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Research Article

Shifted primes with large prime power divisors

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Abstract

We obtain significant lower bounds for the number of shifted prime numbers having a relatively large prime power divisor, where being large has various quantifications. For any given $k \geq 2$, our results show the existence of infinitely many prime numbers p that lie over certain admissible arithmetic progressions, and of the form $p = q^k s + a$ for suitable positive integers a, where q is prime and s is forced to be genuinely small with respect to p. We prove the existence of such prime numbers over progressions both unconditionally, and then conditionally by either assuming the nonexistence of Siegel zeros or weaker forms of the Riemann hypothesis for Dirichlet L-functions. Our approach allows us to provide considerable uniformity regarding the size of the modulus of the progressions, where the sought primes belong to, and the shift parameter a by restricting the size of s at the same time. Finally, assuming the validity of a conjecture about the distribution of prime numbers along progressions with very large modulus, we demonstrate how it is possible to go beyond by showing that $s \leq (p - a)^{\epsilon}$ for every $\epsilon > 0$ when $k = 2$.

Mathematics Subject Classification (2020). 11N05, 11N25, 11N13

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1. Introduction

In this paper, we estimate from below the number of shifted prime numbers having a large prime power divisor, where being large will have different quantifications. As usual, a shifted prime number is assumed to be an integer of the form $p - a$ with p being a prime number and a is a given positive integer which is referred to as the shift parameter. Large prime power factors of consecutive integers were studied by Sander [13]. A related problem about the connection between prime power values of a polynomial and its irreducibility was treated by Bonciocat, Bonciocat and Zaharescu [2]. Specifically, it is shown here that there are infinitely many prime numbers over certain arithmetic progressions with the property that many of the corresponding translated primes always admit a relatively large prime power divisor, namely a divisor of the form q^k with q being a prime number and $k \geq 2$. Beyond Dirichlet's theorem on the abundance of prime numbers in arithmetic progressions, finding infinitely many prime numbers on nonlinear polynomial evaluations turns out to be a notoriously difficult problem in general. This is clearly spelled out in

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the following long standing conjecture which is a special but important subcase of the Hardy-Littlewood conjectures [5], [6].

Conjecture 1. There exist infinitely many prime numbers $p \equiv 1 \pmod 4$ *such that* $\frac{p-1}{4}$ *is the square of a prime number.*

This prototypical conjecture forms a foundational motivation for our approach, and consequently, we seek to find infinitely many prime numbers p subject to similar but weaker conditions such as the one

$$p = q^k s + 1, \tag{1.1}$$

for $k \geq 2$, where q is prime and s is comparably small with respect to p (we definitely would like to have $s = o(p)$), or equivalently, the translated prime $p - 1$ has a relatively large prime power divisor. At this point, it is worth stressing that since there are only finitely many prime numbers of the forms $q^2 + 1, 2q^2 + 1$ and $3q^2 + 1$ when q ranges over all prime numbers, Conjecture 1 can be restated in an equivalent way that relates better to the size of s in (1.1).

Conjecture 2. There exist infinitely many prime numbers of the form $p = q^2 s + 1$*, where* q *is a prime number and* $s \leq 4$*.*

In light of the above remarks, establishing analogs of Conjecture 2 under weaker conditions such as the ones

$$p = q^k s + 1, \quad s \leq \kappa(p)$$

for any given $k \geq 2$, where $\kappa(p) = o(p)$ is a small function of p, could be a fruitful task. As a first attempt on this, note that by Dirichlet's theorem, there are infinitely many primes of the form $p = q^k s + 1$ when s ranges over positive integers, where q prime and $k \geq 2$ are given. Consequently, we have

$$s \leq \kappa(p) := \frac{p-1}{q^k}.$$

However, despite the fact that q can be taken as large as we please, we still can not make $\kappa(p) = o(p)$ with this attempt. Although Conjecture 1 has resisted all efforts for a proof so far, there were striking developments in the literature representing infinitely many prime numbers by other types of nonlinear conditions, the most curious example of this being due to Mills [9] who proved the existence of a real number A such that

$$\left[A^{3^n}\right]$$

happens to be a prime number for every positive integer n, $[x]$ denoting the greatest integer not exceeding x. Our results in this paper can be viewed as a contribution towards Conjecture 2 by restricting the size of s in (1.1) considerably. At the same time, we aim to complement Conjecture 2 with a generalization to any higher degree nonlinearity structure among prime numbers (see Theorems 1.1–1.5 below). This is indeed achieved in a stronger sense by providing both significant lower bounds on the number of such primes and uniformity regarding the size of the modulus of the progressions, where the sought primes belong to, and the shift parameter a. We further explore connections to the nonexistence of Siegel zeros (see Theorems 1.2, 1.3) and, assuming the Riemann hypothesis for Dirichlet L-functions, we show for any given $k \geq 2$ that (see Theorem 1.4), there are quantitatively many primes $p \leq x$ satisfying

$$p = q^k s + a, \quad p \equiv h \pmod m$$

with q prime, where $a \leq h \leq m \leq x^\mu$ for suitable $\mu > 0$, $(h, m) = 1 = (h - a, m)$ and

$$s \leq (p-a)^{\frac{1}{2}+\epsilon}$$

for any given $\epsilon > 0$. Lastly, we indicate how to surpass the $1/2$ exponent for reducing the size of s conditionally, leading in particular to the more pleasant bound

$$s \leq (p-a)^{\epsilon}$$

for any given $\epsilon > 0$ when $k = 2$ (see Theorem 1.5). Let us now state our first contribution. Throughout, the notation exp is used to denote the exponential function, and ϕ is Euler's function.

Theorem 1.1. *Let a, h, m be positive integers such that $a \leq h \leq m$ and $(h, m) = 1 = (h-a, m)$. For $k \geq 2$ and $\alpha > 0$, let $\pi_{a,k,\alpha}(x, m, h)$ be the number of prime numbers $p \leq x$, $p \equiv h \pmod{m}$ satisfying*

$$p = n + a = q^k s + a$$

with q being a prime number and

$$s \leq \frac{n}{(\log n)^{k\alpha}}.$$

Let

$$D(x) := \max_{r \leq x} d(r), \tag{1.2}$$

where $d(r)$ is the number of divisors of r. If $m \leq (\log x)^B$ for some $B > 0$ and a is fixed, then as x tends to infinity, we have

$$\liminf \left(\frac{\pi_{a,k,\alpha}(x, m, h)\phi(m)D(x)(\log x)^{1+(k-1)\alpha} \log\log x}{x} \right) \geq \frac{C_k}{\alpha}, \tag{1.3}$$

where

$$C_k := \frac{1}{k-1}\left(1 - \frac{1}{2^{k-1}}\right) \tag{1.4}$$

for any $k \geq 2$. In particular, for any $\lambda > 0$, we have

$$\pi_{a,k,\alpha}(x, m, h) \geq x \, \exp\left(-\frac{(1+\lambda)\log 2 \log x}{\log\log x}\right) \tag{1.5}$$

when x is large enough in terms of B, k, λ and α. Moreover, if $\alpha \geq B$, then (1.3) *and* (1.5) *continue to hold uniformly for all a with $a \leq h \leq m \leq (\log x)^B$ and $(h, m) = 1 = (h-a, m)$.*

Note that by taking $a = h = m = 1$ and $k = 2$ in Theorem 1.1, we obtain a quantitative form of a weaker version of Conjecture 2. Moreover, the range of the modulus m in Theorem 1.1 is at the same quality as the range of the modulus in the Siegel-Walfisz theorem (see [17] and Section 22 of [3]) which offers the strongest unconditional uniformity in the modulus aspect for the number of prime numbers belonging to an arithmetic progression having that modulus. As our next goal, we demonstrate how it is feasible to restrict the size of s and to extend the range of the modulus m at the same time in Theorem 1.1 with the help of the assumption on the nonexistence of Siegel zeros for Dirichlet L-functions defined by

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

for $\Re(s) > 1$ and χ being a Dirichlet character. For a discussion of Siegel zeros and the exceptional real characters, the reader is referred to Section 14 of [3]. Siegel [15] showed for any nonprincipal real character χ modulo q that $L(s, \chi) \neq 0$ whenever

$$s > 1 - \frac{C(\delta)}{q^{\delta}}$$

for every $\delta > 0$ and some noneffective (not computable) positive constant $C(\delta)$. For improvements and variations on this, we recommend the work of Sarnak and Zaharescu [14]. For applications of the nonexistence of Siegel zeros, see [1] and [8].

Theorem 1.2. *Let a, h, m be positive integers such that $a \le h \le m$ and $(h, m) = 1 = (h - a, m)$. For $k \ge 2$ and $0 < \alpha < 1$, let $\pi_{a,k,\alpha}(x, m, h)$ be the number of prime numbers $p \le x$, $p \equiv h \pmod m$ satisfying*

$$p = n + a = q^k s + a$$

with q being a prime number and

$$s \le ne^{-k(\log n)^\alpha}.$$

If $m \le (\log x)^B$ for some $B > 0$, then assuming that there are no Siegel zeros for Dirichlet L-functions, we have

$$\liminf \left(\frac{\pi_{a,k,\alpha}(x, m, h)\phi(m)D(x)e^{(k-1)(\log x)^\alpha}(\log x)^{\alpha+1}}{x} \right) \ge C_k \tag{1.6}$$

as x tends to infinity, where $D(x)$ is defined as in (1.2) and C_k is defined as in (1.4). In particular, for any $\lambda > 0$,

$$\pi_{a,k,\alpha}(x, m, h) \ge x \exp\left(-\frac{(1+\lambda)\log 2 \log x}{\log\log x} \right) \tag{1.7}$$

holds when x is large enough in terms of B, k, λ and α.

Let us now turn our attention to extending the uniformity of the modulus m further while maintaining the limitation on the size of s.

Theorem 1.3. *Let a, h, m be positive integers such that $a \le h \le m$ and $(h, m) = 1 = (h - a, m)$. For $k \ge 2$, assuming that there are no Siegel zeros for Dirichlet L-functions, there exists an absolute constant $c_1 > 0$ such that if $\pi_{a,k}(x, m, h)$ is the number of prime numbers $p \le x$, $p \equiv h \pmod m$ satisfying*

$$p = n + a = q^k s + a$$

with q being a prime number,

$$s \le ne^{-c_1 k\sqrt{\log n}} \text{ and } m = o\left(\frac{e^{c_1\sqrt{\log x}}}{\sqrt{\log x}} \right),$$

then we have

$$\liminf \left(\frac{\pi_{a,k}(x, m, h)\phi(m)D(x)e^{c_1(k-1)\sqrt{\log x}}(\log x)^{\frac{3}{2}}}{x} \right) \ge \frac{C_k}{c_1} \tag{1.8}$$

as x tends to infinity, where $D(x)$ is defined as in (1.2) and C_k is defined as in (1.4). In particular, for any $\lambda > 0$,

$$\pi_{a,k}(x, m, h) \ge x \exp\left(-\frac{(1+\lambda)\log 2 \log x}{\log\log x} \right) \tag{1.9}$$

holds when x is large enough in terms of λ and k.

Assuming weaker versions of the Riemann hypothesis for Dirichlet L-functions, one can further restrict the size of s and extend the range of m in a much more uniform manner.

Theorem 1.4. *Assume that all zeros of all Dirichlet L-functions have real part $\le \theta$ for some $1/2 \le \theta < 1$. Let a, h, m be positive integers such that $a \le h \le m$ and $(h, m) = 1 = (h - a, m)$. For $k \ge 2$ and*

$$0 \le \mu < \frac{1-\theta}{k},$$

let $\pi_{a,k,\theta,\mu,f}(x, m, h)$ be the number of prime numbers $p \le x$, $p \equiv h \pmod m$ satisfying

$$p = n + a = q^k s + a$$

with q being a prime number and

$$s \leq n^{\theta+k\mu}(\log n)^{3k} f(n)^k,$$

where $f(x)$ is any function tending to infinity (arbitrarily slowly). If $m \leq x^\mu$ (if $\mu = 0$, then we take $m = 1$, and consequently the progression condition $p \equiv h \pmod m$ drops) and a is fixed, then as x tends to infinity, we have

$$\liminf \left(\frac{\pi_{a,k,\theta,\mu,f}(x,m,h)\phi(m)D(x)}{x^{\frac{1+(k-1)\theta}{k}+(k-1)\mu}(\log x)^{3k-5} f(x)^{k-1}} \right) \geq \frac{C_k}{\frac{1-\theta}{k}-\mu}, \tag{1.10}$$

where $D(x)$ is defined as in (1.2) *and C_k is defined as in* (1.4). *In particular, for any $\lambda > 0$,*

$$\pi_{a,k,\theta,\mu,f}(x,m,h) \geq x^{\frac{1+(k-1)\theta}{k}+(k-2)\mu} \exp\left(-\frac{(1+\lambda)\log 2 \log x}{\log\log x} \right) \tag{1.11}$$

holds when x is large enough in terms of $\theta, \mu, \lambda, k, f$. Moreover, if

$$0 \leq \mu < \frac{1-\theta}{2k},$$

then (1.10) *and* (1.11) *continue to hold uniformly for all a with $a \leq h \leq m \leq x^\mu$ and $(h,m) = 1 = (h-a,m)$.*

Since $\theta + k\mu < 1$, Theorem 1.4 puts a more severe limitation on the size of s which is considerably smaller than the ones obtained in Theorem 1.1, Theorem 1.2 and Theorem 1.3. Besides, concerning the exponent of x on the right hand side of (1.11), one observes that

$$\frac{1}{2} < \frac{1+(k-1)\theta}{k} + (k-2)\mu < 1-\mu.$$

Therefore, (1.11) quantifies the abundance of prime numbers belonging to sparser progressions and subject to the required nonlinear conditions. Moreover, assuming the Riemann hypothesis for Dirichlet L-functions, we may take $\theta = 1/2$ and $\mu > 0$ small enough (in terms of k and $\epsilon > 0$) to get

$$s \leq n^{\frac{1}{2}+\epsilon}.$$

Finally, if $\mu = 0$, then we can even deduce the more precise bound

$$s \leq n^{\frac{1}{2}}(\log n)^{3k} f(n)^k$$

at the cost of giving up on the progression condition for p. Perhaps somewhat paradoxical, arriving at such nonlinear conditions satisfied by prime numbers in the above theorems heavily depends on the finer distribution of them along arithmetic progressions with large modulus. Motivated by this, it would be interesting to test the best possible case scenario predicted on the distribution of prime numbers along arithmetic progressions and study its impact in regards to diminishing the size of s. To this end, recall the definition of Chebyshev's function

$$\psi(x,q,a) := \sum_{\substack{n \leq x \\ n \equiv a \pmod q}} \Lambda(n),$$

where $\Lambda(n)$ is von Mangoldt's function and $(a,q) = 1$. Then we have the widely believed conjecture:

Conjecture 3. Uniformly for all $q \leq x$, we have for every $\epsilon > 0$,

$$\psi(x,q,a) = \frac{x}{\phi(q)} + O_\epsilon\left(\frac{x^{\frac{1}{2}+\epsilon}}{\sqrt{q}} \right),$$

where the implied constant depends only on ϵ.

Based on the validity of Conjecture 3, it is pleasant to show that a much more superior estimate than the above Theorems holds for the size of s which in turn brings us closer to the state of Conjecture 2.

Theorem 1.5. *For $k \geq 2$, $1 > \epsilon > \frac{k-2}{2k-2}$ and an integer $1 \leq a \leq x^{\frac{1-\epsilon}{k}}$, let $\pi_{a,k,\epsilon}(x)$ be the number of prime numbers $p \leq x$ such that*

$$p = n + a = q^k s + a$$

with q being a prime number and $s \leq n^\epsilon$. Assuming Conjecture 3, we have

$$\liminf \left(\frac{\pi_{a,k,\epsilon}(x) D(x) (\log x)^2}{x^{\frac{1+(k-1)\epsilon}{k}}} \right) \geq \frac{kC_k}{1-\epsilon} \tag{1.12}$$

as x tends to infinity, where $D(x)$ is defined as in (1.2) *and C_k is defined as in* (1.4). *In particular, for any $\lambda > 0$,*

$$\pi_{a,k,\epsilon}(x) \geq x^{\frac{1+(k-1)\epsilon}{k}} \exp\left(-\frac{(1+\lambda)\log 2 \log x}{\log\log x} \right) \tag{1.13}$$

holds when x is large enough in terms of ϵ, λ and k.

It is worth mentioning that, in contrast with Theorems 1.1–1.4, Theorem 1.5 improves the $1/2$ barrier for the exponent bounding the size of s, since

$$\frac{k-2}{2k-2} < \frac{1}{2}$$

holds for all $k \geq 2$. In particular, to record the strongest results with respect to s separately, we know from (1.13) that there are quantitatively many prime numbers $p = n+a = q^k s + a$ for each of the values $k = 2, 3, 4$ with q prime when

$$s \leq n^\epsilon, \quad s \leq n^{\frac{1}{4}+\epsilon} \text{ and } s \leq n^{\frac{1}{3}+\epsilon},$$

respectively, for each $\epsilon > 0$, provided Conjecture 3 holds.

2. Preliminary results

In this section, we collect all of the preliminary lemmas that will be essential for the proofs. Our first result is needed for the determination of C_k in (1.4).

Lemma 2.1. *For any real number $\beta > 1$, we have*

$$\sum_{x<p\leq 2x} \frac{1}{p^{\beta-1}(p-1)} = (1+o(1)) \frac{C_\beta}{x^{\beta-1}\log x} \tag{2.1}$$

as x tends to infinity, where

$$C_\beta = \frac{1}{\beta-1}\left(1 - \frac{1}{2^{\beta-1}}\right)$$

and the sum on the left hand side of (2.1) *is over all prime numbers belonging to the interval* $(x, 2x]$.

Proof. To begin with, we have by partial summation that

$$\sum_{p>x} \frac{1}{p^\beta} = -\frac{\pi(x)}{x^\beta} + \beta \int_x^\infty \frac{\pi(t)}{t^{\beta+1}}\, dt, \tag{2.2}$$

where $\pi(x)$ is the number of prime numbers $\leq x$. By the prime number theorem with classical error term, we know that

$$\pi(t) = \mathrm{li}(t) + O(te^{-c_0\sqrt{\log t}}) \tag{2.3}$$

for some constant $c_0 > 0$, where

$$\mathrm{li}(x) := \int_2^x \frac{1}{\log t}\, dt$$

is the logarithmic integral. With the help of (2.3), we may write

$$\int_x^\infty \frac{\pi(t)}{t^{\beta+1}}\, dt = \int_x^\infty \frac{\mathrm{li}(t)}{t^{\beta+1}}\, dt + O\left(\int_x^\infty \frac{1}{t^\beta e^{c_0\sqrt{\log t}}}\, dt\right). \tag{2.4}$$

Next we have

$$\int_x^\infty \frac{1}{t^\beta e^{c_0\sqrt{\log t}}}\, dt = O_\beta\left(\int_x^{x^2} \frac{1}{t^\beta e^{c_0\sqrt{\log t}}}\, dt\right) = O_\beta\left(\frac{1}{x^{\beta-1} e^{c_1\sqrt{\log x}}}\right) \tag{2.5}$$

for some constant $c_1 > 0$. We can combine (2.4) and (2.5) to obtain

$$\int_x^\infty \frac{\pi(t)}{t^{\beta+1}}\, dt = \int_x^\infty \frac{\mathrm{li}(t)}{t^{\beta+1}}\, dt + O_\beta\left(\frac{1}{x^{\beta-1} e^{c_1\sqrt{\log x}}}\right). \tag{2.6}$$

On the other hand, by partial integration, we get

$$\int_x^\infty \frac{\mathrm{li}(t)}{t^{\beta+1}}\, dt = \frac{\mathrm{li}(x)}{\beta x^\beta} + \frac{1}{\beta}\int_x^\infty \frac{1}{t^\beta \log t}\, dt. \tag{2.7}$$

Feeding (2.7) into (2.6), one arrives at the formula

$$\int_x^\infty \frac{\pi(t)}{t^{\beta+1}}\, dt = \frac{\mathrm{li}(x)}{\beta x^\beta} + \frac{1}{\beta}\int_x^\infty \frac{1}{t^\beta \log t}\, dt + O_\beta\left(\frac{1}{x^{\beta-1} e^{c_1\sqrt{\log x}}}\right). \tag{2.8}$$

Assembling (2.2) and (2.8), we infer that

$$\sum_{p>x} \frac{1}{p^\beta} = \frac{\mathrm{li}(x) - \pi(x)}{x^\beta} + \int_x^\infty \frac{1}{t^\beta \log t}\, dt + O_\beta\left(\frac{1}{x^{\beta-1} e^{c_1\sqrt{\log x}}}\right). \tag{2.9}$$

Again by (2.3), we have

$$\frac{\mathrm{li}(x) - \pi(x)}{x^\beta} = O\left(\frac{1}{x^{\beta-1} e^{c_0\sqrt{\log x}}}\right).$$

Thus (2.9) can be written in the form

$$\sum_{p>x} \frac{1}{p^\beta} = \int_x^\infty \frac{1}{t^\beta \log t}\, dt + O_\beta\left(\frac{1}{x^{\beta-1} e^{c_1\sqrt{\log x}}}\right) \tag{2.10}$$

for some constant $c_1 > 0$. As a consequence of (2.10), we have

$$\sum_{x<p\le 2x} \frac{1}{p^\beta} = \int_x^{2x} \frac{1}{t^\beta \log t}\, dt + O_\beta\left(\frac{1}{x^{\beta-1} e^{c_1\sqrt{\log x}}}\right). \tag{2.11}$$

Define the function

$$g(x) := \int_x^{2x} \frac{1}{t^\beta \log t}\, dt.$$

Note that $g(x) \to 0$ as x tends to infinity and

$$g'(x) = \frac{1}{2^{\beta-1} x^\beta \log 2x} - \frac{1}{x^\beta \log x}. \tag{2.12}$$

To complete the proof, we show that

$$g(x) x^{\beta-1} \log x$$

is asymptotic to a positive constant as x tends to infinity. For the explicit determination of this constant, note that we have by (2.12) and L'Hôpital's rule that

$$\lim_{x\to\infty} \frac{g(x)}{\frac{1}{x^{\beta-1}\log x}} = \frac{1}{\beta-1} - \frac{1}{2^{\beta-1}} \lim_{x\to\infty} \frac{\log^2 x}{\log 2x((\beta-1)\log x + 1)}. \tag{2.13}$$

Clearly, we also have

$$\lim_{x\to\infty} \frac{\log^2 x}{\log 2x((\beta-1)\log x+1)} = \frac{1}{\beta-1}. \tag{2.14}$$

From (2.13) and (2.14), one justifies that

$$\int_x^{2x} \frac{1}{t^\beta \log t}\,dt = (1+o(1))\frac{1}{(\beta-1)}\left(1-\frac{1}{2^{\beta-1}}\right)\frac{1}{x^{\beta-1}\log x} \tag{2.15}$$

as x tends to infinity. Combining (2.11) and (2.15), we are led to the formula

$$\sum_{x<p\le 2x} \frac{1}{p^\beta} = (1+o(1))\frac{C_\beta}{x^{\beta-1}\log x} + O_\beta\left(\frac{1}{x^{\beta-1}e^{c_1\sqrt{\log x}}}\right) \tag{2.16}$$

as x tends to infinity, where C_β is defined as in the statement of Lemma 2.1. Finally, we observe that

$$\sum_{x<p\le 2x} \frac{1}{p^{\beta-1}(p-1)} = \sum_{x<p\le 2x} \frac{1}{p^\beta} + \sum_{x<p\le 2x} \frac{1}{p^\beta(p-1)} \tag{2.17}$$

and

$$\sum_{x<p\le 2x} \frac{1}{p^\beta(p-1)} = O\left(\sum_{x<p\le 2x} \frac{1}{p^{\beta+1}}\right) = O\left(\frac{1}{x^\beta}\right). \tag{2.18}$$

Proof of Lemma 2.1 is now complete from (2.16)–(2.18). □

Next we have a series of technical lemmas designed to cope with the necessary estimates for Theorems 1.1–1.4 whose proofs require to control weighted averages of discrepancy of prime counting functions along arithmetic progressions with varying large moduli.

Lemma 2.2. *Assume $k \ge 2$ and $m \le (\log x)^B$ for some $B > 0$. Let $X = x/m$. Then we have*

$$\sum_{q\le X^{\frac{1}{k}}} q^{k-1-\epsilon} \max_{(a,mq)=1}\left|\psi(x,mq^k,a) - \frac{x}{\phi(mq^k)}\right| \ll \frac{x}{(\log x)^A} \tag{2.19}$$

for any fixed $\epsilon > 0$ and $A > 0$.

Proof. For any fixed $A > 0$, we decompose the sum on the left hand side of (2.19) as

$$\sum_{q\le(\log x)^A} + \sum_{(\log x)^A<q\le X^{\frac{1}{k}}}. \tag{2.20}$$

When $q \le (\log x)^A$, $mq^k \le (\log x)^{kA+B}$ so that by the Siegel-Walfisz theorem,

$$\begin{aligned}\sum_{q\le(\log x)^A} q^{k-1-\epsilon} \max_{(a,mq)=1}\left|\psi(x,mq^k,a) - \frac{x}{\phi(mq^k)}\right|&\\ \ll xe^{-c_1\sqrt{\log x}} \sum_{q\le(\log x)^A} q^{k-1-\epsilon} \ll xe^{-c_1\sqrt{\log x}}(\log x)^{kA} &\ll xe^{-c_1\sqrt{\log x}}\end{aligned} \tag{2.21}$$

for some constant $c_1 > 0$ which may be different at each appearance. Next, using $\phi(mq^k) \ge \phi(m)\phi(q^k) \ge \phi(q^k)$, and employing the trivial estimate

$$\left|\psi(x,mq^k,a) - \frac{x}{\phi(mq^k)}\right| \le \psi(x,mq^k,a) + \frac{x}{\phi(mq^k)} \ll \frac{x\log x}{q^k} + \frac{x}{q^k-q^{k-1}} \ll \frac{x\log x}{q^k},$$

we see that

$$\sum_{(\log x)^A < q \le X^{\frac{1}{k}}} q^{k-1-\epsilon} \max_{(a,mq)=1} \left| \psi(x, mq^k, a) - \frac{x}{\phi(mq^k)} \right|$$
$$\ll x \log x \sum_{q > (\log x)^A} \frac{1}{q^{1+\epsilon}} \ll \frac{x}{(\log x)^A} \tag{2.22}$$

holds for any fixed $A > 0$ which may be different at each appearance. The proof of Lemma 2.2 is complete from (2.20)–(2.22). □

Let us remark that a strengthening of Lemma 2.2 to the case $\epsilon = 0$ would have remarkable consequences. For example, this could serve as a replacement of the Elliot-Halberstam conjecture (see [4]) adapted to arithmetic progressions having modulus that are perfect powers.

Lemma 2.3. *Assume $k \ge 2$ and $m \le (\log x)^B$ for some $B > 0$. Let $X = x/m$. If there are no Siegel zeros for Dirichlet L-functions, then for any fixed $M > 1$ and $0 < \nu < 1$, we have*

$$\sum_{1 < q \le X^{\frac{1}{k}}} \frac{q^{k-1}}{(\log q)^M} \max_{(a,mq)=1} \left| \psi(x, mq^k, a) - \frac{x}{\phi(mq^k)} \right| \ll \frac{x}{(\log x)^{(M-1)\nu - 1}}. \tag{2.23}$$

Proof. For $0 < \nu < 1$, let us decompose the sum on the left hand side of (2.23) in the form

$$\sum_{q \le e^{(\log x)^\nu}} + \sum_{e^{(\log x)^\nu} < q \le X^{\frac{1}{k}}}. \tag{2.24}$$

By results of Rodosskii [11], [12] and Tatuzawa [16] (see Chapter 9, Satz 2.1 of [10]), when $(a, q) = 1$, we know under the absence of Siegel zeros that

$$\psi(x, q, a) = \frac{x}{\phi(q)} + O\left(\frac{x}{\phi(q)} \exp\left(-c_1 \frac{\log x}{\triangle} \right) \right) \tag{2.25}$$

holds uniformly for

$$q \le \exp\left(\frac{c_0 \log x}{\log\log x} \right),$$

where the constant $c_1 > 0$ depends only on $c_0 > 0$,

$$\triangle = \max(\log q, (\log x \log\log x)^{\frac{3}{7}}). \tag{2.26}$$

To treat the first sum in (2.24), note that since

$$mq^k \le (\log x)^B e^{k(\log x)^\nu} \le \exp\left(\frac{c_0 \log x}{\log\log x} \right)$$

for all sufficiently large x, we are permitted to apply (2.25) and (2.26) to deduce that

$$\sum_{1 < q \le e^{(\log x)^\nu}} \frac{q^{k-1}}{(\log q)^M} \max_{(a,mq)=1} \left| \psi(x, mq^k, a) - \frac{x}{\phi(mq^k)} \right|$$
$$\ll x e^{-c_1 (\log x)^{\nu'}} \sum_{1 < q \le e^{(\log x)^\nu}} \frac{q^{k-1}}{(\log q)^M \phi(mq^k)} \ll x e^{-c_1 (\log x)^{\nu'}} \tag{2.27}$$

for any $0 < \nu' < \min(1 - \nu, 4/7)$ as $M > 1$, and consequently the series

$$\sum_{q > 1} \frac{q^{k-1}}{(\log q)^M \phi(mq^k)}$$

is convergent. However, again by the trivial estimate, we also have

$$\sum_{e^{(\log x)^{\nu}}<q\le X^{\frac{1}{k}}} \frac{q^{k-1}}{(\log q)^M} \max_{(a,mq)=1} \left|\psi(x,mq^k,a)-\frac{x}{\phi(mq^k)}\right| \ll x\log x \sum_{q>e^{(\log x)^{\nu}}} \frac{1}{q(\log q)^M} \ll \frac{x}{(\log x)^{(M-1)\nu-1}}. \tag{2.28}$$

Assembling (2.24), (2.27) and (2.28), one completes the proof of Lemma 2.3. □

Lemma 2.4. *Assume $k\ge 2$ and $m\le e^{\alpha\sqrt{\log x}}$ for any fixed $\alpha>0$. Let $X=x/m$. If there are no Siegel zeros for Dirichlet L-functions, then for any $\epsilon>0$, we have*

$$\sum_{q\le X^{\frac{1}{k}}} q^{k-1-\epsilon} \max_{(a,mq)=1} \left|\psi(x,mq^k,a)-\frac{x}{\phi(mq^k)}\right| \ll xe^{-c'\sqrt{\log x}} \tag{2.29}$$

for some constant $c'>0$ depending only on ϵ.

Proof. We decompose the sum on the left hand side of (2.29) as

$$\sum_{q\le e^{\sqrt{\log x}}} + \sum_{e^{\sqrt{\log x}}<q\le X^{\frac{1}{k}}}. \tag{2.30}$$

To treat the first sum in (2.30), note that when x is large enough

$$mq^k \le e^{(\alpha+k)\sqrt{\log x}} \le \exp\left(\frac{c_0\log x}{\log\log x}\right)$$

for $c_0>0$, and we may again use (2.25) and (2.26) under the absence of Siegel zeros to obtain that

$$\sum_{q\le e^{\sqrt{\log x}}} q^{k-1-\epsilon} \max_{(a,mq)=1} \left|\psi(x,mq^k,a)-\frac{x}{\phi(mq^k)}\right| \ll xe^{-c_1'\sqrt{\log x}} \sum_{q\le e^{\sqrt{\log x}}} \frac{q^{k-1-\epsilon}}{\phi(mq^k)} \ll xe^{-c_1'\sqrt{\log x}} \tag{2.31}$$

for some constant $c_1'>0$, since the series

$$\sum_{q\ge 1} \frac{q^{k-1-\epsilon}}{\phi(mq^k)}$$

is convergent. On the other hand, employing the trivial estimate, we have

$$\sum_{e^{\sqrt{\log x}}<q\le X^{\frac{1}{k}}} q^{k-1-\epsilon} \max_{(a,mq)=1} \left|\psi(x,mq^k,a)-\frac{x}{\phi(mq^k)}\right| \ll x\log x \sum_{q>e^{\sqrt{\log x}}} \frac{1}{q^{1+\epsilon}} \ll xe^{-\frac{\epsilon}{2}\sqrt{\log x}} \tag{2.32}$$

Finally, combining (2.30)–(2.32), one verifies (2.29). □

Lemma 2.5. *Assume that all zeros of all Dirichlet L-functions have real part $\le\theta$ for some $1/2\le\theta<1$. Let $X=x/m$. If $k\ge 2$ and $m\le x^{\mu}$ for some $0\le\mu<\theta$, then for any $\epsilon>0$, we have*

$$\sum_{q\le X^{\frac{1}{k}}} q^{k-1-\epsilon} \max_{(a,mq)=1} \left|\psi(x,mq^k,a)-\frac{x}{\phi(mq^k)}\right| \ll x^{1-\frac{(1-\theta)\epsilon}{k}}(\log x)^2 \tag{2.33}$$

Proof. For $0 < \lambda < \frac{1-\mu}{k}$, we write the sum on the left hand side of (2.33) in the form

$$\sum_{q \le x^\lambda} + \sum_{x^\lambda < q \le X^{\frac{1}{k}}} . \tag{2.34}$$

Using the hypothesis that all zeros of all Dirichlet L-functions have real part $\le \theta$ for some $1/2 \le \theta < 1$, we know that

$$\max_{(a,mq)=1} \left| \psi(x, mq^k, a) - \frac{x}{\phi(mq^k)} \right| \ll x^\theta (\log x)^2 \tag{2.35}$$

holds uniformly for all $q \le X^{\frac{1}{k}}$. Thus (2.35) leads to the estimate

$$\sum_{q \le x^\lambda} q^{k-1-\epsilon} \max_{(a,mq)=1} \left| \psi(x, mq^k, a) - \frac{x}{\phi(mq^k)} \right| \ll x^\theta (\log x)^2 \sum_{q \le x^\lambda} q^{k-1-\epsilon} \ll x^{\theta + \lambda(k-\epsilon)} (\log x)^2. \tag{2.36}$$

But by the trivial estimate, one also obtains

$$\sum_{x^\lambda < q \le X^{\frac{1}{k}}} q^{k-1-\epsilon} \max_{(a,mq)=1} \left| \psi(x, mq^k, a) - \frac{x}{\phi(mq^k)} \right| \ll x \log x \sum_{q > x^\lambda} \frac{1}{q^{1+\epsilon}} \ll x^{1-\lambda\epsilon} \log x. \tag{2.37}$$

Choosing $\lambda = \frac{1-\theta}{k} < \frac{1-\mu}{k}$, we verify that (2.33) follows from (2.34), (2.36) and (2.37). $\square$

3. Proof of Theorem 1.1

Given positive integers a, h, m such that $a \le h \le m \le (\log x)^B$, $(h, m) = 1 = (h-a, m)$, and $k \ge 2$, we first consider the sum

$$\mathcal{S}_{x,y} := \sum_{\substack{n \le x-a \\ n \equiv h-a \pmod m}} \Lambda(n+a) \sum_{\substack{y < q \le 2y \\ q^k \mid n \\ q \text{ prime}}} 1, \tag{3.1}$$

where y is a parameter that will be chosen in terms of x later. Clearly, (3.1) can be rewritten as

$$\mathcal{S}_{x,y} = \sum_{\substack{y < q \le 2y \\ q \text{ prime}}} \left(\sum_{\substack{u \le x \\ u \equiv h \pmod m \\ u \equiv a \pmod{q^k}}} \Lambda(u) \right). \tag{3.2}$$

Note that we should have $(m, q) = 1$, since otherwise using the fact that n is divisible by q^k, we have $(m, n) > 1$ and consequently $(h-a, m) > 1$, contrary to our assumption. Moreover, $(a, q) = 1$, since otherwise by the fact that q is prime and a is divisible by q, we have

$$y < q \le a \le h \le m \le (\log x)^B$$

which is not possible as soon as $y \ge a$. Thus to obtain the desired uniformity over a, it is necessary to take y at least as big as m. It follows that we may further assume in (3.2) without loss of generality that $(m, q) = 1 = (a, q)$. At the same time, the congruences

$$u \equiv h \pmod m, \ u \equiv a \pmod{q^k}$$

can be combined to a single congruence of the form

$$u \equiv r_{a,h} \pmod{mq^k}$$

for some $(r_{a,h}, mq^k) = 1$. According to the above observations, we obtain

$$\mathcal{S}_{x,y} = \sum_{\substack{y<q\le 2y \\ q \text{ prime}}} \psi(x, mq^k, r_{a,h}) = \frac{x}{\phi(m)} \sum_{\substack{y<q\le 2y \\ q \text{ prime}}} \frac{1}{q^{k-1}(q-1)} + \sum_{\substack{y<q\le 2y \\ q \text{ prime}}} \left(\psi(x, mq^k, r_{a,h}) - \frac{x}{\phi(mq^k)}\right). \tag{3.3}$$

First, applying Lemma 2.1 with $\beta = k$, we get

$$\frac{x}{\phi(m)} \sum_{\substack{y<q\le 2y \\ q \text{ prime}}} \frac{1}{q^{k-1}(q-1)} = \frac{(1+o(1))C_k x}{\phi(m) y^{k-1} \log y} \tag{3.4}$$

as x tends to infinity. Using (2.19) in Lemma 2.2 with $\epsilon = 1$, one deduces that

$$y^{k-2} \left| \sum_{\substack{y<q\le 2y \\ q \text{ prime}}} \left(\psi(x, mq^k, r_{a,h}) - \frac{x}{\phi(mq^k)}\right) \right| \le \sum_{y<q\le 2y} q^{k-2} \max_{(r,mq)=1} \left| \psi(x, mq^k, r) - \frac{x}{\phi(mq^k)} \right| \ll \frac{x}{(\log x)^A} \tag{3.5}$$

for every $A > 0$ when $m \le (\log x)^B$ and $2y \le (x/m)^{\frac{1}{k}}$. It is evident from (3.5) that

$$\left| \sum_{\substack{y<q\le 2y \\ q \text{ prime}}} \left(\psi(x, mq^k, r_{a,h}) - \frac{x}{\phi(mq^k)}\right) \right| \ll \frac{x}{y^{k-2}(\log x)^A}, \tag{3.6}$$

Putting (3.3)–(3.6) together, we arrive at the formula

$$\mathcal{S}_{x,y} = \frac{(1+o(1))C_k x}{\phi(m) y^{k-1} \log y} + O\left(\frac{x}{y^{k-2}(\log x)^A}\right) \tag{3.7}$$

for every $A > 0$. Given any $\alpha > 0$, we take $y = (\log x)^\alpha$. If $A > \alpha + B$, then

$$\phi(m) y \log y = o((\log x)^A)$$

and (3.7) reduces to

$$\mathcal{S}_{x,y} = \frac{(1+o(1))C_k x}{\phi(m) y^{k-1} \log y} = \frac{(1+o(1))C_k x}{\alpha \phi(m) (\log x)^{(k-1)\alpha} \log\log x} \tag{3.8}$$

for fixed a and any given $\alpha > 0$ when x tends to infinity. Of course (3.8) uniformly holds for all a with $a \le h \le m \le (\log x)^B$ with $(h,m) = 1 = (h-a, m)$, provided we take $\alpha \ge B$ so that $y = (\log x)^\alpha \ge m$. This choice of y is feasible and obviously satisfies $2y \le (x/m)^{\frac{1}{k}}$ when x is large enough. To justify the relevance of (3.1) to our problem, we know that the number of prime numbers $p \le x$, $p \equiv h \pmod m$ with $p = n + a = q^k s + a$ for some prime number $q \ge (\log n)^\alpha$, so equivalently that

$$s \le \frac{n}{(\log n)^{k\alpha}},$$

is at least

$$\sum_{\substack{n\le x-a\\ n+a\equiv h \pmod m\\ n+a \text{ prime}}} \left(\frac{1}{d(n)} \sum_{\substack{q\ge(\log n)^\alpha\\ q^k\mid n\\ q \text{ prime}}} 1\right) \ge$$

$$\frac{1}{D(x)\log x}\left(\sum_{\substack{n\le x-a\\ n+a\equiv h \pmod m}} \Lambda(n+a) \sum_{\substack{q\ge(\log n)^\alpha\\ q^k\mid n\\ q \text{ prime}}} 1 - \sum_{\substack{n\le x-a\\ n+a\equiv h \pmod m\\ n+a \text{ not prime}}} \Lambda(n+a) \sum_{\substack{q\ge(\log n)^\alpha\\ q^k\mid n\\ q \text{ prime}}} 1\right). \tag{3.9}$$

Clearly we have

$$\sum_{\substack{n\le x-a\\ n\equiv h-a \pmod m}} \Lambda(n+a) \sum_{\substack{y<q\le 2y\\ q^k\mid n\\ q \text{ prime}}} 1 \le \sum_{\substack{n\le x-a\\ n+a\equiv h \pmod m}} \Lambda(n+a) \sum_{\substack{q\ge(\log n)^\alpha\\ q^k\mid n\\ q \text{ prime}}} 1. \tag{3.10}$$

Moreover, by Chebyshev estimates, it holds that

$$\begin{aligned}\sum_{\substack{n\le x-a\\ n+a\equiv h \pmod m\\ n+a \text{ not prime}}} \Lambda(n+a) \sum_{\substack{q\ge(\log n)^\alpha\\ q^k\mid n\\ q \text{ prime}}} 1 &\le \sum_{\substack{n\le x-a\\ n+a\equiv h \pmod m\\ n+a \text{ not prime}}} \Lambda(n+a)d(n)\\ &\le D(x)\log x \sum_{\substack{n\le x-a\\ n+a=r^k, k\ge 2\\ r \text{ prime}}} 1 \ll \sqrt{x}D(x). \end{aligned} \tag{3.11}$$

Combining (3.9)–(3.11), we justify the importance of (3.1) with

$$\pi_{a,k,\alpha}(x,m,h) \ge \frac{\mathcal{S}_{x,y}}{D(x)\log x} + O\left(\frac{\sqrt{x}}{\log x}\right). \tag{3.12}$$

From (3.8) and (3.12), we derive the lower bound

$$\pi_{a,k,\alpha}(x,m,h) \ge \frac{(1+o(1))C_k x}{\alpha\phi(m)D(x)(\log x)^{1+(k-1)\alpha}\log\log x} \tag{3.13}$$

as x tends to infinity. Note that (1.3) is another way of expressing (3.13). For all large r in terms of $\lambda > 0$, it is well-known that (see Theorem 317 in [7])

$$d(r) \le \exp\left(\frac{(1+\lambda)\log 2\log r}{\log\log r}\right).$$

It follows from this that

$$D(x) \le \exp\left(\frac{(1+\lambda)\log 2\log x}{\log\log x}\right) \tag{3.14}$$

for all large x in terms of λ. Since the upper bound for $D(x)$ in (3.14) is the most dominant term apart from x when placed on the righthand side of (3.13) instead of $D(x)$, we deduce (1.5) from (3.13) and (3.14). This completes the proof.

4. Proof of Theorem 1.2

We again consider $\mathcal{S}_{x,y}$ from (3.1). Using $m \le (\log x)^B$ and Lemma 2.1, we have

$$\mathcal{S}_{x,y} = \frac{(1+o(1))C_k x}{\phi(m)y^{k-1}\log y} + \sum_{\substack{y<q\le 2y \\ q \text{ prime}}} \left(\psi(x, mq^k, r_{a,h}) - \frac{x}{\phi(mq^k)}\right) \tag{4.1}$$

as x tends to infinity, where we require $(r_{a,h}, mq^k) = 1$ and $y \ge m$. Assuming the nonexistence of Siegel zeros, we are allowed to apply (2.23) in Lemma 2.3 to get that

$$\left| \sum_{\substack{y<q\le 2y \\ q \text{ prime}}} \left(\psi(x, mq^k, r_{a,h}) - \frac{x}{\phi(mq^k)}\right) \right| \ll \frac{(\log y)^M x}{y^{k-1}(\log x)^{(M-1)\nu-1}} \tag{4.2}$$

for any $M > 1$ and $0 < \nu < 1$ provided $2y \le (x/m)^{\frac{1}{k}}$. From (4.1) and (4.2), we obtain

$$\mathcal{S}_{x,y} = \frac{(1+o(1))C_k x}{\phi(m)y^{k-1}\log y} + O\left(\frac{(\log y)^M x}{y^{k-1}(\log x)^{(M-1)\nu-1}}\right). \tag{4.3}$$

For any fixed $0 < \alpha < 1$, we take $y = e^{(\log x)^\alpha}$ so that $y \ge m$ and $2y \le (x/m)^{\frac{1}{k}}$ hold when x is large enough since $m \le (\log x)^B$. If $\alpha < \nu < 1$, then

$$(M+1)\alpha < (M-1)\nu - B - 1$$

is satisfied when M is large enough. This inequality guarantees that

$$\phi(m)(\log y)^{M+1} = o\left((\log x)^{(M-1)\nu-1}\right),$$

and consequently, (4.3) becomes

$$\mathcal{S}_{x,y} = \frac{(1+o(1))C_k x}{\phi(m)e^{(k-1)(\log x)^\alpha}(\log x)^\alpha} \tag{4.4}$$

as x tends to infinity. Exactly as in (3.12), one derives

$$\pi_{a,k,\alpha}(x, m, h) \ge \frac{\mathcal{S}_{x,y}}{D(x)\log x} + O\left(\frac{\sqrt{x}}{\log x}\right), \tag{4.5}$$

where $\pi_{a,k,\alpha}(x, m, h)$ is the number of prime numbers $p \le x$, $p \equiv h \pmod m$ with $p = n + a = q^k s + a$ for some prime number $q \ge e^{(\log n)^\alpha}$, so equivalently that

$$s \le n e^{-k(\log n)^\alpha}.$$

We infer from (4.4) and (4.5) that

$$\pi_{a,k,\alpha}(x, m, h) \ge \frac{(1+o(1))C_k x}{\phi(m)D(x)e^{(k-1)(\log x)^\alpha}(\log x)^{\alpha+1}} \tag{4.6}$$

as x tends to infinity and (1.6) holds as a consequence of (4.6). Using $\phi(m) \le (\log x)^B$, the upper bound for $D(x)$ in (3.14) is the most dominant term besides x in (4.6), and therefore (1.7) follows. This completes the proof of Theorem 1.2.

5. Proof of Theorem 1.3

The proof of Theorem 1.3 proceeds similarly as in the proof of Theorem 1.2. Using Lemma 2.1, one gets

$$\mathcal{S}_{x,y} = \frac{(1+o(1))C_k x}{\phi(m)y^{k-1}\log y} + \sum_{\substack{y<q\le 2y \\ q \text{ prime}}} \left(\psi(x, mq^k, r_{a,h}) - \frac{x}{\phi(mq^k)}\right) \tag{5.1}$$

when x tends to infinity, where $(r_{a,h}, mq^k) = 1$ and $y \geq m$. Since there are no Siegel zeros by assumption, we are permitted to apply (2.29) in Lemma 2.4 with $\epsilon = 1$ and obtain that

$$\left| \sum_{\substack{y<q\leq 2y \\ q \text{ prime}}} \left(\psi(x, mq^k, r_{a,h}) - \frac{x}{\phi(mq^k)} \right) \right| \ll \frac{x}{y^{k-2} e^{c_2\sqrt{\log x}}} \tag{5.2}$$

for some absolute constant $c_2 > 0$ provided $2y \leq (x/m)^{\frac{1}{k}}$. It follows from (5.1) and (5.2) that

$$\mathcal{S}_{x,y} = \frac{(1+o(1))C_k x}{\phi(m) y^{k-1} \log y} + O\left(\frac{x}{y^{k-2} e^{c_2\sqrt{\log x}}} \right). \tag{5.3}$$

We take $c_1 = c_2/2$ and $y = e^{c_1\sqrt{\log x}}$. Since $m = o\left(\frac{e^{c_1\sqrt{\log x}}}{\sqrt{\log x}} \right)$, $y \geq m$ and $2y \leq (x/m)^{\frac{1}{k}}$ are satisfied when x is large enough, we have

$$\phi(m) y \log y = o(e^{c_2\sqrt{\log x}}).$$

Thus (5.3) can be brought to the form

$$\mathcal{S}_{x,y} = \frac{(1+o(1))C_k x}{c_1 \phi(m) e^{c_1(k-1)\sqrt{\log x}} \sqrt{\log x}} \tag{5.4}$$

as x tends to infinity. As before, we also have

$$\pi_{a,k}(x, m, h) \geq \frac{\mathcal{S}_{x,y}}{D(x) \log x} + O\left(\frac{\sqrt{x}}{\log x} \right), \tag{5.5}$$

where $\pi_{a,k}(x, m, h)$ is the number of prime numbers $p \leq x$, $p \equiv h \pmod m$ with $p = n + a = q^k s + a$ for some prime number $q \geq e^{c_1\sqrt{\log n}}$, so equivalently that

$$s \leq n e^{-c_1 k\sqrt{\log n}}.$$

One deduces from (5.4) and (5.5) that

$$\pi_{a,k}(x, m, h) \geq \frac{(1+o(1))C_k x}{c_1 \phi(m) D(x) e^{c_1(k-1)\sqrt{\log x}} (\log x)^{\frac{3}{2}}} \tag{5.6}$$

when x tends to infinity and (1.8) is immediate from (5.6). Using (3.14) and

$$\phi(m) \leq \frac{e^{c_1\sqrt{\log x}}}{\sqrt{\log x}}$$

when x is large enough, we also verify (1.9) from (5.6). This completes the proof of Theorem 1.3.

6. Proof of Theorem 1.4

Once again, by Lemma 2.1, we have

$$\mathcal{S}_{x,y} = \frac{(1+o(1))C_k x}{\phi(m) y^{k-1} \log y} + \sum_{\substack{y<q\leq 2y \\ q \text{ prime}}} \left(\psi(x, mq^k, r_{a,h}) - \frac{x}{\phi(mq^k)} \right) \tag{6.1}$$

when x tends to infinity, where $(r_{a,h}, mq^k) = 1$ and $y \geq m$. Applying (2.33) from Lemma 2.5 with $\epsilon = 1$, the estimate

$$\left| \sum_{\substack{y<q\leq 2y \\ q \text{ prime}}} \left(\psi(x, mq^k, r_{a,h}) - \frac{x}{\phi(mq^k)} \right) \right| \ll \frac{x^{1-\frac{1-\theta}{k}} (\log x)^2}{y^{k-2}} \tag{6.2}$$

follows, where $2y \le (x/m)^{\frac{1}{k}}$, $m \le x^{\mu}$ with $0 \le \mu < \theta$ and by our assumption $1/2 \le \theta < 1$ is such that $\Re(s) \le \theta$ for any zero s of any Dirichlet L-function. But then assembling (6.1) and (6.2), we see that

$$\mathcal{S}_{x,y} = \frac{(1+o(1))C_k x}{\phi(m) y^{k-1} \log y} + O\left(\frac{x^{1-\frac{1-\theta}{k}}(\log x)^2}{y^{k-2}}\right). \tag{6.3}$$

For $0 \le \mu < \frac{1-\theta}{k}$ (note that this already implies the condition $0 \le \mu < \theta$ arising from Lemma 2.5), we are allowed to take

$$y = \frac{x^{\frac{1-\theta}{k}-\mu}}{(\log x)^3 f(x)}, \tag{6.4}$$

where $f(x)$ is a function tending to infinity however slowly. Then note that when x is large enough,

$$2y \le x^{\frac{1-\mu}{k}} \le (x/m)^{\frac{1}{k}}$$

is satisfied and our results will be uniform in the shift parameter a as soon as $y \ge m$ and therefore when $0 < \mu < \frac{1-\theta}{2k}$ as $m \le x^{\mu}$. Consequently, (6.4) gives that

$$\phi(m) y \log y = o\left(\frac{x^{\frac{1-\theta}{k}}}{(\log x)^2}\right)$$

and (6.3) becomes

$$\mathcal{S}_{x,y} = \frac{(1+o(1))C_k x^{\frac{1+(k-1)\theta}{k}+(k-1)\mu}(\log x)^{3k-4} f(x)^{k-1}}{\left(\frac{1-\theta}{k}-\mu\right)\phi(m)} \tag{6.5}$$

as x tends to infinity. Moreover, we have

$$\pi_{a,k,\theta,\mu,f}(x,m,h) \ge \frac{\mathcal{S}_{x,y}}{D(x)\log x} + O\left(\frac{\sqrt{x}}{\log x}\right), \tag{6.6}$$

where $\pi_{a,k,\theta,\mu,f}(x,m,h)$ is the number of prime numbers $p \le x$, $p \equiv h \pmod{m}$ with $p = n + a = q^k s + a$ for some prime number

$$q \ge \frac{n^{\frac{1-\theta}{k}-\mu}}{(\log n)^3 f(n)}$$

so equivalently that

$$s \le n^{\theta + k\mu}(\log n)^{3k} f(n)^k.$$

Note that

$$\frac{x^{\frac{1+(k-1)\theta}{k}+(k-1)\mu}}{\phi(m)} \ge x^{\frac{1+(k-1)\theta}{k}+(k-2)\mu}$$

and the exponent

$$\frac{1+(k-1)\theta}{k} + (k-2)\mu$$

exceeds a half. Therefore the O-term on the righthand side of (6.6) can be neglected, and we obtain from (6.5) and (6.6) that

$$\pi_{a,k,\theta,\mu,f}(x,m,h) \ge \frac{(1+o(1))C_k x^{\frac{1+(k-1)\theta}{k}+(k-1)\mu}(\log x)^{3k-5} f(x)^{k-1}}{\left(\frac{1-\theta}{k}-\mu\right)\phi(m) D(x)}. \tag{6.7}$$

Thus (1.10) holds by (6.7). Since the upper bound for $D(x)$ in (3.14) is the most dominant term on the righthand side of (6.7) apart from $x^{\frac{1+(k-1)\theta}{k}+(k-1)\mu}$ and $\phi(m)$, (1.11) follows. This completes the proof of Theorem 1.4.

7. Proof of Theorem 1.5

We start by considering the relevant sum

$$\mathcal{S}^*_{x,y} := \sum_{n\le x-a}\Lambda(n+a)\sum_{\substack{y<q\le 2y\\ q^k\mid n\\ q\ \text{prime}}}1 = \sum_{\substack{y<q\le 2y\\ q\ \text{prime}}}\left(\sum_{\substack{u\le x\\ u\equiv a \pmod{q^k}}}\Lambda(u)\right). \tag{7.1}$$

Note that as $a \le x^{\frac{1-\epsilon}{k}}$, we see that $(a,q)=1$ when $y \ge x^{\frac{1-\epsilon}{k}}$. Thus, applying Lemma 2.1, (7.1) becomes

$$\mathcal{S}^*_{x,y} = \frac{(1+o(1))C_k x}{y^{k-1}\log y} + \sum_{\substack{y<q\le 2y\\ q\ \text{prime}}}\left(\psi(x,q^k,a) - \frac{x}{\phi(q^k)}\right). \tag{7.2}$$

Assuming Conjecture 3, we have for every $\epsilon>0$ that

$$\left|\sum_{\substack{y<q\le 2y\\ q\ \text{prime}}}\left(\psi(x,q^k,a) - \frac{x}{\phi(q^k)}\right)\right| \ll_\epsilon x^{\frac{1}{2}+\frac{\epsilon}{3}}\sum_{\substack{y<q\le 2y\\ q\ \text{prime}}}\frac{1}{q^{\frac{k}{2}}} \tag{7.3}$$

provided $2y \le x^{\frac{1}{k}}$. If $k\ge 3$, then we obtain from (2.16) that

$$\sum_{\substack{y<q\le 2y\\ q\ \text{prime}}}\frac{1}{q^{\frac{k}{2}}} \ll \frac{1}{y^{\frac{k}{2}-1}\log y}. \tag{7.4}$$

Moreover, if $k=2$, then by the Mertens formula

$$\sum_{\substack{y<q\le 2y\\ q\ \text{prime}}}\frac{1}{q} = \log\log 2y - \log\log y + O\left(\frac{1}{\log y}\right) \ll \frac{1}{\log y}. \tag{7.5}$$

It follows from (7.3)–(7.5) that

$$\left|\sum_{\substack{y<q\le 2y\\ q\ \text{prime}}}\left(\psi(x,q^k,a) - \frac{x}{\phi(q^k)}\right)\right| \ll_\epsilon \frac{x^{\frac{1}{2}+\frac{\epsilon}{3}}}{y^{\frac{k}{2}-1}\log y} \tag{7.6}$$

holds for every $\epsilon>0$ and $k\ge 2$ when $2y\le x^{\frac{1}{k}}$. Gathering (7.2) and (7.6), we see that

$$\mathcal{S}^*_{x,y} = \frac{(1+o(1))C_k x}{y^{k-1}\log y} + O\left(\frac{x^{\frac{1}{2}+\frac{\epsilon}{3}}}{y^{\frac{k}{2}-1}\log y}\right), \tag{7.7}$$

provided $x^{\frac{1-\epsilon}{k}} \le y \le \frac{x^{\frac{1}{k}}}{2}$. Note that, choosing

$$y = x^{\frac{1-\epsilon}{k}},$$

(7.7) becomes

$$\mathcal{S}^*_{x,y} = \frac{(1+o(1))kC_k x^{\frac{1+(k-1)\epsilon}{k}}}{(1-\epsilon)\log x}. \tag{7.8}$$

Similarly, as above, we can show that

$$\pi_{a,k,\epsilon}(x) \ge \frac{1}{D(x)\log x}\sum_{n\le x-a}\Lambda(n+a)\sum_{\substack{q\ge n^{\frac{1-\epsilon}{k}}\\ q^k\mid n\\ q\ \text{prime}}}1 + O\left(\frac{\sqrt{x}}{\log x}\right). \tag{7.9}$$

The O-term in (7.9) can be neglected if we impose the condition

$$\frac{1+(k-1)\epsilon}{k} > \frac{1}{2}. \tag{7.10}$$

Clearly, (7.10) is equivalent to

$$\epsilon > \frac{k-2}{2k-2}.$$

Assembling (7.8)–(7.10), one obtains

$$\pi_{a,k,\epsilon}(x) \geq \frac{(1+o(1))kC_k x^{\frac{1+(k-1)\epsilon}{k}}}{(1-\epsilon)D(x)(\log x)^2}. \tag{7.11}$$

Finally, (1.12) and (1.13) are easily verified from (7.11) with the use of (3.14). This completes the proof.

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