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SOME NEW STABILITY RESULTS OF VOLTERRA INTEGRAL EQUATIONS ON TIME SCALES

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Abstract. In this paper, we generalize two types of Volterra integral equations given on time scales and examine their Hyers-Ulam and Hyers-Ulam-Rassias stabilities. We also prove these stability results for the non-homogeneous nonlinear Volterra integral equation on time scales and provide an example to support these results. Moreover, we show that the general Volterra type integral equation given on time scales has the Hyers-Ulam-Rassias stability. Our results extend and improve some recent developments announced in the current literature.

1. Introduction

In 1940, the famous stability theory of the linear functional equation was introduced by Ulam [25]. Since then a series of mathematical questions related to this stability theory was collected in the book [25] and studied by Hyers [15] and improved by Rassias [17]. From then on, stabilities of many functional, differential and integral equations have been investigated, see [1, 4, 5, 9, 14, 16, 19, 20], and references therein.

A time scale $\mathbb{T}$ is an arbitrary non-empty closed subset of the real numbers $\mathbb{R}$. The theory of time scales analysis has been rising fast and has attracted much interest. Therefore, many researchers have studied this issue [2, 11, 21, 22, 24]. The pioneer of this theory was Hilger [12]. He introduced this theory in 1988 with the inspiration to unify continuous and discrete calculus. Also, the stability analysis of dynamic equations has become an important topic both theoretically and practically because dynamic equations occur in many areas such as mechanics, physics and economics. For the introduction to the calculus on time scales and to the theory of dynamic equations on time scales, we recommend the books [6] and [7] by Bohner and Peterson.

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To the best of our knowledge, the first ones who pay attention to Hyers-Ulam stability for Volterra integral equations on time scales are Andras and Meszaros [4] and Hua, Li, Feng [13]. However they restricted their research to the case when integrand satisfies Lipschitz conditions with some Lipschitz constants.

Yaseen [26] investigated the Hyers-Ulam-Rassias stability for the following class of Volterra integral equations on time scales

$$x(t) = \int_{t_0}^{t} f\left(t, s, x(s)\right) \Delta s, \qquad t \in \mathbb{T}. \tag{1.1}$$

In 2017, Hamza and Ghallab [3] showed that the non-homogeneous Volterra integral equation of the first kind on time scales

$$x(t) = f(t) + \int_{a}^{t} K\left(t, s\right) x(s) \Delta s, \qquad t \in [a, b] \cap \mathbb{T} \tag{1.2}$$

has the Hyers-Ulam stability and Hyers-Ulam-Rassias stability.

Gachpazan and Baghani [10] discussed the Hyers-Ulam stability of the following non-homogeneous nonlinear Volterra integral equation

$$x(t) = f(t) + \varphi\left(\int_{a}^{t} K\left(t, s, x(s)\right) ds\right), \quad t \in [a, b]. \tag{1.3}$$

Finally, in 2020, Reinfelds and Christian [18] studied Hyers–Ulam stability of general Volterra type integral equations on bounded time scales

$$x(t) = f\left(t, x(t), x(\sigma(t)), \int_{a}^{t} K(t, s, x(s), x(\sigma(s))\Delta s\right), \qquad t \in [a, \infty) \cap \mathbb{T}. \tag{1.4}$$

Motivated by the above papers, we generalize two equations (1.1) and (1.2) given on time scales and examine their Hyers-Ulam and Hyers-Ulam-Rassias stabilities. We prove the existence and uniqueness of the solution and the stability results for equation (1.3) on time scales, and also provide an example to support these results. After that, we showed that equation (1.4), which is given on a time scale, has the Hyers-Ulam-Rassias stability.

2. Preliminaries on Time Scales

In this section, we present some basic notations, definitions and properties concerning the calculus on time scales, for more details the reader is referred to [6, 7].

As we said above, a time scale $\mathbb{T}$ is an arbitrary non-empty closed subset of the real numbers $\mathbb{R}$. Since a time scale may or may not be connected, the concept of jump operator is useful to describe the structure of the time scale under consideration and is also used in defining the delta derivative. The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$, while the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$.

The jump operators allow the classification of points in a time scale $\mathbb{T}$. A point $t \in \mathbb{T}$ is said to be right dense if $\sigma(t) = t$, right scattered if $\sigma(t) > t$, left dense if $\rho(t) = t$, left scattered if $\rho(t) < t$, isolated if $\rho(t) < t < \sigma(t)$, and dense if

$\rho(t) = t = \sigma(t)$. If $\mathbb{T}$ has a left scattered maximum n, then $\mathbb{T}^\kappa = \mathbb{T}\backslash\{n\}$, otherwise $\mathbb{T}^\kappa = \mathbb{T}$.

A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at every right dense points in $\mathbb{T}$ and its left sided limits exist (finite) at every left dense points in $\mathbb{T}$. The set of all rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by $C_{rd}(\mathbb{T}, \mathbb{R})$.

The graininess function $\mu : \mathbb{T} \to [0, +\infty)$ is defined by $\mu(t) = \sigma(t) - t$. The function $f : \mathbb{T} \to \mathbb{R}$ is regressive if

$$1 + \mu(t)f(t) \neq 0 \qquad \text{for all } t \in \mathbb{T}^\kappa.$$

Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and fix $t \in \mathbb{T}^\kappa$. The delta derivative (also Hilger derivative) $f^\Delta(t)$ exists if for every $\epsilon > 0$ there exists a neighbourhood $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$ such that

$$\left|(f(\sigma(t)) - f(s)) - f^\Delta(t)(\sigma(t) - s)\right| \leq \epsilon \left|\sigma(t) - s\right| \qquad \text{for all } s \in U.$$

If f is rd-continuous, then there is a function F such that $F^\Delta(t) = f(t)$ (see [6, 7]). In this case, we define the (Cauchy) delta integral by

$$\int\limits_r^s f(t)\Delta t = F(s) - F(r) \quad \text{for all } r, s \in \mathbb{T}.$$

Let $\beta : \mathbb{T} \to \mathbb{R}$ be a regressive and rd-continuous function. The Cauchy initial value problem for linear equation

$$x^\Delta = \beta(t)x, \quad x(a) = 1, \quad a \in \mathbb{T}$$

has the unique solution $e_\beta(., a) : \mathbb{T} \to \mathbb{R}$ [6, 7]. More explicitly, the exponential function $e_\beta(., a)$ is given by

$$e_\beta(t, a) = \exp\left(\int_a^t \xi_{\mu(s)}(\beta(s))\Delta s\right) \quad \text{for} \quad a, t \in \mathbb{T},$$

where

$$\xi_h(z) = \begin{cases} z, & h = 0; \\ \frac{1}{h}\log(1 + hz), & h > 0. \end{cases}$$

Let $|.|$ denote the Euclidean norm on $\mathbb{R}^n$. We will consider the linear space of continuous functions $C(I_\mathbb{T}, \mathbb{R}^n)$ such that

$$\sup_{t \in I_\mathbb{T}} \frac{|x(t)|}{e_\beta(t, a)} < \infty,$$

and denote it by $C_\beta(I_\mathbb{T}, \mathbb{R}^n)$. The space $C_\beta(I_\mathbb{T}, \mathbb{R}^n)$, endowed with the Bielecki type norm

$$\|x\|_\beta = \sup_{t \in I_\mathbb{T}} \frac{|x(t)|}{e_\beta(t, a)},$$

is a Banach space (see [8, 23]).

3. Main results

Firstly, we generalize equation (1.1) and prove that it has the Hyers-Ulam-Rassias stability on time scales.

Consider the following non-linear Volterra integral equation

$$x(t) = f(t) + \int_a^t K\left(t, s, x(s), x(\sigma(s))\right) \Delta s, \qquad s, t \in I_{\mathbb{T}} = [a, \infty) \cap \mathbb{T}, \tag{3.1}$$

where $f \in C_{rd}(I_{\mathbb{T}}, \mathbb{R}^n)$, $K \in C_{rd}(I_{\mathbb{T}} \times I_{\mathbb{T}} \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$ and $x : I_{\mathbb{T}} \to \mathbb{R}^n$ is the unknown function.

Theorem 3.1. *Let k_1, k_2, L_1 and L_2 are positive constants and assume that $K : I_{\mathbb{T}} \times I_{\mathbb{T}} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function which additionally satisfies*

$$|K(t, s, p, q) - K(t, s, \overline{p}, \overline{q})| \leq L_1 |p - \overline{p}| + L_2 |q - \overline{q}|$$

for $t, s \in I_{\mathbb{T}}$ and $p, q, \overline{p}, \overline{q} \in C_{rd}(I_{\mathbb{T}}, \mathbb{R}^n)$. If a function $g \in C_{rd}(I_{\mathbb{T}}, \mathbb{R}^n)$ satisfies

$$\left| g(t) - f(t) - \int_a^t K\left(t, s, g(s), g(\sigma(s))\right) \Delta s \right| \leq \theta(t), \qquad t \in I_{\mathbb{T}}, \tag{3.2}$$

where $\theta \in C_{rd}(I_{\mathbb{T}}, \mathbb{R})$ with

$$\int_a^t e_{L_1}\left(t, \sigma(s)\right) \theta(s) \Delta s \leq k_1 \theta(t) \ \text{ and } \int_a^t e_{L_2}\left(t, \sigma\left(\sigma(s)\right)\right) \theta(s) \Delta s \leq k_2 \theta(t),$$

then there exists a unique solution $u \in C_{rd}(I_{\mathbb{T}}, \mathbb{R}^n)$ of equation (3.1) such that

$$|g(t) - u(t)| \leq (1 + k_1 L_1 + k_2 L_2)\theta(t), \qquad t \in I_{\mathbb{T}}.$$

Proof. Set

$$l(t) = g(t) - f(t) - \int_a^t K\left(t, s, g(s), g(\sigma(s))\right) \Delta s, \qquad t \in I_{\mathbb{T}}.$$

Then, by (3.2), we have

$$|l(t)| \leq \theta(t).$$

Let $u \in C_{rd}(I_{\mathbb{T}}, \mathbb{R}^n)$ be the unique solution of equation (3.1). Hence, we obtain

$$\begin{aligned}
&|g(t)-u(t)| \\
= \quad & \left| l(t)+f(t)+\int_a^t K(t,s,g(s),g(\sigma(s)))\,\Delta s \ -f(t)-\int_a^t K(t,s,u(s),u(\sigma(s)))\,\Delta s \right| \\
\leq \quad & |l(t)|+\left| \int_a^t [K(t,s,g(s),g(\sigma(s)))-K(t,s,u(s),u(\sigma(s)))]\Delta s \right| \\
\leq \quad & \theta(t)+\int_a^t |K(t,s,g(s),g(\sigma(s)))-K(t,s,u(s),u(\sigma(s)))|\,\Delta s \\
\leq \quad & \theta(t)+L_1\int_a^t |g(s)-u(s)|\,\Delta s+L_2\int_a^t |g(\sigma(s))-u(\sigma(s))|\,\Delta s.
\end{aligned}$$

By using Gronwall's Inequality in [6, Section 6.1], we have that

$$\begin{aligned}
|g(t)-u(t)| \quad &\leq \quad \theta(t)+L_1\int_a^t e_{L_1}(t,\sigma(s))\,\theta(s)\Delta s+L_2\int_a^t e_{L_2}(t,\sigma(\sigma(s)))\,\theta(s)\Delta s \\
&\leq \quad \theta(t)+L_1k_1\theta(t)+L_2k_2\theta(t) \\
&\leq \quad (1+k_1L_1+k_2L_2)\theta(t).
\end{aligned}$$

This shows that equation (3.1) has the Hyers-Ulam-Rassias stability.

Corollary 3.2. *Under the assumptions of Theorem 3.1, if we take θ as a constant function then we say that equation (3.1) has the Hyers-Ulam stability.*

Secondly, by generalizing equation (1.2), the Hyers-Ulam and Hyers-Ulam-Rassias stabilities on time scales are proved. The Volterra integral equation examined at this stage is as follows:

$$x(t)=f(t)+\lambda\int_a^t K(t,s)\,x(s)\Delta s, \qquad t\in I_{\mathbb{T}}=[a,b]\cap\mathbb{T}, \tag{3.3}$$

where $f\in C_{rd}(I_{\mathbb{T}},\mathbb{R})$, $\lambda\in\mathbb{R}$, $K\in C_{rd}(I_{\mathbb{T}}\times I_{\mathbb{T}},\mathbb{R})$ and $x: I_{\mathbb{T}}\to\mathbb{R}$ is the unknown function.

Theorem 3.3. *The integral equation (3.3) on $I_{\mathbb{T}}$ has the Hyers-Ulam-Rassias stability, that is, for a fixed function $\omega\in C_{rd}(I_{\mathbb{T}},\mathbb{R})$ we have that for every $\psi\in C_{rd}(I_{\mathbb{T}},\mathbb{R})$ with*

$$\left|\psi(t)-f(t)-\lambda\int_a^t K(t,s)\,\psi(s)\Delta s\right|\leq\omega(t),$$

for which there exist constants P and M with $|\lambda MP|<1$ such that $\int_a^t \omega(s)\Delta s\leq P\omega(t)$ and $|K(t,s)|\leq M$, $\forall t,s\in I_{\mathbb{T}}$, then there exists a unique $\varphi\in C_{rd}(I_{\mathbb{T}},\mathbb{R})$ such that $[\Psi\varphi](t)=\varphi(t)$ and $|\psi(t)-\varphi(t)|\leq C\omega(t)$ for some $C>0$.

Proof. Consider the following iterative scheme

$$\psi_n(t) := f(t) + \lambda \int_a^t K(t,s)\,\psi_{n-1}(s)\Delta s, \qquad n = 1, 2, 3, ... \tag{3.4}$$

for $t \in I_{\mathbb{T}}$ with $\psi_0(t) = \psi(t)$. We prove that $\{\psi_n(t)\}_{n\in\mathbb{N}}$ converges uniformly to the unique solution of equation (3.3) on $I_{\mathbb{T}}$. We write $\psi_n(t)$ as a telescoping sum

$$\psi_n(t) = \psi_0(t) + \sum_{i=1}^{n} \left[\psi_i(t) - \psi_{i-1}(t)\right],$$

so

$$\lim_{n\to\infty} \psi_n(t) = \psi_0(t) + \sum_{i=1}^{\infty} \left[\psi_i(t) - \psi_{i-1}(t)\right], \qquad \forall t \in I_{\mathbb{T}}. \tag{3.5}$$

By mathematical induction, it is easy to see that the following estimate

$$|\psi_n(t) - \psi_{n-1}(t)| \leq (\lambda MP)^{n-1}\omega(t) \tag{3.6}$$

holds for each $n \in \mathbb{N}$ and all $t \in I_{\mathbb{T}}$. For $n = 1$, we have

$$|\psi_1(t) - \psi(t)| \leq \omega(t).$$

Hence, estimate (3.6) holds for $n = 1$. Assume that estimate (3.6) is true for $n = k \geq 1$. We have

$$\begin{aligned} |\psi_{k+1}(t) - \psi_k(t)| &\leq \lambda \int_a^t |K(t,s)|\,|\psi_k(s) - \psi_{k-1}(s)|\,\Delta s \\ &\leq \lambda M \int_a^t (\lambda MP)^{k-1}\omega(s)\Delta s \\ &\leq (\lambda MP)^k \omega(t), \end{aligned}$$

hence estimate (3.6) is valid for $n = k + 1$. This shows that estimate (3.6) is true for all $n \geq 1$ on $I_{\mathbb{T}}$. We see that

$$|\psi_i(t) - \psi_{i-1}(t)| \leq (\lambda MP)^{i-1}\omega(t),$$

and

$$\sum_{i=1}^{\infty} (\lambda MP)^{i-1}\omega(t) = \frac{\omega(t)}{1 - \lambda MP}.$$

Applying Weierstrass M-Test, we conclude that the infinite series

$$\sum_{i=1}^{\infty} \left[\psi_i(t) - \psi_{i-1}(t)\right]$$

converges uniformly on $t \in I_{\mathbb{T}}$. Thus from (3.5), the sequence $\{\psi_n(t)\}_{n\in\mathbb{N}}$ converges uniformly on $I_{\mathbb{T}}$ to some $\varphi(t) \in C_{rd}(I_{\mathbb{T}}, \mathbb{R})$. Next, we show that the limit of the sequence $\varphi(t)$ is the exact solution of (3.3). For all $t \in I_{\mathbb{T}}$ and each $n \geq 1$, we have

$$\left| \int_a^t K(t,s)\,\psi_n(s)\Delta s - \int_a^t K(t,s)\,\varphi(s)\Delta s \right| \leq M \int_a^t |\psi_n(s) - \varphi(s)|\,\Delta s.$$

Taking the limits as $n \to \infty$ we see that the right hand side of the above inequality tends to zero and so

$$\lim_{n\to\infty} \int_a^t K(t,s)\,\psi_n(s)\Delta s = \int_a^t K(t,s)\,\varphi(s)\Delta s, \qquad \forall\ t \in I_{\mathbb{T}}.$$

By letting $n \to \infty$ on both sides of (3.4) we conclude that $\varphi(t)$ is the unique solution of (3.3). Then there exists a positive integer N such that $|\psi_N(t) - \varphi(t)| \leq \omega(t)$. Hence

$$\begin{aligned}
|\psi - \varphi| &\leq |\psi(t) - \psi_N(t)| + |\psi_N(t) - \varphi(t)| \\
&\leq |\psi(t) - \psi_1(t)| + |\psi_1(t) - \psi_2(t)| + ... + |\psi_{N-1}(t) - \psi_N(t)| + |\psi_N(t) - \varphi(t)| \\
&\leq \sum_{i=1}^{N} |\psi_{i-1}(t) - \psi_i(t)| + |\psi_N(t) - \varphi(t)| \\
&\leq \sum_{i=1}^{N} (\lambda MP)^{i-1}\omega(t) + |\psi_N(t) - \varphi(t)| \\
&\leq \sum_{i=1}^{\infty} (\lambda MP)^{i-1}\omega(t) + \omega(t) \\
&\leq \frac{1}{1-\lambda MP}\omega(t) + \omega(t) = \left(1 + \frac{1}{1-\lambda MP}\right)\omega(t) \ = C.\omega(t),
\end{aligned}$$

which shows that (3.3) has Hyers-Ulam-Rassias stability on $I_{\mathbb{T}}$.

Corollary 3.4. *Under the assumptions of Theorem 3.1, if we take ω as a constant function then we obtain the Hyers-Ulam stability result of equation (3.3).*

Thirdly, we generalize equation (1.3) to the time scale and show that it has Hyers-Ulam and Hyers-Ulam-Rassias stabilities. Consider the following Volterra integral equation

$$x(t) = f(t) + \varphi\left(\int_a^t K(t,s,x(s))\,\Delta s\right), \qquad t \in I_{\mathbb{T}} = [a,b] \cap \mathbb{T}, \tag{3.7}$$

where $f \in C_{rd}(I_{\mathbb{T}}, \mathbb{R}^n)$, $K \in C_{rd}(I_{\mathbb{T}} \times I_{\mathbb{T}} \times \mathbb{R}^n, \mathbb{R}^n)$, $x : I_{\mathbb{T}} \to \mathbb{R}^n$ is the unknown function and φ is a bounded linear transformation on $I_{\mathbb{T}}$.

Theorem 3.5. *Let $f \in C_{rd}(I_{\mathbb{T}}, \mathbb{R}^n)$, $K : I_{\mathbb{T}} \times I_{\mathbb{T}} \times \mathbb{R}^n \to \mathbb{R}^n$ be jointly continuous in its first and third variables and rd-continuous in its second variable, $L : I_{\mathbb{T}} \to \mathbb{R}$ be rd-continuous, $\gamma > 1$ and $\beta(s) = L(s)\gamma$, where $\gamma > \|\varphi\|$. If*

$$|K(t,s,p) - K(t,s,q)| \leq L(s)\,|p-q|\,, \qquad p,q \in \mathbb{R}^n, \qquad s < t,$$

$$m = \sup_{t\in I_{\mathbb{T}}} \frac{1}{e_\beta(t,a)} \left| f(t) + \varphi\left(\int_a^t K(t,s,0)\,\Delta s\right)\right| < \infty,$$

then the integral equation (3.7) has a unique solution $x \in C_{rd}(I_{\mathbb{T}}, \mathbb{R}^n)$.

Proof. Consider the Banach space $C_\beta(I_\mathbb{T}, \mathbb{R}^n)$. To prove the result, we define an operator $F : C_\beta(I_\mathbb{T}, \mathbb{R}^n) \to C_\beta(I_\mathbb{T}, \mathbb{R}^n)$ by

$$[Fx](t) = f(t) + \varphi\left(\int_a^t K(t, s, x(s))\,\Delta s\right). \tag{3.8}$$

The fixed point of F will be solution to (3.7). Thus we want to prove that there exists a unique x such that $Fx = x$. For any $u, v \in C_\beta(I_\mathbb{T}, \mathbb{R}^n)$, we obtain

$$\begin{aligned}
\|Fu - Fv\|_\beta &= \sup_{t\in I_\mathbb{T}} \frac{|[Fu](t) - [Fv](t)|}{e_\beta(t,a)} \\
&\leq \sup_{t\in I_\mathbb{T}} \frac{1}{e_\beta(t,a)} \left|\varphi\left(\int_a^t [K(t,s,u(s)) - K(t,s,v(s))]\Delta s\right)\right| \\
&\leq \sup_{t\in I_\mathbb{T}} \frac{1}{e_\beta(t,a)} \|\varphi\| \int_a^t L(s)\,|u(s) - v(s)|\,\Delta s \\
&= \|\varphi\| \sup_{t\in I_\mathbb{T}} \frac{1}{e_\beta(t,a)} \int_a^t L(s) e_\beta(s,a) \frac{|u(s) - v(s)|}{e_\beta(s,a)} \Delta s \\
&\leq \|\varphi\|\,\|u - v\|_\beta \sup_{t\in I_\mathbb{T}} \frac{1}{e_\beta(t,a)} \int_a^t L(s) e_\beta(s,a) \Delta s \\
&\leq \frac{\|\varphi\|}{\gamma} \|u - v\|_\beta \sup_{t\in I_\mathbb{T}} \frac{1}{e_\beta(t,a)} \int_a^t \beta(s) e_\beta(s,a) \Delta s \\
&= \frac{\|\varphi\|}{\gamma} \|u - v\|_\beta \sup_{t\in I_\mathbb{T}} \frac{1}{e_\beta(t,a)} \int_a^t e_\beta^\Delta(s,a) \Delta s \\
&= \frac{\|\varphi\|}{\gamma} \|u - v\|_\beta \sup_{t\in I_\mathbb{T}} \left[1 - \frac{1}{e_\beta(t,a)}\right] \\
&\leq \frac{\|\varphi\|}{\gamma} \|u - v\|_\beta\,.
\end{aligned}$$

Next, we show that $F : C_\beta(I_\mathbb{T}, \mathbb{R}^n) \to C_\beta(I_\mathbb{T}, \mathbb{R}^n)$. Let $x \in C_\beta(I_\mathbb{T}, \mathbb{R}^n)$. Taking norms, we get

$$\begin{aligned}
\|Fx\|_\beta &= \|Fx - F0 + F0\|_\beta \leq \|Fx - F0\|_\beta + \|F0\|_\beta \\
&\leq \frac{\|x\|_\beta}{\gamma} \|\varphi\| + m < \infty.
\end{aligned}$$

As $\frac{\|\varphi\|}{\gamma} < 1$, we see that F is a contraction self mapping on $C_\beta(I_\mathbb{T}, \mathbb{R}^n)$ and so Banach's fixed point theorem applies, yielding the existence of a unique fixed point x of F.

Theorem 3.6. *Under the assumptions of Theorem* 3.5 *the equation $Fx = x$, where $F \in C_{rd}(I_\mathbb{T}, \mathbb{R}^n)$ is defined as in (3.8) has the Hyers-Ulam-Rassias stability, that is, for a fixed function $\psi \in C_{rd}(I_\mathbb{T}, \mathbb{R})$ we have that for every $x \in C_{rd}(I_\mathbb{T}, \mathbb{R}^n)$ with*

$\|x - Fx\|_\beta \le \psi(t)$ there exists a unique $x_0 \in C_{rd}(I_{\mathbb{T}}, \mathbb{R}^n)$ such that $Fx_0 = x_0$ and $\|x - x_0\|_\beta \le C\psi(t)$.

Proof. From Theorem 3.5, there exists a unique $x_0 \in C_{rd}(I_{\mathbb{T}}, \mathbb{R}^n)$ such that $Fx_0 = x_0$. Then, we have

$$\begin{aligned} \|x - x_0\|_\beta &\le \|x - Fx\|_\beta + \|Fx - x_0\|_\beta \\ &\le \psi(t) + \|Fx - Fx_0\|_\beta \\ &\le \psi(t) + \frac{\|x - x_0\|_\beta}{\gamma} \|\varphi\|. \end{aligned}$$

Hence, we obtain

$$\|x - x_0\|_\beta \le C\psi(t),$$

where $C = \left[1 - \frac{\|\varphi\|}{\gamma}\right]^{-1} > 0$.

Remark. *If we take $\varphi(x)$ as a constant function and consider the complete metric space $(C[a,b], \|.\|_\infty)$, then we get Theorem 4.1 in* [10].

Example 3.1. *Consider the scalar integral equation*

$$x(t) = t^4 + \varphi\left(\int_a^t (s + \sigma(s))\left[x(s)^2 + 1\right]^{\frac{1}{2}} \Delta s\right), \qquad a, t \in I_{\mathbb{T}}, \quad a \ge 0.$$

We claim that this integral equation has a unique solution for arbitrary $\mathbb{T}$ and the equation $Fx = x$ has the Hyers-Ulam Rassias stability.

Proof. We will use Theorem 3.5 and make use of the fact that $K(t,s,p) = (s + \sigma(s))\left[p^2 + 1\right]^{\frac{1}{2}}$ has a bounded partial derivative with respect to p everywhere. Consider

$$\begin{aligned} |K(t,s,p) - K(t,s,q)| &= \left|(s + \sigma(s))\left[p^2 + 1\right]^{\frac{1}{2}} - (s + \sigma(s))\left[q^2 + 1\right]^{\frac{1}{2}}\right| \\ &\le (s + \sigma(s)) \sup_{r \in \mathbb{R}} \left|\frac{r}{\left[r^2 + 1\right]^{\frac{1}{2}}}\right| |p - q| \\ &\le (s + \sigma(s))\,|p - q|. \end{aligned}$$

We here used the mean value theorem. So, we have that $L(s) = s + \sigma(s)$. For choices of $\gamma = 2$ and $\|\varphi\| = \frac{2}{3}$, we have $\beta(s) = 2\,(s + \sigma(s))$. Using Bernoulli's Inequality in [18, p.42] we get

$$e_\beta(t,a) \ge 1 + t^2 - a^2,$$

which is followed by estimate $m < \infty$. The result now follows from Theorem 3.5.

If a function $x \in C_\beta(I_{\mathbb{T}}, \mathbb{R}^n)$ satisfies the inequality $\|x - Fx\|_\beta \le \psi(t)$ where $\psi(t)$ non-negative function then Theorem 3.6 implies that there exists a unique x_0 such that $Fx_0 = x_0$ and

$$\|x - x_0\|_\beta \le \frac{3}{2}\psi(t).$$

From Theorem 3.6, we obtain the following result related to the Hyers-Ulam stability.

Corollary 3.7. *If ψ is a constant function, then equation (3.7) has the Hyers-Ulam stability.*

Finally, we showed Hyers-Ulam-Rassias stability of equation (1.4) on time scales.

Theorem 3.8. *If $x_0 \in C_\beta(I_\mathbb{T}, \mathbb{R}^n)$ is a solution of the Volterra type integral equation (1.4) and*

$$M\left(1 + \frac{1+q\gamma}{1-r\gamma} + \frac{1}{\gamma}\right) < 1,$$

then the Volterra type integral equation (1.4) is Hyers-Ulam-Rassias stable, that is, for a fixed function $\psi \in C_\beta(I_\mathbb{T}, \mathbb{R})$ we have that for every $x \in C_\beta(I_\mathbb{T}, \mathbb{R}^n)$ with

$$\sup_{t \in I_\mathbb{T}} \frac{\left| x(t) - f\left(t, x(t), x(\sigma(t)), \int_a^t K(t,s,x(s),x(\sigma(s))\Delta s\right)\right|}{e_\beta(t,a)} = \|x - Fx\|_\beta \le \psi(t),$$

there exists a unique $x_0 \in C_\beta(I_\mathbb{T}, \mathbb{R}^n)$ such that $Fx_0 = x_0$ and $\|x - x_0\|_\beta \le C\psi(t)$.

Proof. According to Theorem 3.1 in [18], there is a unique solution x_0 to the Volterra type integral equation (1.4) in the Banach space $C_\beta(I_\mathbb{T}, \mathbb{R}^n)$. Therefore we have

$$\begin{aligned} \|x - x_0\|_\beta &\le \|x - Fx\|_\beta + \|Fx - Fx_0\|_\beta \\ &\le \psi(t) + M\left(1 + \frac{1+q\gamma}{1-r\gamma} + \frac{1}{\gamma}\right) \|x - x_0\|_\beta . \end{aligned}$$

So we get

$$\|x - x_0\|_\beta \le C\psi(t),$$

where

$$C = \left(1 - M\left(1 + \frac{1+q\gamma}{1-r\gamma} + \frac{1}{\gamma}\right)\right)^{-1}.$$

Remark. *Choosing $\psi(t) = \epsilon$ in Theorem 3.8, yields Theorem 3.3 in* [18].

4. Conclusion

In this paper, the Hyers-Ulam stability and Hyers-Ulam-Rassias stability theorems for four types of Volterra integral equations on time scales were investigated. Additionally, a numerical example to support the study was given. The theorems proved here generalize some recent results given in [3, 10, 18, 26].

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