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Broadening the convergence domain of Seventh-order method satisfying Lipschitz and Hölder conditions

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Abstract

The local convergence analysis of a seventh order algorithm for solving nonlinear equations is presented in the current discussion by assuming that the first-order Fréchet derivative belongs to the Lipschitz class. This approach yields radii of convergence ball, error bound and uniqueness of the solution. Further, generalization of the study extended by considering Hölder continuity condition. At last, we estimated the radii of the convergence balls using a variety of numerical examples, including a nonlinear Hammerstein equation.

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1. Introduction

One of the most crucial problems of numerical analysis concerns with finding efficiently and precisely the approximate locally unique solution x^* of the nonlinear equation having the form of

$$T(x) = 0, \tag{1}$$

where $T : D \subseteq X \to Y$ such as T is defined on a convex open subset D of a Banach space X with values in a Banach space Y. Many problems in various fields of computational science, engineering and other disciplines involves optimization, differential equations, integral equations, radiative transfer theory, can be brought in

a form like (1) using mathematical modeling. Analytical methods of solving such type of problems are very scarce or almost non existent. Therefore, many researchers only rely on iterative methods and they have proposed a plethora of iterative methods.

The practice of numerical functional analysis for finding such solutions are widely and substantially connected to Newton-like methods which is defined as follows:

$$x_{n+1} = x_n - [T'(x_n)]^{-1}T(x_n), \quad n \geq 0, \tag{2}$$

is frequently used by various researchers as it has quadratic convergence (can be seen in the ref. [1]). The other properties of Newton's method are established in the article [2]. Moreover, in some applications involving stiff systems, high-order methods are useful. Therefore, it is important to study high-order methods.

The local convergence analysis of iterative procedures is based on the information around a solution, to find estimates of the radii of the convergence balls. There exists many studies which deals with the local and the semilocal convergence analysis of Newton-like methods. In the last decades, many iterative methods of increasing order of convergence have been developed and have shown their efficiency, in numerical terms, like third-order [[3], [4]], fourth-order [5] and fifth-order [[6], [7]] etc.

Another issue of equal importance is obtaining the radius of convergence ball as well as developing a theory to extend the convergence domain. Numerous authors have studied the local convergence analysis using Taylor's series but have not obtained the radii of convergence ball for the solution, as shown in the references [[8],[9]]. In terms of computing cost, it has a number of drawbacks. Many authors, for example discussed these techniques as evidenced by the reference [[1], [10]].Using the considered approach, it is possible to compute the convergence radii and the error estimate.

However, lower-order classical methods have been usually re-examined for only trying to increase its order of convergence. In particular, Sharma and Gupta [11] constructed three steps method of order five, defined as follows:

$$
\begin{aligned}
y_n &= x_n - \frac{1}{2}\Gamma_n T(x_n), \\
z_n &= x_n - [T'(y_n)]^{-1}T(x_n), \\
x_{n+1} &= z_n - [2[T'(y_n)]^{-1} - \Gamma_n]T(z_n),
\end{aligned}
\tag{3}
$$

where, $\Gamma_n = [T'(x_n)]^{-1}$. The local convergence of above multi-step Homeier's-like method has been studied by Panday and Jaiswal [12] with the help of Lipschitz and Hölder continuity conditions. In the extension of three-step Homeier's method, Xiao and Yin [13] developed a fourth-step seventh order convergent method, which is given as:

$$
\begin{aligned}
y_n &= x_n - \frac{1}{2}\Gamma_n T(x_n), \\
z_n^{(1)} &= x_n - [T'(y_n)]^{-1}T(x_n), \\
z_n^{(2)} &= z_n^{(1)} - [2[T'(y_n)]^{-1} - \Gamma_n]T(z_n^{(1)}), \\
x_{n+1} &= z_n^{(2)} - [2[T'(y_n)]^{-1} - \Gamma_n]T(z_n^{(2)}).
\end{aligned}
\tag{4}
$$

This method requires the evaluation of three functions, two first order derivatives and two matrix inversions per iteration. In this article, we will provide the answers with the range of initial guess x^*, which shows us how near the initial guess should be necessary for the method (4) to be guaranteed to be convergent. The semillocal convergence of the aforesaid algorithm has been also discussed under various continuity conditions [[14]-[16]]

In finding out the existence of the solution, we require the computation of the first-order Fréchet derivative. However, the analysis of convergence is established by applying the Taylor series approach based on higher-order derivatives. These techniques, which need the higher-order derivatives, limit the applicability of the algorithm. As an motivational illustration[18], define a function f on $D = [-\frac{1}{2}, \frac{5}{2}]$ by

$$
f(x) = \begin{cases} x^3 log(x^2) + x^5 - x^4, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}
\tag{5}
$$

It is clearly notable that f''' is unbounded on D. Hence, the theory based on higher-order derivatives [17] fail to solve the above problem. Also, one get no idea about the domain of convergence [18]. The local convergence study gives valuable information regarding the radius of convergence ball.

In this presented paper, motivated by the foregoing observations and ongoing work in this area, we discuss the local convergence for the method (4), by following the approach based on f' to stay away from the evaluation of higher-order Fréchet derivatives. Thereby, we enlarge the utility of the method (4) by using hypotheses only on the first-order derivative of the function T and Lipschitz and Hölder continuity conditions for which earlier studies can not be used due to the computation of higher-order Fréchet derivatives.

The arrangement of the whole text is as follows: the Local convergence analysis of the method (4) under Lipschitz condition" section deals with the local convergence results of the method (4). Similarly, the "Local convergence analysis of the method (4) under Hölder condition" section deals with the local convergence results of the method (4) under weaker assumptions followed by the "Applications with Numerical Examples" section. The "Conclusions" section is placed in the last section.

2. Local convergence analysis of the method (4) under Lipschitz condition

In this section, we describe the local convergence analysis of the method (4) which is centered on some parameters and scalar functions. Considering $\psi_0 > 0$ and $\psi > 0$ be two parameters with $\psi_0 \leq \psi$, we define the functions $\eta_1, \eta_2, \eta_3, \eta_4, p, H_1, H_2, H_3$ and H_4 on interval $[0, \frac{1}{\psi_0})$ by

$$
\begin{aligned}
\eta_4(a) = & \left[\frac{\psi\eta_3(a)a}{2(1 - \psi_0\eta_3(a)a)} \right. \\
& + \frac{1}{1 - \psi_0\eta_3(a)a}\psi[\eta_1(a) + \eta_3(a)] \times \left(\frac{1}{1 - p(a)}\left[1 + \frac{\psi_0}{2}\eta_3(a)a\right]a \right) \\
& + \left. \left(\frac{1}{1 - \psi_0 a}.\frac{\psi[a + \eta_1(a)a]}{1 - p(a)} \times \left[1 + \frac{\psi_0}{2}\eta_3(a)a\right] \right) \right]\eta_3(a),
\end{aligned}
\tag{6}
$$

where

$$
\eta_1(a) = \frac{1}{1 - \psi_0 a}\left[\frac{\psi a}{2} + \frac{1 + (\psi_0/2)a}{2} \right],
\tag{7}
$$

$$
\eta_2(a) = \frac{1}{1 - \psi_0 a}\left[\frac{\psi a}{2} + \frac{\psi[1 + \eta_1(a)][\psi_0/2a + 1]a}{1 - p(a)} \right],
\tag{8}
$$

$$
\begin{aligned}
\eta_3(a) = & \left[\frac{\psi\eta_2(a)a}{2(1 - \psi_0\eta_2(a)a)} \right. \\
& + \frac{1}{1 - \psi_0\eta_2(a)a}\psi[\eta_1(a) + \eta_2(a)] \times \left(\frac{1}{1 - p(a)}\left[1 + \frac{\psi_0}{2}\eta_2(a)a\right]a \right) \\
& + \left. \left(\frac{1}{1 - \psi_0 a}.\frac{\psi[a + \eta_1(a)a]}{1 - p(a)} \times \left[1 + \frac{\psi_0}{2}\eta_2(a)a\right] \right) \right]\eta_2(a),
\end{aligned}
\tag{9}
$$

and

$$
p(a) = \psi_0\eta_1(a)a.
\tag{10}
$$

Let

$$
H_1(a) = \eta_1(a) - 1, \quad H_2(a) = \eta_2(a) - 1,
\tag{11}
$$
$$
H_3(a) = \eta_3(a) - 1, \quad H_4(a) = \eta_4(a) - 1,
\tag{12}
$$

and

$$
\rho_1 = \frac{2}{2\psi + 5\psi_0} < \frac{1}{\psi_0}.
\tag{13}
$$

We observe that $\eta_1(\rho_1) = 1$ and can attain

$$0 \le \eta_1(a) < 1 \; for \; a \in [0, \rho_1), \quad 0 < \rho_1 < 1/\psi_0. \tag{14}$$

Now, we have that $H_1(0) = H_2(0) = H_3(0) = H_4(0) < 0$ and $H_1(a) \to +\infty$ or a positive constant as $a \to (\frac{1}{\psi_0})^-$ (here $a \to (\frac{1}{\psi_0})^-$ means that $a \to (\frac{1}{\psi_0})$ with all values of a satisfying $a < \frac{1}{\psi_0}$).The intermediate value theorem confirms the existence of the zeros of the function $H_2(a)$ in the interval $(0, \frac{1}{\psi_0})$. We denote the smallest zero of $H_2(a)$ as ρ_2. Also, $h_2(\rho_1) > 0$ and $\rho_1 < \frac{1}{\psi_0}$, which follows from that

$$0 < \rho_2 < \rho_1, \quad 0 < \eta_2(a) < 1, for \; a \in [0, \rho_2). \tag{15}$$

Following this procedure, there comes the existence of zeros of functions $H_i, \; i = 1, 2, 3, 4$ in the interval $(0, \rho_0)$. Denote by ρ_i, respectively, the smallest solution of functions $H_i's$. Define the radius of convergence ρ and for each $a \in [0, \rho)$ by

$$\rho = min\{\rho_i\}, \; i = 1, 2, 3, 4; \quad 0 \le \eta_i(a) < 1. \tag{16}$$

Let $B(x^*, \rho)$, $\overline{B(x^*, \rho)}$ stand, respectively for the open and closed ball in X such that $x^* \in X$ and of radius $\rho > 0$. Next, we present the local convergence analysis of method (4) using the preceding notations and Lipschitz conditions.

Theorem 2.1. *Suppose that $T : D \subseteq X \to Y$ be a continuously first order Fréchet differentiable operator. Let $\psi_0 > 0$ and $\psi > 0$ be two given parameters. Assume that there exists $x^* \in D$ for all $x, y \in D$ and fulfill the below conditions:*

$$T(x^*) = 0, \; [T'(x^*)]^{-1} \in L(Y, X), \tag{17}$$

where, $L(X, Y)$ is the set of bounded linear operators from X to Y,

$$\|[T'(x^*)]^{-1}(T'(x) - T'(x^*))\| \quad \le \quad \psi_0\|x - x^*\|, \tag{18}$$

$$\|[T'(x^*)]^{-1}(T'(x) - T'(y))\| \quad \le \quad \psi\|x - y\|, \tag{19}$$
$$B(x^*, \rho) \quad \subseteq \quad D, \tag{20}$$

where ρ is defined by equation (16). Then, the sequence $\{x_n\}$ generated by the method (4) for $x_0 \in B(x^*, \rho)\backslash\{x^*\}$ is well defined in $B(x^*, \rho)$ remains in $B(x^*, \rho)$ for each $n = 0, 1, 2, \cdots$ and converges to x^*. Consequently, the following relations holds for $n = 0, 1, 2, \cdots$:

$$\|y_n - x^*\| \le \eta_1(\|x_n - x^*\|)\|x_n - x^*\| \le \|x_n - x^*\| < \rho, \tag{21}$$

$$\|z_n^{(1)} - x^*\| \le \eta_2(\|x_n - x^*\|)\|x_n - x^*\| \le \|x_n - x^*\| < \rho, \tag{22}$$

$$\|z_n^{(2)} - x^*\| \le \eta_3(\|x_n - x^*\|)\|x_n - x^*\| \le \|x_n - x^*\| < \rho, \tag{23}$$

and

$$\|x_{n+1} - x^*\| \le \eta_4(\|x_n - x^*\|)\|x_n - x^*\| \le \|x_n - x^*\| < \rho, \tag{24}$$

where the functions $\eta_i, i = 1, 2, 3, 4$ are defined by the expressions (6) -(9). Moreover, if there exists $\varrho \in [\rho, \frac{1}{\psi_0})$ such that $\overline{B(x^*, \varrho)} \subseteq D$. Then the limit point x^* is the only solution of equation $T(x) = 0$ in $D_1 = D \cap \overline{B(x^*, \varrho)}$.

Proof. We shall show by mathematical induction that sequence $\{x_n\}$ is well defined and converges to x^*. Using the hypotheses, $x_0 \in B(x^*, \rho) \backslash \{x^*\}$, equation (13) and inequality (18), we have

$$\|[T'(x^*)]^{-1}(T'(x_0) - T'(x^*))\| \le \psi_0\|x_0 - x^*\| \le \psi_0\rho < 1. \tag{25}$$

It follows from the above and the Banach lemma on invertible operator [20] that $[T'(x_0)]^{-1} \in L(Y, X)$ or $T'(x_0)$ is invertible and

$$\|[T'(x_0)]^{-1}T'(x^*)\| \le \frac{1}{1 - \psi_0\|x_0 - x^*\|}. \tag{26}$$

Now, y_0 is well defined by the first sub-step of the scheme (4) and for $n = 0$,

$$\begin{aligned} y_0 - x^* &= x_0 - x^* - \frac{1}{2}[[T'(x_0)]^{-1}T(x_0)] \\ &= \frac{1}{2}[T'(x_0)]^{-1}T(x_0)] + [T'(x_0)]^{-1}[T'(x_0)(x_0 - x^*) - T(x_0) + T(x^*)]. \end{aligned} \tag{27}$$

Expanding $T(x_0)$ along x^* and taking the norm, we get

$$\begin{aligned} \|y_0 - x^*\| &\le \left\|\frac{1}{2}[T'(x_0)]^{-1}T'(x^*)\right\|\left[\left[\int_0^1 \|[T'(x^*)]^{-1}[T'(x^* + t(x_0 - x^*)]dt\|\|x_0 - x^*\|\right] \right. \\ &\quad \left. + \|[T'(x_0)]^{-1}T'(x^*)\|\int_0^1 \|[T'(x^*)]^{-1}[T'(x_0) - T'(x^* + t(x_0 - x^*))]dt\|\|x_0 - x^*\|. \end{aligned} \tag{28}$$

Thus, we get

$$\begin{aligned} \|y_0 - x^*\| &\le \frac{1}{1 - \psi_0\|x_0 - x^*\|}\left[\frac{1}{2}\left[\frac{\psi_0}{2}\|x_0 - x^*\| + 1\right]\right. \\ &\quad \left. + \frac{\psi}{2}\|x_0 - x^*\|\right]\|x_0 - x^*\| \le \eta_1(\|x_0 - x^*\|)\|x_0 - x^*\| < \rho. \end{aligned} \tag{29}$$

From the inequalities (18) and (29), we have

$$\begin{aligned} \|[T'(x^*)]^{-1}[T'(y_0) - T'(x^*)]\| &\le \psi_0\|y_0 - x^*\| \\ &\le \psi_0\eta_1(\|x_0 - x^*\|)\|x_0 - x^*\| \\ &= p(\|x_0 - x^*\|) < 1. \end{aligned} \tag{30}$$

Thus, by Banach lemma,

$$\|[T'(y_0)]^{-1}T'(x^*)\| \le \frac{1}{1 - p(\|x_0 - x^*\|)}. \tag{31}$$

From the second sub-step of the method (4), we have

$$\begin{aligned} z_0^{(1)} - x^* &= x_0 - x^* - [T'(y_0)^{-1}T(x_0)] \\ &= x_0 - x^* - [T'(x_0)]^{-1}T(x_0) + [T'(x_0)]^{-1}[T'(y_0) - T'(x_0)]T'(y_0)^{-1}T(x_0). \end{aligned} \tag{32}$$

On taking norm of the equation (32), we get

$$\begin{aligned} \|z_0^{(1)} - x^*\| &\le \|x_0 - x^* - [T'(x_0)]^{-1}T(x_0)\| + \|[T'(x_0)]^{-1}T'(x^*)\|. \\ &\quad \|[T'(x^*)]^{-1}[T'(y_0) - T'(x_0)]\|\|[T'(y_0)]^{-1}T'(x^*)\|\|[T'(x^*)]^{-1}T(x_0)\| \\ &\le \frac{1}{1 - \psi_0\|x_0 - x^*\|}\left[\frac{\psi}{2}\|x_0 - x^*\|\right. \\ &\quad \left. + \frac{[\psi\|y_0 - x^*\| + \psi\|x_0 - x^*\|][\frac{\psi_0}{2}\|x_0 - x^*\| + 1]}{1 - p(\|x_0 - x^*\|)}\right]\|x_0 - x^*\|. \end{aligned} \tag{33}$$

Thus, we get

$$\|z_0^{(1)} - x^*\| \le \eta_2(\|x_0 - x^*\|)\|x_0 - x^*\| \le \|x_0 - x^*\| < \rho. \tag{34}$$

From the next sub-step of the method (4), we have

$$
\begin{aligned}
z_0^{(2)} - x^* &= z_0^{(1)} - x^* - (2[T'(y_0)]^{-1} - [T'(x_0)]^{-1})T(z_0^{(1)}) \\
&= (z_0^{(1)} - x^* - [T'(z_0^{(1)})]^{-1}T'(z_0^{(1)})) + [T'(z_0^{(1)})]^{(-1)}T'(x^*)T'(x^*)^{-1}[T'(y_0) - T'(z_0^{(1)})] \\
&\quad .[T'(y_0)]^{(-1)}T'(x^*)[T'(x^*)]^{-1}T(z_0^{(1)}) + [T'(x_0)]^{(-1)}T'(x^*)[T'(x^*)]^{-1}[T'(y_0) - T'(x_0)] \\
&\quad .[T'(y_0)]^{(-1)}T'(x^*)[T'(x^*)]^{-1}T(z_0^{(1)}).
\end{aligned}
\tag{35}
$$

On expanding $T(z_0^{(1)})$ along x^* and taking norm of the equation (35), we get

$$
\begin{aligned}
\|z_0^{(2)} - x^*\| &\leq \frac{1}{1 - \psi_o\|z_0^{(1)} - x^*\|}\frac{\psi}{2}\|z_0^{(1)} - x^*\|.\|z_0^{(1)} - x^*\| + \frac{[\psi\|y_0 - x^*\| + \psi\|z_0^{(1)} - x^*\|]}{1 - \psi_0\|z_0^{(1)} - x^*\|} \\
&\quad \times \|[T'(x^*)]^{-1}T(z_0^{(1)})\| + \frac{1}{1 - \psi_0\|x_0 - x^*\|}[\psi\|y_0 - x^*\| + \psi\|x_0 - x^*\|] \\
&\quad \times \|[T'(x^*)]^{-1}T(z_0^{(1)})\| \\
&\leq \frac{1}{1 - \psi_0\|z_0^{(1)} - x^*\|}\frac{\psi}{2}\|z_0^{(1)} - x^*\|.\|z_0^{(1)} - x^*\| + \frac{[\psi\|y_0 - x^*\| + \psi\|z_0^{(1)} - x^*\|]}{1 - \psi_0\|z_0^{(1)} - x^*\|} \\
&\quad \times \frac{1}{1 - p(\|x_0 - x^*\|)}\left(1 + \frac{\psi_0}{2}\|z_0^{(1)} - x^*\|\right)\|z_0^{(1)} - x^*\| + \frac{[\psi\|y_0 - x^*\| + \psi\|x_0 - x^*\|]}{1 - \psi_0\|x_0 - x^*\|} \\
&\quad \times \frac{1}{1 - p(\|x_0 - x^*\|)}\left(1 + \frac{L_0}{2}\|z_0^{(1)} - x^*\|\right)\|z_0^{(1)} - x^*\|.
\end{aligned}
\tag{36}
$$

Thus, we have

$$
\|z_0^{(2)} - x^*\| \leq \eta_3(\|x_0 - x^*\|)\|x_0 - x^*\| < \rho.
\tag{37}
$$

Now, from the last sub-step of the method (4), we have

$$
\begin{aligned}
x_1 - x^* &= z_0^{(2)} - x^* - (2[T'(y_0)]^{-1} - [T'(x_0)]^{-1})T(z_0^{(2)}) \\
&= (z_0^{(2)} - x^* - [T'(z_0^{(2)})]^{-1}T'(z_0^{(2)})) + [T'(z_0^{(2)})]^{(-1)}T'(x^*)T'(x^*)^{-1}[T'(y_0) - T'(z_0^{(1)})] \\
&\quad .[T'(y_0)]^{(-1)}T'(x^*)[T'(x^*)]^{-1}T(z_0^{(2)}) + [T'(x_0)]^{(-1)}T'(x^*)[T'(x^*)]^{-1}[T'(y_0) - T'(x_0)] \\
&\quad .[T'(y_0)]^{(-1)}T'(x^*)[T'(x^*)]^{-1}T(z_0^{(2)}).
\end{aligned}
\tag{38}
$$

On expanding $T(z_0^{(2)})$ along x^* and taking norm of the equation (38), we get

$$
\begin{aligned}
\|x_1 - x^*\| &\leq \frac{1}{1 - \psi_0\|z_0^{(2)} - x^*\|}\frac{\psi}{2}\|z_0^{(2)} - x^*\|.\|z_0^{(2)} - x^*\| + \frac{[\psi\|y_0 - x^*\| + \psi\|z_0^{(2)} - x^*\|]}{1 - \psi_0\|z_0^{(2)} - x^*\|} \\
&\quad \times \|[T'(x^*)]^{-1}T(z_0^{(2)})\| + \frac{1}{1 - \psi_0\|x_0 - x^*\|}[\psi\|y_0 - x^*\| + \psi\|x_0 - x^*\|] \\
&\quad \times \|[T'(x^*)]^{-1}T(z_0^{(2)})\| \\
&\leq \frac{1}{1 - \psi_0\|z_0^{(2)} - x^*\|}\frac{\psi}{2}\|z_0^{(2)} - x^*\|.\|z_0^{(2)} - x^*\| + \frac{[\psi\|y_0 - x^*\| + \psi\|z_0^{(2)} - x^*\|]}{1 - \psi_0\|z_0^{(2)} - x^*\|} \\
&\quad \times \frac{1}{1 - p(\|x_0 - x^*\|)}\left(1 + \frac{\psi_0}{2}\|z_0^{(2)} - x^*\|\right)\|z_0^{(2)} - x^*\| + \frac{[\psi\|y_0 - x^*\| + \psi\|x_0 - x^*\|]}{1 - \psi_0\|x_0 - x^*\|} \\
&\quad \times \frac{1}{1 - p(\|x_0 - x^*\|)}\left(1 + \frac{\psi_0}{2}\|z_0^{(2)} - x^*\|\right)\|z_0^{(2)} - x^*\|.
\end{aligned}
\tag{39}
$$

Thus, we have

$$
\|x_1 - x^*\| \leq \eta_4(\|x_0 - x^*\|)\|x_0 - x^*\| < \rho,
\tag{40}
$$

which shows that for $n = 0$, $x_1 \in B(x^*, \rho)$. The function $H_4(a) = \eta_4(a) - 1$ gives $H_4(0) < 0$ and $H_4(\rho_3) > 0$. Hence, $H_4(t)$ has at least one root in $(0, \rho_3)$. Let ρ be the smallest root of $H_4(t)$ in $(0, \rho_3)$. Then, we have

$$
0 < \rho < \rho_3 < \rho_2 < \rho_1 < 1/\psi_0,
\tag{41}
$$

and

$$
0 < \eta_4(a) < 1, for\ a \in [0, \rho).
\tag{42}
$$

By simply replacing $x_0, y_0, z_0^{(1)}, z_0^{(2)}, x_1$ by $x_n, y_n, z_n^{(1)}, z_n^{(2)}, x_{n+1}$ in the preceding estimates, we arrive at inequalities $(21) - (24)$. By the estimate,

$$
\|x_{n+1} - x^*\| \leq \eta_4(\|x_0 - x^*\|)\|x_n - x^*\| < \rho.
\tag{43}
$$

We conclude that $\lim_{n\to\infty} x_n = x^*$ and $x_{n+1} \in B(x^*, \rho)$. Finally, to prove the uniqueness, let $y^* \in B(x^*, \rho)$ where $y^* \neq x^*$ with $T(y^*) = 0$. Define $F = \int_0^1 T'(x^* + t(y^* - x^*))dt$. On expanding $T(y^*)$ along x^* and using inequality (18), we obtain

$$\|[T'(x^*)]^{-1} \int_0^1 [T'(x^* + t(y*-x^*) - T'(x^*)]dt\|$$

$$\leq \frac{\psi_0}{2}\|y^* - x^*\| \leq \frac{\psi_0}{2}\varrho < 1. \tag{44}$$

So, by Banach lemma, $\int_0^1 [T'(x^*)]^{-1}[T'(x^* + t(y^* - x^*))]dt$ exists and invertible leading to the conclusion $x^* = y^*$, which completes the uniqueness part of the proof. $\qquad\square$

3. Local convergence analysis of the method (4) under Hölder condition

In this section, we move forward to present the local convergence analysis using Hölder condition because there are numerous nonlinear equations for which the assumptions based on Lipschitz condition fails to solve without using higher-order derivatives. As an illustration, we review the following problem given in the article[19].

$$T(x)(s) = x(s) - \int_0^1 G(s,t)\left(x(t)^{\frac{5}{2}} + \frac{x(t)^2}{2}\right)dt, \tag{45}$$

where $T : C[0,1] \to C[0,1]$ and the kernel G is the Green's function defined on the interval $[0,1] \times [0,1]$ by

$$G(s,t) = \begin{cases} (1-s)t & t \leq s, \\ s(1-t), & s \leq t. \end{cases}$$

Note that,

$$\left\|\int_0^1 G(s,t)\right\| \leq \frac{1}{8}.$$

Then,

$$\|G'(x) - G'(y))\| \leq \frac{1}{8}\left(\frac{5}{2}\|x-y\|^{\frac{3}{2}} + \|x-y\|\right). \tag{46}$$

Clearly, it can be seen that G' does not satisfy the Lipschitz continuity condition. However, G' is Hölder continuous. For such kind of examples, we also derive the local convergence results. This analysis also generalizes the local convergence analysis presented in the previous section. Supposing $q \in (0,1]$, $\kappa_0 > 0$ and $\kappa > 0$ be two parameters with $\kappa_0 \leq \kappa$, we define the functions $\mu_1, \mu_2, \mu_3, \mu_4, p, M_1, M_2, M_3$ and M_4 on interval $\left[0, \left(\frac{1}{\kappa_0}\right)^{\frac{1}{q}}\right)$ by

$$
\begin{aligned}
\mu_4(a) \quad = \quad & \Big[\frac{\kappa\mu_3(a)^q.a^q}{(q+1)(1-\kappa_0\mu_3(a)^q.a^q)} \\
& + \frac{1}{1-\kappa_0\mu_3(a^q)a^q}\kappa[\mu_1(a)^q + \mu_3(a)^q]a^q \times \left(\frac{1}{1-p(a)}\Big[1 + \frac{\kappa_0}{q+1}\mu_3(a)^q.a^q\Big]\right) \\
& + \left(\frac{1}{1-\kappa_0 a^q} . \frac{\kappa[1+\mu_1(a)^q]a^q}{1-p(a)} \times \Big[1 + \frac{\kappa_0}{q+1}\mu_3(a)^q.a^q\Big]\right)\Big]\mu_3(a),
\end{aligned} \tag{47}
$$

where

$$\mu_1(a) = \frac{1}{1-\kappa_0 a^q}\left[\frac{\kappa a^q}{q+1} + \frac{1 + \frac{\kappa_0}{(q+1)}.a^q}{2}\right], \tag{48}$$

$$\mu_2(a) = \frac{1}{1-\kappa_0 a^q}\left[\frac{\kappa a^q}{q+1} + \frac{\kappa[1+\mu_1(a)^q][\frac{\kappa_0}{(q+1)}.a^q + 1].a^q}{1-p(a)}\right], \tag{49}$$

$$
\begin{aligned}
\mu_3(a) \;=\; & \Big[\frac{\kappa \mu_2(a)^q . a^q}{(q+1)(1 - \kappa_0 \mu_2(a)^q . a^q)} \\
& + \frac{1}{1 - \kappa_0 \mu_2(a^q) a^q} \kappa [\mu_1(a)^q + \mu_2(a)^q] a^q \times \left(\frac{1}{1 - p(a)} \Big[1 + \frac{\kappa_0}{q+1} \mu_2(a)^q . a^q \Big] \right) \\
& + \left(\frac{1}{1 - \kappa_0 a^q} . \frac{\kappa [1 + \mu_1(a)^q] a^q}{1 - p(a)} \times \Big[1 + \frac{\kappa_0}{q+1} \mu_2(a)^q . a^q \Big] \right) \Big] \mu_2, (a),
\end{aligned}
\tag{50}
$$

and

$$
p(a) = \kappa_0 \mu_1(a)^q a^q.
\tag{51}
$$

Let

$$
M_1(a) = \mu_1(a) - 1, \quad M_2(a) = \mu_2(a) - 1,
\tag{52}
$$
$$
M_3(a) = \mu_3(a) - 1, \quad M_4(a) = \mu_4(a) - 1.
\tag{53}
$$

and

$$
\rho_1 = \left(\frac{q+1}{2\kappa + \kappa_0(3 + 2q)} \right)^{\frac{1}{q}} < \left(\frac{1}{\kappa_0} \right)^{\frac{1}{q}}.
\tag{54}
$$

We observe that $\mu_1(\rho_1) = 1$ and

$$
0 \le \mu_1(a) < 1 \; for \; a \in [0, \rho_1), \quad 0 < \rho_1 < (1/\kappa_0)^{\frac{1}{q}}.
\tag{55}
$$

We denote the smallest zero of $M_2(a)$ as ρ_2. Also, $M_2(\rho_1) > 0$ and $\rho_1 < \left(\frac{1}{\kappa_0} \right)^{\frac{1}{q}}$, which follows from that

$$
0 < \rho_2 < \rho_1, \quad 0 < \mu_2(a) < 1, for \; a \in [0, \rho_2).
\tag{56}
$$

Following this procedure, there comes the existence of zeros of functions M_i, $i = 1, 2, 3, 4$ in the interval $(0, \rho_0)$. Denote by ρ_i, respectively, the smallest solution of functions $M'_i s$. Define the radius of convergence ρ and for each $a \in [0, \rho)$ by

$$
\rho = min\{\rho_i\}, \; i = 1, 2, 3, 4, \quad 0 \le \mu_i(a) < 1.
\tag{57}
$$

Also, we assume that there exists $x^* \in D$ for all $x, y \in D$ and fulfill the below conditions along with the assumption (17):

$$
\|[T'(x^*)]^{-1}(T'(x) - T'(x^*))\| \;\le\; \kappa_0 \|x - x^*\|^q.
\tag{58}
$$

$$
\|[T'(x^*)]^{-1}(T'(x) - T'(y))\| \;\le\; \kappa \|x - y\|^q,
\tag{59}
$$
$$
B(x^*, \rho) \;\subseteq\; D.
\tag{60}
$$

Lemma 3.1. *If T satisfies the assumptions* (58) *and* (59), *consequently the inequalities given below hold for $x \in D$, $q = (0, 1]$ and $t \in [0, 1]$:*

$$
\|[T'(x^*)]^{-1} T'(x)\| \;\le\; 1 + \kappa_0 \|x - x^*\|^q,
\tag{61}
$$

$$
\|[T'(x^*)]^{-1}(T'(x^* + t(x - x^*)))\| \;\le\; 1 + \kappa_0 t^q \|x - x^*\|^q,
\tag{62}
$$

$$
\|[T'(x^*)]^{-1} T(x)\| \;\le\; \left(1 + \frac{\kappa_0}{q+1} \|x - x^*\|^q \right) . \|x - x^*\|.
\tag{63}
$$

Proof. By considering the hypothesis (58), we get

$$
\begin{aligned}
\|[T'(x^*)]^{-1}T'(x)\| &\leq 1 + \|[T'(x^*)]^{-1}T'(x) - [T'(x^*)]^{-1}T'(x^*)\| \\
&\leq 1 + \kappa_0\|x - x^*\|^q.
\end{aligned}
\tag{64}
$$

In the similar manner, we can derive

$$
\begin{aligned}
\|[T'(x^*)]^{-1}(T'(x^* + t(x - x^*)))\| &\leq 1 + \kappa_0\|x^* + t(x - x^*) - x^*\|^q \\
&\leq 1 + \kappa_0 t^q\|x - x^*\|^q.
\end{aligned}
\tag{65}
$$

Next,

$$
\begin{aligned}
\|[T'(x^*)]^{-1}T(x)\| &\leq \int_0^1 [\kappa_0\|x^* + t(x - x^*) - x^*\|^q + 1].\|x - x^*\|dt \\
&\leq \left(1 + \frac{\kappa_0}{q+1}\|x - x^*\|^q\right).\|x - x^*\|.
\end{aligned}
\tag{66}
$$

$\square$

Theorem 3.2. *Suppose that there exists $x^* \in D$ and the Fréchet differentiable operator $T : D \subseteq X \to Y$ satisfies the assumptions (58) and (59). Then, the sequence $\{x_n\}$ generated by the method (4) for $x_0 \in B(x^*, \rho)\backslash\{x^*\}$ is well defined in $B(x^*, \rho)$ remains in $B(x^*, \rho)$ for each $n = 0, 1, 2, \cdots$ and converges to x^*. Henceforward, the following measures holds for $n = 0, 1, 2, \cdots$:*

$$
\|y_n - x^*\| \leq \mu_1(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\| < \rho,
\tag{67}
$$

$$
\|z_n^{(1)} - x^*\| \leq \mu_2(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\| < \rho,
\tag{68}
$$

$$
\|z_n^{(2)} - x^*\| \leq \mu_3(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\| < \rho,
\tag{69}
$$

and

$$
\|x_{n+1} - x^*\| \leq \mu_4(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\| < \rho,
\tag{70}
$$

where the functions $\mu_i, i = 1, 2, 3, 4$ are defined by the expressions (47) - (50). Moreover, if there exists $\varrho \in \left[\rho, \left(\frac{1+q}{\kappa_0}\right)^{\frac{1}{q}}\right]$ such that $\overline{B(x^*, \varrho)} \subseteq D$. Then the limit point x^* is the only solution of equation $T(x) = 0$ in $D_1 = D \cap \overline{B(x^*, \varrho)}$.

Proof. Assume that $\|x_0 - x^*\|^q < \frac{1}{\kappa_0}$ and using the hypotheses, $x_0 \in B(x^*, \rho)\backslash\{x^*\}$ and inequality (58), we have that

$$
\|[T'(x^*)]^{-1}(T'(x_0) - T'(x^*))\| \leq \kappa_0\|x_0 - x^*\|^q < 1.
\tag{71}
$$

It follows from the above and the Banach lemma on invertible operator [20] that $[T'(x_0)]^{-1} \in L(Y, X)$ or $T'(x_0)$ is invertible and

$$
\|[T'(x_0)]^{-1}T'(x^*)\| \leq \frac{1}{1 - \kappa_0\|x_0 - x^*\|^q}.
\tag{72}
$$

Now, y_0 is well defined by the first sub-step of the scheme (4) and for $n = 0$ we get the approximation (27). Expanding $T(x_0)$ along x^* and taking the norm, we get the inequality (28). Finally, using the assumption (58) and the inequality (72), we get

$$
\begin{aligned}
\|y_0 - x^*\| &\leq \frac{1}{1 - \kappa_0\|x_0 - x^*\|^q}\left[\frac{1}{2}\left[\frac{\kappa_0}{q+1}\|x_0 - x^*\|^q + 1\right]\right. \\
&\quad + \left.\frac{\kappa}{q+1}\|x_0 - x^*\|^q\right]\|x_0 - x^*\| \\
&\leq \mu_1(\|x_0 - x^*\|)\|x_0 - x^*\| < \rho.
\end{aligned}
\tag{73}
$$

From the inequalities (58) and (73), we have

$$
\begin{aligned}
\|[T'(x^*)]^{-1}[T'(y_0) - T'(x^*)]\| &\leq \kappa_0 \|y_0 - x^*\|^q \\
&\leq \kappa_0 \mu_1 (\|x_0 - x^*\|)^q \|x_0 - x^*\|^q \\
&= p(\|x_0 - x^*\|) < 1.
\end{aligned}
\tag{74}
$$

Again, from the second sub-step of the method (4), we find the approximation (32). On taking the norm, we get

$$
\begin{aligned}
\|z_0^{(1)} - x^*\| &\leq \|x_0 - x^* - [T'(x_0)]^{-1}T(x_0)\| + \|[T'(x_0)]^{-1}T'(x^*)\|. \\
&\quad \|[T'(x^*)]^{-1}[T'(y_0) - T'(x_0)]\|\|[T'(y_0)]^{-1}T'(x^*)\|\|[T'(x^*)]^{-1}T(x_0)\| \\
&\leq \frac{1}{1 - \kappa_0\|x_0 - x^*\|^q}\left[\frac{\kappa}{q+1}\|x_0 - x^*\|^q\right. \\
&\quad + \left.\frac{[\kappa\|y_0 - x^*\|^q + \kappa(\|x_0 - x^*\|^q)][\frac{\kappa_0}{q+1}\|x_0 - x^*\|^q + 1]}{1 - p(\|x_0 - x^*\|)}\right]\|x_0 - x^*\|.
\end{aligned}
\tag{75}
$$

Thus, we get

$$
\|z_0^{(1)} - x^*\| \leq \mu_2(\|x_0 - x^*\|)\|x_0 - x^*\| \leq \|x_0 - x^*\| < \rho.
\tag{76}
$$

From the next sub-step of the method (4), we have the approximation (35). On expanding $T(z_0^{(1)})$ along x^* and taking norm, we get

$$
\begin{aligned}
\|z_0^{(2)} - x^*\| &\leq \frac{1}{1 - \kappa_0\|z_0^{(1)} - x^*\|^q}\frac{\kappa}{q+1}\|z_0^{(1)} - x^*\|^q.\|z_0^{(1)} - x^*\| + \frac{[\kappa\|y_0 - x^*\|^q + \kappa\|z_0^{(1)} - x^*\|^q]}{1 - \kappa_0\|z_0^{(1)} - x^*\|^q} \\
&\quad \times \|[T'(x^*)]^{-1}T(z_0^{(1)})\| + \frac{1}{1 - \kappa_0\|x_0 - x^*\|^q}[\kappa\|y_0 - x^*\|^q + \kappa\|x_0 - x^*\|^q] \\
&\quad \times \|[T'(x^*)]^{-1}T(z_0^{(1)})\| \\
&\leq \frac{1}{1 - \kappa_0\|z_0^{(1)} - x^*\|^q}\frac{\kappa}{q+1}\|z_0^{(1)} - x^*\|^q.\|z_0^{(1)} - x^*\| + \frac{[\kappa\|y_0 - x^*\|^q + \kappa\|z_0^{(1)} - x^*\|^q]}{1 - \kappa_0\|z_0^{(1)} - x^*\|^q} \\
&\quad \times \frac{1}{1 - p(\|x_0 - x^*\|)}\left(1 + \frac{\kappa_0}{q+1}\|z_0^{(1)} - x^*\|^q\right)\|z_0^{(1)} - x^*\| + \frac{[\kappa\|y_0 - x^*\|^q + \kappa\|x_0 - x^*\|^q]}{1 - \kappa_0\|x_0 - x^*\|^q} \\
&\quad \times \frac{1}{1 - p(\|x_0 - x^*\|^q)}\left(1 + \frac{\kappa_0}{q+1}\|z_0^{(1)} - x^*\|^q\right)\|z_0^{(1)} - x^*\|.
\end{aligned}
\tag{77}
$$

Thus, we have

$$
\|z_0^{(2)} - x^*\| \leq \mu_3(\|x_0 - x^*\|)\|x_0 - x^*\| < \rho,
\tag{78}
$$

Finally, from the last sub-step of the method (4) we have the approximation (38). On expanding $T(z_0^{(2)})$ along x^* and taking norm, we get

$$
\begin{aligned}
\|x_1 - x^*\| &\leq \frac{1}{1 - \kappa_0\|z_0^{(2)} - x^*\|^q}\frac{\kappa}{q+1}\|z_0^{(2)} - x^*\|^q.\|z_0^{(2)} - x^*\| + \frac{[\kappa\|y_0 - x^*\|^q + \kappa\|z_0^{(2)} - x^*\|^q]}{1 - \kappa_0\|z_0^{(2)} - x^*\|^q} \\
&\quad \times \|[T'(x^*)]^{-1}T(z_0^{(2)})\| + \frac{1}{1 - \kappa_0\|x_0 - x^*\|^q}[\kappa\|y_0 - x^*\|^q + \kappa\|x_0 - x^*\|^q] \\
&\quad \times \|[T'(x^*)]^{-1}T(z_0^{(2)})\| \\
&\leq \frac{1}{1 - \kappa_0\|z_0^{(2)} - x^*\|^q}\frac{\kappa}{q+1}\|z_0^{(2)} - x^*\|^q.\|z_0^{(2)} - x^*\| + \frac{[\kappa\|y_0 - x^*\|^q + \kappa\|z_0^{(2)} - x^*\|^q]}{1 - \kappa_0\|z_0^{(2)} - x^*\|^q} \\
&\quad \times \frac{1}{1 - p(\|x_0 - x^*\|)}\left(1 + \frac{\kappa_0}{q+1}\|z_0^{(2)} - x^*\|^q\right)\|z_0^{(2)} - x^*\| + \frac{[\kappa\|y_0 - x^*\|^q + \kappa\|x_0 - x^*\|^q]}{1 - \kappa_0\|x_0 - x^*\|^q} \\
&\quad \times \frac{1}{1 - p(\|x_0 - x^*\|)}\left(1 + \frac{\kappa_0}{q+1}\|z_0^{(2)} - x^*\|^q\right)\|z_0^{(2)} - x^*\|.
\end{aligned}
\tag{79}
$$

Thus, we have

$$
\|x_1 - x^*\| \leq \mu_4(\|x_0 - x^*\|)\|x_0 - x^*\| < \rho,
\tag{80}
$$

which shows that for $n = 0$, $x_1 \in B(x^*, \rho)$. The function $M_4(a) = \mu_4(a) - 1$ gives $M_4(0) < 0$ and $M_4(\rho_3) > 0$. Hence, $M_4(t)$ has at least one root in $(0, \rho_3)$. Let ρ be the smallest root of $M_4(t)$ in $(0, \rho_3)$. Then, we have

$$
0 < \rho < \rho_3 < \rho_2 < \rho_1 < \left(\frac{1}{\kappa_0}\right)^{\frac{1}{q}}.
\tag{81}
$$

and

$$0 < \mu_4(a) < 1, for \ a \in [0, \rho). \tag{82}$$

By simply replacing $x_0, y_0, z_0^{(1)}, z_0^{(2)}, x_1$ by $x_n, y_n, z_n^{(1)}, z_n^{(2)}, x_{n+1}$ in the preceding estimates, we arrive at inequalities $(67) - (70)$. By the estimate

$$\|x_{n+1} - x^*\| \le \mu_4(\|x_0 - x^*\|)\|x_n - x^*\| < \rho. \tag{83}$$

We conclude that $\lim_{n\to\infty} x_n = x^*$ and $x_{n+1} \in B(x^*, \rho)$. Finally, to prove the uniqueness, let $y^* \in B(x^*, \rho)$ where $y^* \ne x^*$ with $T(y^*) = 0$. Define $F = \int_0^1 T'(x^* + t(y^* - x^*))dt$. On expanding $T(y^*)$ along x^* and using inequality (58), we obtain

$$\|[T'(x^*)]^{-1} \int_0^1 [T'(x^* + t(y * -x^*) - T'(x^*)]dt\|$$
$$\le \frac{\kappa_0}{q+1}\|y^* - x^*\|^q \le \frac{\kappa_0}{q+1}\varrho^q < 1. \tag{84}$$

So, by Banach lemma, $\int_0^1 [T'(x^*)]^{-1}[T'(x^* + t(y^* - x^*))]dt$ exists and invertible leading to the conclusion $x^* = y^*$, which completes the uniqueness part of the proof. $\qquad\square$

4. Applications with numerical examples

In this section, two numerical examples are worked out to demonstrate the efficiency of our local convergence analysis by giving the radii of convergence for the scheme (4). We obtain better results using our technique.

Example 4.1. *[12] Returning back to the illustration example given in the introduction of this study of Lipschitz, The unique solution is $x^* = 1$. The consecutive derivatives of f are*

$$\begin{aligned} f'(x) &= 3x^2 log x^2 + 5x^4 - 4x^3 + 2x^2, \\ f''(x) &= 6x log x^2 + 20x^3 - 12x^2 + 10x, \\ f'''(x) &= 6 log x^2 + 60x^2 - 24x + 22. \end{aligned}$$

It can be easily visible that f''' is unbounded on D. Nevertheless, all the assumptions of the Theorem (2.1) for the iterative method (4) are satisfied and hence applying the convergence results with $x^* = 1$, we obtain $\psi_0 = \psi = 96.6628$.

*Here, we will use the iterative method (4) and compare it with the scheme given by Cordero et. al [24] and we denote it by CHMT. From the above Table (1) , we can see that the radius ρ of convergence computed for the method (4) seems to be finer than the method CHMT in the reference [25]. As a result, the approach being evaluated is more powerful. We use "η" functions to calculate value of ρ (Fig.1). Also, with the help of **Mathematica 9** software for numerical computation, we have calculated the execution time taken by η_4 to give final value of ρ is 0.171443 seconds.*

Example 4.2. *[26] In order to show the applicability of the results presented in this paper, we consider the following Planck's radiation law problem [27]:*

$$\phi(\lambda) = \frac{8\pi cP\lambda^{-5}}{e^{\frac{cP}{\lambda BT}} - 1},$$

which calculates the energy density within an isothermal blackbody, where

- *λ is the wavelength of the radiation*

Table 1: Comparison of convergence radius (Example 4.1)

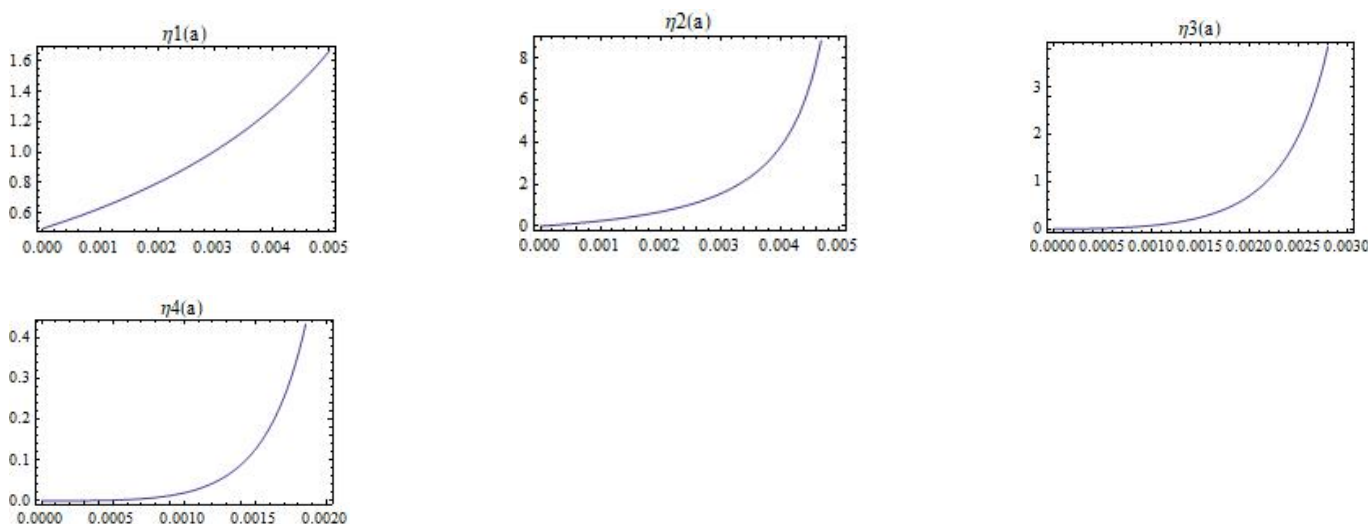| $Radius$	Method (4)	CHMT($\theta = -2$)
ρ_1	0.00295578	0.006689
ρ_2	0.00246894	0.005750
ρ_3	0.00217353	0.003001
ρ_4	0.00208131	0.001943
ρ	0.00208131	0.001943

Figure 1: "η" functions for Example 1

- T is the absolute temperature of the blackbody

- B is Boltzmann's constant

- P is the Planck's constant

- c is the speed of light.

Suppose, we would like to determine wavelength λ which corresponds to maximum energy density $\phi(\lambda)$. Therefore, solving for maxima we define

$$f(x) = e^{-x} - 1 + \frac{x}{5}.$$

We have $x^* = 4.965114$ and $F'(x^*) = 0.193023$. Then, on using assumptions $(58) - (59)$ we have that, $q = 1$, $\kappa_0 = 0.0608658 < \kappa = 0.094888$. Table (2) displays the radius ρ of convergence by the discussed method (4) along with the existing scheme given by Khan et al. [26]. We discovered that when compared the provided method enlarges the radius of the convergence ball. We get the value of ρ using "η" functions (Fig.2). Also, with the help of **Mathematica 9** *software for numerical computation, we have calculated the execution time taken by η_4 to give final value of ρ is 0.166638 seconds.*

5. Conclusions

In the presented work, we have analyzed the local convergence analysis of the efficient seventh order method for solving the nonlinear equation in Banach spaces. A convergence theorem for existence and uniqueness of the solution has been established followed by its error bounds giving the benefit that the

Table 2: Comparison of convergence radius (Example 4.2)

$Radius$	Method (4)	KFS
ρ_1	4.04772	9.23282
ρ_2	2.99797	2.40532
ρ_3	2.58569	1.11454
ρ_4	2.45972	-
ρ	2.45972	1.11454
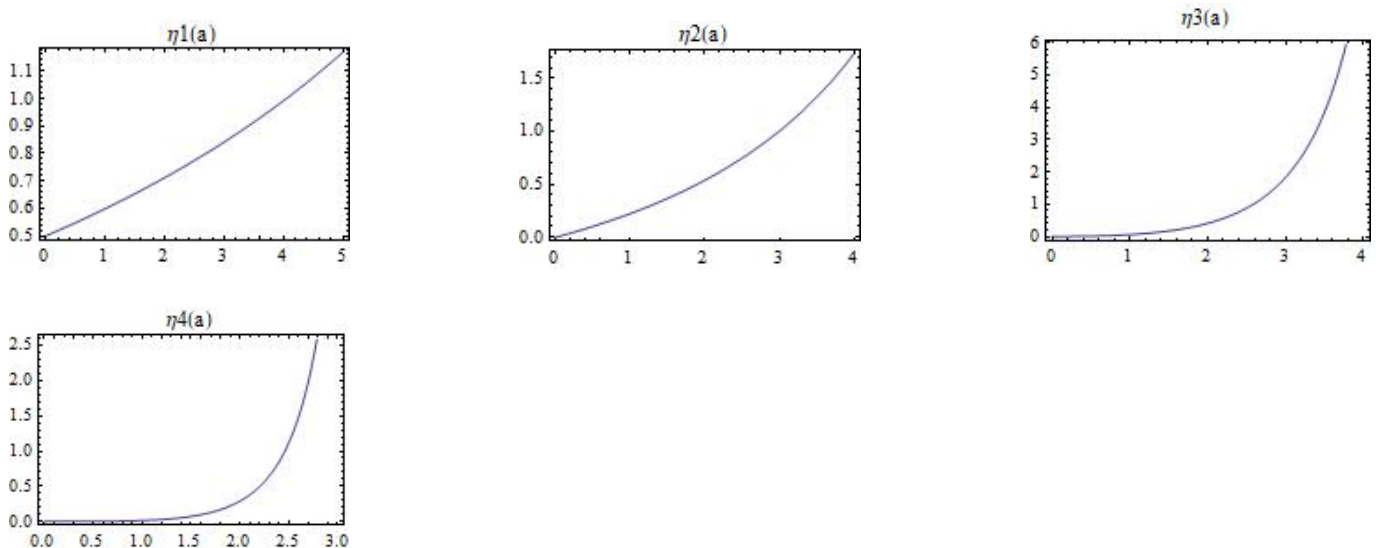
Figure 2: "η" functions for Example 2

iterative method always converges to the solution. This local convergence analysis is applicable in solving such problems for which higher-order derivative based previous studies fail. Later, we relaxed these assumptions and derived convergence results under Hölder condition for solving different types of nonlinear integral equations, which are not solvable by the previous approach.

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