\partial t^{\alpha}} := \lim_{\epsilon \to 0} \frac{f(t + \epsilon t^{1-\alpha}) - f(t)}{\epsilon} \quad \text{in } B,$$

for each $t > 0$. Some more details on conformable derivative can be found in [1, 2, 8, 14, 9, 18]. It is easy to see that $\alpha = s = 1$, Problem (1.1)-(1.2) becomes the classical heat equation.

We mention now some previous results for conformable derivative. In [12], an inverse problem for second boundary with conformable diffusion is shown. In [13], Bayrak and his colleagues investigated approximate solution of the time-fractional Fisher equation with small delay. In [10], the authors studied some nonlinear partial differential equations with conformable derivative. In [14], the authors focused on a mild solution of conformable fractional abstract initial value problem.

The well-posedness of Problem (1.1)-(1.2) was established in [11]. Indeed, the paper [11] derived more clearly the existence and the regularity of the mild solution of Problem (1.1)-(1.2). One of the highlights of our problem is the occurrence the fractional Laplacian $(-\Delta)^s$ for any $0 < s \le 1$. Our main goal in this paper is to study the limit problem of the mild solution when $s \to 1^-$. Up to now, there has not been any literature surveying the mentioned issue. Our paper is the first result concerned with the limit problem for the fractional diffusion equation with conformable derivatives. In order to overcome some complicated evaluations, we use some new techniques for the computations for exponential functions.

2. Initial value problem

2.1. Premilinaries

Let us recall that the spectral problem

$$\begin{cases} (-\Delta)^s e_n(x) = \lambda_n^s e_n(x), & x \in \mathcal{D}, \\ e_n(x) = 0, & x \in \partial D, \end{cases}$$

admits the eigenvalues $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \le \ldots$ with $\lambda_n \to \infty$ as $n \to \infty$. The corresponding eigenfunctions are $e_n \in H_0^1(\Omega)$.

Next, Let a given positive number $\sigma \ge 0$. Let us also define the Hilbert scale space as follows

$$\mathbb{H}^{\sigma}(\Omega) = \left\{ \psi \in L^2(\Omega) : \sum_{n=1}^{\infty} \lambda_n^{2\sigma} \langle \psi, e_n \rangle^2 < +\infty \right\}, \tag{2.1}$$

with the following norm $\|\psi\|_{\mathbb{H}^{\sigma}(\Omega)} = \left(\sum_{n=1}^{\infty} \lambda_n^{2\sigma} \langle \psi, e_n \rangle^2 \right)^{\frac{1}{2}}$.

2.2. The linear case

In this section, we focus the following initial value problem (1.1) under the linear case with the initial condition (1.2). Here v_0 and source function F are defined later.

In order to find a precise formulation for solutions, we consider the spectral decomposition

$$v_s(x,t) = \sum_{n=1}^{\infty} \left(\int_{\Omega} v_s(x,t) e_n(x) dx \right) e_n(x).$$

Thanks for the work [11], we get the following equality

$$\int_\Omega v_s(x,t)e_n(x)dx = \exp\Big(-\alpha^{-1}t^\alpha\lambda_n^s\Big)\int_\Omega v_0(x)e_n(x)dx + \int_0^t \theta^{\alpha-1}\exp\Big(-\alpha^{-1}(t^\alpha-\theta^\alpha)\lambda_n^s\Big)\Big(\int_\Omega F(x,\theta)e_n(x)dx\Big)d\theta. \tag{2.2}$$

Then the mild solution to problem (1.1)–(1.2) is defined by

$$v_s(x,t) = \sum_{n=1}^\infty \exp\Big(-\alpha^{-1}t^\alpha\lambda_n^s\Big)\Big(\int_\Omega v_0(x)e_n(x)dx\Big)e_n(x) + \sum_{n=1}^\infty\Big(\int_0^t \theta^{\alpha-1}\exp\Big(-\alpha^{-1}(t^\alpha-\theta^\alpha)\lambda_n^s\Big)\Big(\int_\Omega F(x,\theta)e_n(x)dx\Big)d\theta\Big)e_n(x). \tag{2.3}$$

Here $0 < s < 1$.

Lemma 1. *Let $0 < s < 1$. Then we have the following inequality*

$$\Big|\lambda_n^s - \lambda_n\Big| \le C(m)\lambda_n^{s-m}(1-s)^m, \quad \lambda_n \le 1 \tag{2.4}$$

and

$$\Big|\lambda_n^s - \lambda_n\Big| \le C(m)\lambda_n^{1-m}(1-s)^m, \quad \lambda_n > 1. \tag{2.5}$$

Proof. If $\lambda_n \le 1$ then we get that the following bound

$$\Big|\lambda_n^s - \lambda_n\Big| = \lambda_n^s - \lambda_n = \lambda_n^s\Big(1-\exp\big(-(1-s)\log(\frac{1}{\lambda_n})\big)\Big) \le C(m)\lambda_n^s(1-s)^m\log^m(\frac{1}{\lambda_n}) \le C(m)\lambda_n^{s-m}(1-s)^m, \tag{2.6}$$

where we have used the inequality $1-e^{-z} \le C(m)z^m$.
If $\lambda_n \ge 1$ then we get that the following bound

$$\Big|\lambda_n^s - \lambda_n\Big| = \lambda_n - \lambda_n^s = \lambda_n\Big(1-\exp\big(-(1-s)\log(\frac{1}{\lambda_n})\big)\Big) \le C(m)\lambda_n(1-s)^m\log^m(\frac{1}{\lambda_n}) \le C(m)\lambda_n^{1-m}(1-s)^m. \tag{2.7}$$

□

Theorem 1. *Let $v_0 \in \mathbb{H}^{1-m+p}(\Omega)$ and $F \in L^\infty(0,T;\mathbb{H}^{1+p-m}(\Omega))$ for any $p > 0$. Then we get*

$$\Big\|v_s - v^{**}\Big\|_{L^\infty(0,T;\mathbb{H}^p(\Omega))} \le C(m,T,\alpha)(1-s)^m\Big(\Big\|v_0\Big\|_{\mathbb{H}^{1-m+p}(\Omega)} + \Big\|F\Big\|_{L^\infty(0,T;\mathbb{H}^{1+p-m}(\Omega))}\Big) \tag{2.8}$$

for any $p \ge 0$.

Proof. The mild solution of Problem (1.1)-(1.2) with $s = 1$ is given by

$$v^{**}(x,t) = \sum_{n=1}^\infty \exp\Big(-\alpha^{-1}t^\alpha\lambda_n\Big)\Big(\int_\Omega v_0(x)e_n(x)dx\Big)e_n(x) + \sum_{n=1}^\infty\Big(\int_0^t \theta^{\alpha-1}\exp\Big(-\alpha^{-1}(t^\alpha-\theta^\alpha)\lambda_n\Big)\Big(\int_\Omega F(x,\theta)e_n(x)dx\Big)d\theta\Big)e_n(x). \tag{2.9}$$

By subtracting both sides of the two expressions above, we get the following difference

$$
\begin{aligned}
& v_s(x,t) - v^{**}(x,t) \\
&= \sum_{n=1}^{\infty} \Big[\exp\Big(-\alpha^{-1} t^{\alpha} \lambda_n^s \Big) - \exp\Big(-\alpha^{-1} t^{\alpha} \lambda_n \Big) \Big] \Big(\int_\Omega v_0(x) e_n(x) dx \Big) e_n(x) \\
&+ \sum_{n=1}^{\infty} \bigg(\int_0^t \theta^{\alpha-1} \Big[\exp\Big(-\alpha^{-1} (t^\alpha - \theta^\alpha) \lambda_n^s - \exp\Big(-\alpha^{-1} (t^\alpha - \theta^\alpha) \lambda_n \Big) \Big] \Big(\int_\Omega F(x,\theta) e_n(x) dx \Big) d\theta \bigg) e_n(x) \\
&= B_1(x,t) + B_2(x,t).
\end{aligned} \tag{2.10}
$$

Let us first treat the quantity B_1. Using Parseval's equality and the inequality $|e^{-a} - e^{-b}| \le |a - b|$ for any $a, b > 0$, we find that

$$
\begin{aligned}
\Big\| B_1(.,t) \Big\|^2_{\mathbb{H}^p(\Omega)} &= \sum_{n=1}^{\infty} \lambda_n^{2p} \Big[\exp\Big(-\alpha^{-1} t^\alpha \lambda_n^s \Big) - \exp\Big(-\alpha^{-1} t^\alpha \lambda_n \Big) \Big]^2 \Big(\int_\Omega v_0(x) e_n(x) dx \Big)^2 \\
&\le \alpha^{-2} t^{2\alpha} \sum_{n=1}^{\infty} \lambda_n^{2p} \Big| \lambda_n^s - \lambda_n \Big|^2 \Big(\int_\Omega v_0(x) e_n(x) dx \Big)^2 \\
&= \alpha^{-2} t^{2\alpha} \sum_{\lambda_n \le 1} \lambda_n^{2p} \Big| \lambda_n^s - \lambda_n \Big|^2 \Big(\int_\Omega v_0(x) e_n(x) dx \Big)^2 \\
&+ \alpha^{-2} t^{2\alpha} \sum_{\lambda_n > 1} \lambda_n^{2p} \Big| \lambda_n^s - \lambda_n \Big|^2 \Big(\int_\Omega v_0(x) e_n(x) dx \Big)^2.
\end{aligned} \tag{2.11}
$$

In view of Lemma 1, we derive that

$$
\begin{aligned}
\Big\| B_1(.,t) \Big\|^2_{\mathbb{H}^p(\Omega)} &\le C(m) \alpha^{-2} t^{2\alpha} (1-s)^{2m} \sum_{\lambda_n \le 1} \lambda_n^{2s-2m+2p} \Big(\int_\Omega v_0(x) e_n(x) dx \Big)^2 \\
&+ C(m) \alpha^{-2} t^{2\alpha} (1-s)^{2m} \sum_{\lambda_n > 1} \lambda_n^{2-2m+2p} \Big(\int_\Omega v_0(x) e_n(x) dx \Big)^2 \\
&\le C(m) \alpha^{-2} t^{2\alpha} (1-s)^{2m} \bigg(\Big\| v_0 \Big\|^2_{\mathbb{H}^{s-m+p}(\Omega)} + \Big\| v_0 \Big\|^2_{\mathbb{H}^{1-m+p}(\Omega)} \bigg).
\end{aligned} \tag{2.12}
$$

Since the fact that

$$
\Big\| v_0 \Big\|_{\mathbb{H}^{s-m+p}(\Omega)} \le C(s,m,p) \Big\| v_0 \Big\|_{\mathbb{H}^{1-m+p}(\Omega)}
$$

we know that the following estimate

$$
\Big\| B_1(.,t) \Big\|_{\mathbb{H}^p(\Omega)} \le C(m,s,p) \alpha^{-1} t^\alpha (1-s)^m \Big\| v_0 \Big\|_{\mathbb{H}^{1-m+p}(\Omega)}. \tag{2.13}
$$

Let us to study the second term $\Big\| B_2(.,t) \Big\|_{\mathbb{H}^p(\Omega)}$. Indeed, we get that

$$
\begin{aligned}
& \Big\| B_2(.,t) \Big\|^2_{\mathbb{H}^p(\Omega)} \\
&= \sum_{n=1}^{\infty} \lambda_n^{2p} \bigg(\int_0^t \theta^{\alpha-1} \Big[\exp\Big(-\alpha^{-1} (t^\alpha - \theta^\alpha) \lambda_n^s - \exp\Big(-\alpha^{-1} (t^\alpha - \theta^\alpha) \lambda_n \Big) \Big] \Big(\int_\Omega F(x,\theta) e_n(x) dx \Big) d\theta \bigg)^2 \\
&\le \sum_{n=1}^{\infty} \lambda_n^{2p} \int_0^t \theta^{\alpha-1} \Big[\exp\Big(-\alpha^{-1} (t^\alpha - \theta^\alpha) \lambda_n^s - \exp\Big(-\alpha^{-1} (t^\alpha - \theta^\alpha) \lambda_n \Big) \Big]^2 \Big(\int_\Omega F(x,\theta) e_n(x) dx \Big)^2 d\theta.
\end{aligned} \tag{2.14}
$$

By applying the inequality $|e^{-a} - e^{-b}| \le |a - b|, \;\; a, b \ge 0$ we derive that

$$
\exp\Big(-\alpha^{-1} (t^\alpha - \theta^\alpha) \lambda_n^s - \exp\Big(-\alpha^{-1} (t^\alpha - \theta^\alpha) \lambda_n \Big) \le \alpha^{-1} (t^\alpha - \theta^\alpha) \Big| \lambda_n^s - \lambda_n \Big|. \tag{2.15}
$$

Hence, we get that

$$\Big\|B_2(.,t)\Big\|^2_{\mathbb{H}^p(\Omega)} \le \frac{\alpha^{-2}T^\alpha}{\alpha}\left(\int_0^t \theta^{\alpha-1}(t^\alpha-\theta^\alpha)^2\sum_{n=1}^\infty \lambda_n^{2p}\Big|\lambda_n^s-\lambda_n\Big|^2\Big(\int_\Omega F(x,\theta)e_n(x)dx\Big)^2 d\theta\right) \tag{2.16}$$

By looking at Lemma 1, we know that

$$\begin{aligned}
&\sum_{n=1}^\infty \lambda_n^{2p}\Big|\lambda_n^s-\lambda_n\Big|^2\Big(\int_\Omega F(x,\theta)e_n(x)dx\Big)^2\\
&=\sum_{\lambda_n\le 1} \lambda_n^{2p}\Big|\lambda_n^s-\lambda_n\Big|^2\Big(\int_\Omega F(x,\theta)e_n(x)dx\Big)^2\\
&+\sum_{\lambda_n> 1} \lambda_n^{2p}\Big|\lambda_n^s-\lambda_n\Big|^2\Big(\int_\Omega F(x,\theta)e_n(x)dx\Big)^2\\
&\le C(m)(1-s)^{2m}\sum_{\lambda_n\le 1}\lambda_n^{2p+2s-2m}\Big(\int_\Omega F(x,\theta)e_n(x)dx\Big)^2\\
&+C(m)(1-s)^{2m}\sum_{\lambda_n> 1}\lambda_n^{2p+2-2m}\Big(\int_\Omega F(x,\theta)e_n(x)dx\Big)^2.
\end{aligned} \tag{2.17}$$

Since the fact that $s<1$, we follows from the latter estimate that

$$\sum_{n=1}^\infty \lambda_n^{2p}\Big|\lambda_n^s-\lambda_n\Big|^2\Big(\int_\Omega F(x,\theta)e_n(x)dx\Big)^2 \le C(m)(1-s)^{2m}\Big\|F(.,\theta)\Big\|^2_{\mathbb{H}^{1+p-m}(\Omega)}. \tag{2.18}$$

Combining (2.16), (2.18), we obtain that the following bound

$$\begin{aligned}
\Big\|B_2(.,t)\Big\|^2_{\mathbb{H}^p(\Omega)} &\le C(m,T,\alpha)(1-s)^{2m}\int_0^t \theta^{\alpha-1}(t^\alpha-\theta^\alpha)^2\Big\|F(.,\theta)\Big\|^2_{\mathbb{H}^{1+p-m}(\Omega)}d\theta\\
&\le C(m,T,\alpha)(1-s)^{2m}\Big\|F\Big\|^2_{L^\infty(0,T;\mathbb{H}^{1+p-m}(\Omega))}\Big(\int_0^t \theta^{\alpha-1}(t^\alpha-\theta^\alpha)^2d\theta\Big).
\end{aligned} \tag{2.19}$$

Set $z=\theta^\alpha$ then $dz=\alpha\theta^{\alpha-1}d\theta$. Then we obtain that

$$\int_0^t \theta^{\alpha-1}(t^\alpha-\theta^\alpha)^2d\theta=\int_0^{t^\alpha}(t^\alpha-z)^2dz=\frac{t^{3\alpha}}{3} \tag{2.20}$$

This implies that

$$\Big\|B_2(.,t)\Big\|^2_{\mathbb{H}^p(\Omega)} \le C(m,T,\alpha)(1-s)^{2m}\Big\|F\Big\|^2_{L^\infty(0,T;\mathbb{H}^{1+p-m}(\Omega))}, \tag{2.21}$$

which implies that

$$\Big\|B_2(.,t)\Big\|_{\mathbb{H}^p(\Omega)} \le C(m,T,\alpha)(1-s)^{m}\Big\|F\Big\|_{L^\infty(0,T;\mathbb{H}^{1+p-m}(\Omega))}. \tag{2.22}$$

Combining (2.13) and (2.22), we derive that

$$\begin{aligned}
\Big\|v_s(.,t)-v^{**}(.,t)\Big\|_{\mathbb{H}^p(\Omega)} &\le \Big\|B_1(.,t)\Big\|_{\mathbb{H}^p(\Omega)}+\Big\|B_2(.,t)\Big\|_{\mathbb{H}^p(\Omega)}\\
&\le C(m,T,\alpha)(1-s)^m\Big(\Big\|v_0\Big\|_{\mathbb{H}^{1-m+p}(\Omega)}+\Big\|F\Big\|_{L^\infty(0,T;\mathbb{H}^{1+p-m}(\Omega))}\Big).
\end{aligned} \tag{2.23}$$

Since the right-hand side of (2.23) is independent of t, we derive that

$$\Big\|v_s-v^{**}\Big\|_{L^\infty(0,T;\mathbb{H}^p(\Omega))} \le C(m,T,\alpha)(1-s)^m\Big(\Big\|v_0\Big\|_{\mathbb{H}^{1-m+p}(\Omega)}+\Big\|F\Big\|_{L^\infty(0,T;\mathbb{H}^{1+p-m}(\Omega))}\Big). \tag{2.24}$$

□

3. Conclusion

In this work, the diffusion equation with conformable derivative, and fractional Laplacian tends to 1^-. We proved the convergence of the mild solution, while the principle techniques is exponential kernels.

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