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UPPER BOUNDS FOR THE BLOW UP TIME FOR THE KIRCHHOFF-TYPE EQUATION

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Abstract. In this research, we take into account the Kirchhoff type equation with variable exponent. The Kirchhoff type equation is known as a kind of evolution equations,namely, PDEs, where t is an independent variable. This type problem can be extensively used in many mathematical models of various applied sciences such as flows of electrorheological fluids, thin liquid films, and so on. This research, we investigate the upper bound for blow up time under suitable conditions.

1. Introduction

In this work, we deal with the upper bounds of blow up time of solutions of the p-Kirchhoff type equation with variable exponent

$$\begin{cases} u_{tt} - M\left(\|\nabla u\|_p^p\right)\Delta_p u + |u_t|^{r(\cdot)-2} u_t = |u|^{q(\cdot)-2} u, & (x,t) \in \Omega \times (0,T), \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), & x \in \Omega, \\ u(x,t) = 0, & \partial\Omega \times (0,T), \end{cases} \tag{1}$$

where Ω is a bounded domain and this bounded has the smooth boundary $\partial\Omega$ in R^n $(n \geq 1)$. The term

$$\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) \quad \text{with} \quad p \geq 2$$

is called and p-Laplacian and

$$M(s) = 1 + s.$$

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The variable exponents $r(\cdot)$ and $q(\cdot)$ are taken as measurable functions on Ω satisfying

$$2 \leq r^- \leq r(x) \leq r^+ < q^- \leq q(x) \leq q^+ \leq q^*, \tag{2}$$

where

$$\begin{cases} r^- = ess\inf_{x\in\Omega} r(\cdot), & r^+ = ess\sup_{x\in\Omega} r(\cdot), \\ q^- = ess\inf_{x\in\Omega} q(\cdot), & q^+ = ess\sup_{x\in\Omega} q(\cdot), \end{cases}$$

and

$$q^* = \begin{cases} \infty, \text{ if } n = 1, 2, \\ \frac{2n}{n-2}, \text{ if } n \geq 3. \end{cases}$$

The problem (1) generalizes the model of Kirchhoff [8]. The Kirchhoff equation as an extension of the classical D'Alembert's wave equation for free vibrations of elastic strings. The equations with variable exponents can appear research fields such as image processing, nonlinear elasticity theory and electrorheological fluids [5,6,19].

In [14], the some problem (1) was studied. The authors proved the stability and the global existence of the solution with positive initial energy.

When $p \equiv 2$, (1) is reduced to following Kirchhoff equation

$$u_{tt} - M\left(\|\nabla u\|_2^2\right)\Delta u + u_t |u_t|^{r(\cdot)-2} = u|u|^{q(\cdot)-2}. \tag{3}$$

In [16], Pişkin studied Eq. (3) for the blow up of solutions.

When $M\left(\|\nabla u\|_2^2\right) \equiv 1$, (3) is reduced to following wave equation

$$u_{tt} - \Delta u + u_t |u_t|^{r(\cdot)-2} = u|u|^{q(\cdot)-2}. \tag{4}$$

In [13], Messaoudi et al. took into consideration equation Eq. (4). The authors discussed the local existence and the blow up of solutions.

Messaoudi and Talahmeh [12] considered the following quasilinear wave equation

$$u_{tt} - \operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) + u_t |u_t|^{r(x)-2} = u|u|^{q(x)-2}. \tag{5}$$

They proved a finite-time blow-up for the solutions with negative initial energy and for certain solutions with positive energy. Later, Li et al. [10] proved the asymptotic stability of solutions (5). Recently, some other authors studied hyperbolic type equations with variable exponents (see [1,2,4,11,17,18,20,21]).

Motivated by the above studies, in this work, we consider the upper bounds of blow up time of the solution (1) under suitable conditions.

This work is outlined as the following: In the section 2, we give some results about the variable exponent Sobolev spaces ($W^{1,p(\cdot)}(\Omega)$) and Lebesgue spaces ($L^{p(\cdot)}(\Omega)$). In the last section, the upper bounds of blow up time will be proved.

2. Preliminaries

In this part, we give some results in relation to the variable exponent spaces ($L^{p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$). For more details, see [6, 7, 9, 15].

Let Ω is a bounded domain of R^n, $p:\Omega\to[1,\infty)$ be a measurable function. We define the variable exponent Lebesgue space by

$$L^{p(\cdot)}(\Omega)=\left\{u:\Omega\to R,\ u \text{ is measurable and } \rho_{p(.)}(\lambda u)\ <\infty,\ \text{ for some } \lambda>0\right\}$$

where

$$\rho_{p(.)}(u)=\int_\Omega |u|^{p(x)}\,dx.$$

The space $L^{p(\cdot)}(\Omega)$ endowed with the norm (Luxemburg norm)

$$\|u\|_{p(\cdot)}=\inf\left\{\lambda>0:\int_\Omega\left|\frac{u}{\lambda}\right|^{p(x)}dx\le 1\right\},$$

$L^{p(x)}(\Omega)$ is a Banach space.

Next we define the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ as follows

$$W^{1,p(\cdot)}(\Omega)=\left\{u\in L^{p(x)}(\Omega):\nabla u \text{ exists and } |\nabla u|\in L^{p(x)}(\Omega)\right\}.$$

It can be seen the Sobolev space with the variable exponent is a Banach space with respect to the norm

$$\|u\|_{1,p(x)}=\|u\|_{p(x)}+\|\nabla u\|_{p(x)}.$$

The space $W_0^{1,p(x)}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$ with respect to the norm $\|u\|_{1,p(x)}$. For $u\in W_0^{1,p(x)}(\Omega)$, we define an equivalent norm

$$\|u\|_{1,p(x)}=\|\nabla u\|_{p(x)}.$$

The variable exponents $p(\cdot)$ and $q(\cdot)$ satisfy the log-Hölder continuity condition:

$$|p(x)-p(y)|\le-\frac{B}{\log|x-y|},\ \text{ for all } x,y\in\Omega \text{ and } |x-y|<\delta, \tag{6}$$

where $B>0$, $0<\delta<1$.

Lemma 1. *[6]. Suppose that (6) holds. Then*

$$\|u\|_{p(\cdot)}\le c\|\nabla u\|_{p(\cdot)},\ \textit{for all } u\in W_0^{1,p(\cdot)}(\Omega),$$

where $c=c(p^-,p^+,|\Omega|)>0$.

The above inequality is known as Poincare inequality.

Lemma 2. *[6]. Let $p(\cdot) \in C(\overline{\Omega})$ and $q : \Omega \to [1, \infty)$ be a measurable function and satisfy*

$$\underset{x \in \overline{\Omega}}{essinf} (p^*(x) - q(x)) > 0.$$

Then the Sobolev embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ is continuous and compact, here

$$p^*(x) = \begin{cases} \frac{np^-}{n-p^-}, & \text{if } p^- < n \\ \infty, & \text{if } p^- \geq n. \end{cases}$$

By combining arguments of [3,13], we have the following local existence theorem.

Theorem 1. *Suppose that (2) and (6) hold, and let $(u_0, u_1) \in W_0^{1,p}(\Omega) \times L^2(\Omega)$, then there exists a unique solution $u(x,t)$ of the problem (1), which satisfies*

$$\begin{aligned} u &\in L^\infty\left([0,T); W_0^{1,p}(\Omega)\right), \\ u_t &\in L^\infty\left([0,T); L^2(\Omega)\right) \cap L^{r(\cdot)}(\Omega \times (0,T)). \end{aligned}$$

3. Upper bounds for blow up time

In this part, we will proved the the upper bounds of blow up of solutions for the problem (1). Firstly, we give the following lemma.

Lemma 3. *[13]. If $q : \Omega \to [1, \infty)$ is a measurable function and*

$$2 \leq q^- \leq q(\cdot) \leq q^+ < \frac{2n}{n-2}; \quad n \geq 3 \tag{7}$$

hold, then we have the following estimates:

i)

$$\rho_{q(.)}^{\frac{s}{q^-}}(u) \leq c\left(\|\nabla u\|^2 + \rho_{q(.)}(u)\right), \tag{8}$$

ii)

$$\|u\|_{q^-}^s \leq c\left(\|\nabla u\|^2 + \|u\|_{q^-}^{q^-}\right), \tag{9}$$

iii)

$$\rho_{q(.)}^{\frac{s}{q^-}}(u) \leq c\left(|F(t)| + \|u_t\|^2 + \rho_{q(.)}(u)\right), \tag{10}$$

iv)

$$\|u\|_{q^-}^s \leq c\left(|F(t)| + \|u_t\|^2 + \|u\|_{q^-}^{q^-}\right), \tag{11}$$

v)

$$c\|u\|_{q^-}^{q^-} \leq \rho_{q(.)}(u) \tag{12}$$

for any $u \in H_0^1(\Omega)$ and $2 \leq s \leq q$, where $c > 1$ a positive constant and $F(t) = -E(t)$.

Now, the main result of this work is given in the following theorem.

Theorem 2. *Let the assumptions of Theorem 1 hold true and suppose that*

$$E(0) < 0.$$

Then the solution of the problem (1) blows up in finite time.

Proof. Multiplying the equation in the problem (1) by u_t and integrating on Ω , we obtain

$$\int_\Omega u_t u_{tt} dx - \int_\Omega u_t M\left(\|\nabla u\|_p^p\right) \Delta_p u dx + \int_\Omega u_t |u_t|^{r(x)-2} u_t dx = \int_\Omega u_t |u|^{q(x)-2} u dx.$$

If each term is calculated separetely and if these found terms are written in their place, we get this equality.

$$\frac{d}{dt}\left[\frac{1}{2}\|u_t\|^2 + \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{2p}\|\nabla u\|_p^{2p} - \int_\Omega \frac{1}{q(x)}|u|^{q(x)} dx\right] = -\int_\Omega |u_t|^{r(x)} dx,$$

$$E'(t) = -\|u_t\|_{r(x)}^{r(x)}, \tag{13}$$

where

$$E(t) = \frac{1}{2}\|u_t\|^2 + \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{2p}\|\nabla u\|_p^{2p} - \int_\Omega \frac{1}{q(x)}|u|^{q(x)} dx. \tag{14}$$

Set

$$F(t) = -E(t),$$

then $E(0) < 0$ and (13) gives $F(t) \geq F(0) > 0$. Also, by the definition $F(t)$, we get

$$\begin{aligned} F(t) &= -\frac{1}{2}\|u_t\|^2 - \frac{1}{p}\|\nabla u\|_p^p - \frac{1}{2p}\|\nabla u\|_p^{2p} + \int_\Omega \frac{1}{q(x)}|u|^{q(x)} dx \\ &\leq \int_\Omega \frac{1}{q(x)}|u|^{q(x)} dx \\ &\leq \frac{1}{q^-}\rho_{q(.)}(u). \end{aligned} \tag{15}$$

Define

$$\Phi(t) = F^{1-\sigma}(t) + \varepsilon \int_\Omega u u_t dx, \tag{16}$$

where ε small to be chosen later and

$$0 < \sigma \leq \min\left\{\frac{q^- - r^+}{(r^+ - 1) q^-}, \frac{q^- - 2}{2q^-}\right\}. \tag{17}$$

Calculation the derivative of (16) and using the equation in the problem (1), we find

$$\Phi'(t) = (1-\sigma) F^{-\sigma}(t) F'(t) + \varepsilon \int_\Omega \left(u_t^2 + u u_{tt}\right) dx$$

$$= (1-\sigma) F^{-\sigma}(t) F'(t) + \varepsilon \|u_t\|^2 - \varepsilon \|\nabla u\|_p^p - \varepsilon \|\nabla u\|_p^{2p} + \varepsilon \int_\Omega |u|^{q(x)} dx - \varepsilon \int_\Omega u u_t |u_t|^{r(x)-2} dx. \quad (18)$$

Use the definition of the $F(t)$, it follows that

$$-\varepsilon q^-(1-\mu) F(t) = \frac{\varepsilon q^-(1-\mu)}{2} \|u_t\|^2 + \frac{\varepsilon q^-(1-\mu)}{p} \|\nabla u\|_p^p + \frac{\varepsilon q^-(1-\mu)}{2p} \|\nabla u\|_p^{2p} - \varepsilon q^-(1-\mu) \int_\Omega \frac{1}{q(x)} |u|^{q(x)} dx, \quad (19)$$

where $0 < \mu < 1$.

Add and subtract (19) into (18), we obtain

$$\Phi'(t) \geq (1-\sigma) F^{-\sigma}(t) F'(t) + \varepsilon q^-(1-\mu) F(t) + \varepsilon \left(\frac{q^-(1-\mu)}{2} + 1 \right) \|u_t\|^2 + \varepsilon \left(\frac{q^-(1-\mu)}{p} - 1 \right) \|\nabla u\|_p^p + \varepsilon \left(\frac{q^-(1-\mu)}{2} - 1 \right) \|\nabla u\|_p^{2p} + \varepsilon \mu \int_\Omega |u|^{q(x)} dx - \varepsilon \int_\Omega u u_t |u_t|^{r(x)-2} dx. \quad (20)$$

Then, for μ small enough, we derive

$$\Phi'(t) \geq \varepsilon \beta \left[F(t) + \|u_t\|^2 + \|\nabla u\|_p^p + \|\nabla u\|_p^{2p} + \rho_{q(.)}(u) \right] + (1-\sigma) F^{-\sigma}(t) F'(t) - \varepsilon \int_\Omega u u_t |u_t|^{r(x)-2} dx, \quad (21)$$

where

$$\beta = \min\left\{ q^-(1-\mu),\ \varepsilon\mu,\ \frac{q^-(1-\mu)}{p} - 1, \frac{q^-(1-\mu)}{2} - 1,\ \frac{q^-(1-\mu)}{2} + 1 \right\} > 0$$

and

$$\rho_{q(\cdot)}(u) = \int_\Omega |u|^{q(\cdot)} dx.$$

By using Young inequality, the last term of (21) yields

$$XY \leq \frac{\delta^k X^k}{k} + \frac{\delta^{-l} Y^l}{l},$$

where $X, Y \geq 0,\ \delta > 0,\ k, l \in R^+$ such that $\frac{1}{k} + \frac{1}{l} = 1$. Hence, we have

$$\int_\Omega |u_t|^{r(x)-1} u dx \leq \int_\Omega \frac{1}{r(x)} \delta^{r(x)} |u|^{r(x)} dx + \int_\Omega \frac{r(x)-1}{r(x)} \delta^{-\frac{r(x)}{r(x)-1}} |u_t|^{r(x)} dx$$

$$\leq \frac{1}{r^-}\int_\Omega \delta^{r(x)}\left|u\right|^{r(x)}dx+\frac{r^+-1}{r^+}\int_\Omega \delta^{-\frac{r(x)}{r(x)-1}}\left|u_t\right|^{r(x)}dx, \quad (22)$$

where δ is constant depending on the time t and specified later. Using the inequality (22), we obtain from (21) that

$$\begin{aligned}\Phi'(t) &\geq \varepsilon\beta\left[F(t)+\|u_t\|^2+\|\nabla u\|_p^p+\|\nabla u\|_p^{2p}+\rho_{q(.)}(u)\right]\\ &\quad+(1-\sigma)F^{-\sigma}(t)F'(t)-\varepsilon\frac{1}{r^-}\int_\Omega \delta^{r(x)}\left|u\right|^{r(x)}dx\\ &\quad-\varepsilon\frac{r^+-1}{r^+}\int_\Omega \delta^{-\frac{r(x)}{r(x)-1}}\left|u_t\right|^{r(x)}dx. \quad (23)\end{aligned}$$

Therefore, by picking δ so that $\delta^{-\frac{r(x)}{r(x)-1}}=bF^{-\sigma}(t)$, where $b>0$ will be determined later, we have

$$\begin{aligned}\Phi'(t) &\geq \varepsilon\beta\left[F(t)+\|u_t\|^2+\|\nabla u\|_p^p+\|\nabla u\|_p^{2p}+\rho_{q(.)}(u)\right]\\ &\quad+(1-\sigma)F^{-\sigma}(t)F'(t)-\varepsilon\frac{1}{r^-}\int_\Omega b^{1-r(x)}F^{\sigma(r(x)-1)}(t)\left|u\right|^{r(x)}dx\\ &\quad-\varepsilon\frac{r^+-1}{r^+}\int_\Omega bF^{-\sigma}(t)\left|u_t\right|^{r(x)}dx\\ &\geq \varepsilon\beta\left[F(t)+\|u_t\|^2+\|\nabla u\|_p^p+\|\nabla u\|_p^{2p}+\rho_{q(.)}(u)\right]\\ &\quad+(1-\sigma)F^{-\sigma}(t)F'(t)-\varepsilon\frac{b^{1-r^-}}{r^-}F^{\sigma\left(r^+-1\right)}(t)\int_\Omega \left|u\right|^{r(x)}dx\\ &\quad-\varepsilon\left(\frac{r^+-1}{r^+}\right)bF^{-\sigma}(t)\int_\Omega \left|u_t\right|^{r(x)}dx\\ &\geq \varepsilon\beta\left[F(t)+\|u_t\|^2+\|\nabla u\|_p^p+\|\nabla u\|_p^{2p}+\rho_{q(.)}(u)\right]\\ &\quad+\left[(1-\sigma)-\varepsilon\left(\frac{r^+-1}{r^+}\right)b\right]F^{-\sigma}(t)F'(t)\\ &\quad-\varepsilon\frac{b^{1-r^-}}{r^-}F^{\sigma\left(r^+-1\right)}(t)\int_\Omega \left|u\right|^{r(x)}dx. \quad (24)\end{aligned}$$

By using (12) and (15), we get

$$F^{\sigma\left(r^+-1\right)}(t)\int_\Omega \left|u\right|^{r(x)}dx \leq F^{\sigma\left(r^+-1\right)}(t)\left[\int_{\Omega_-}\left|u\right|^{r^-}dx+\int_{\Omega_+}\left|u\right|^{r^+}dx\right]$$

$$\begin{aligned}
&\leq F^{\sigma\left(r^{+}-1\right)}(t)\, c\left[\left(\int_{\Omega_{-}} |u|^{q^{-}}\, dx\right)^{\frac{r^{-}}{q^{-}}} + \left(\int_{\Omega_{+}} |u|^{q^{-}}\, dx\right)^{\frac{r^{+}}{q^{-}}}\right] \\
&= F^{\sigma\left(r^{+}-1\right)}(t)\, c\left[\|u\|_{q^{-}}^{r^{-}} + \|u\|_{q^{-}}^{r^{+}}\right] \\
&\leq c\left(\frac{1}{q^{-}}\rho_{q(.)}(u)\right)^{\sigma\left(r^{+}-1\right)}\left[\left(\rho_{q(.)}(u)\right)^{\frac{r^{-}}{q^{-}}} + \left(\rho_{q(.)}(u)\right)^{\frac{r^{+}}{q^{-}}}\right] \\
&= c_1\left[\left(\rho_{q(.)}(u)\right)^{\frac{r^{-}}{q^{-}}+\sigma\left(r^{+}-1\right)} + \left(\rho_{q(.)}(u)\right)^{\frac{r^{+}}{q^{-}}+\sigma\left(r^{+}-1\right)}\right] \qquad (25)
\end{aligned}$$

where $\Omega_{-} = \{x \in \Omega : |u| < 1\}$ and $\Omega_{+} = \{x \in \Omega : |u| \geq 1\}$.

We then use Lemma 3 and (17), for

$$s = r^{-} + \sigma q^{-}\left(r^{+}-1\right) \leq q^{-}$$

and

$$s = r^{+} + \sigma q^{-}\left(r^{+}-1\right) \leq q^{-},$$

to deduce, from (25),

$$F^{\sigma\left(r^{+}-1\right)}(t)\int_{\Omega} |u|^{r(x)}\, dx \leq c_1\left[\|\nabla u\|^{2} + \rho_{q(.)}(u)\right]. \qquad (26)$$

Hence, substituting the inequality (26) into (24), we have

$$\begin{aligned}
\Phi'(t) \geq\ & \varepsilon\left(\beta - \frac{b^{1-r^{-}}}{r^{-}}c_1\right)\left[F(t) + \|u_t\|^{2} + \|\nabla u\|_{p}^{p} + \|\nabla u\|_{p}^{2p} + \rho_{q(.)}(u)\right] \\
& + \left[(1-\sigma) - \varepsilon\left(\frac{r^{+}-1}{r^{+}}\right)b\right]F^{-\sigma}(t)\, F'(t). \qquad (27)
\end{aligned}$$

As for coming step, let b large enough so that $\gamma = \beta - \frac{b^{1-r^{-}}}{r^{-}}c_1 > 0$, and choose ε small enough such that $(1-\sigma) - \varepsilon\left(\frac{r^{+}-1}{r^{+}}\right)b \geq 0$ and

$$\Phi(t) \geq \Phi(0) = F^{1-\sigma}(0) + \varepsilon\int_{\Omega} u_0 u_1 dx > 0,\ \forall t \geq 0. \qquad (28)$$

Consequently, (27) yields

$$\begin{aligned}
\Phi'(t) &\geq \varepsilon\gamma\left[F(t) + \|u_t\|^{2} + \|\nabla u\|_{p}^{p} + \|\nabla u\|_{p}^{2p} + \rho_{q(.)}(u)\right] \\
&\geq \varepsilon\gamma\left[F(t) + \|u_t\|^{2} + \|\nabla u\|_{p}^{p} + \|\nabla u\|_{p}^{2p} + \|u\|_{q^{-}}^{q^{-}}\right], \qquad (29)
\end{aligned}$$

due to (12). Therefore we get,

$$\Phi(t) \geq \Phi(0) > 0,\ \text{for all } t \geq 0.$$

Using the Hölder inequality, we have

$$\begin{aligned}\left|\int_{\Omega} uu_t dx\right|^{\frac{1}{1-\sigma}} &\leq \|u\|^{\frac{1}{1-\sigma}} \|u_t\|^{\frac{1}{1-\sigma}} \\ &\leq C\left(\|u\|_{q^-}^{\frac{1}{1-\sigma}} \|u_t\|^{\frac{1}{1-\sigma}}\right).\end{aligned}$$

Thanks to Young inequality, we get

$$\left|\int_{\Omega} uu_t dx\right|^{\frac{1}{1-\sigma}} \leq C\left(\|u\|_{q^-}^{\frac{\alpha}{1-\sigma}} + \|u_t\|^{\frac{\theta}{1-\sigma}}\right), \tag{30}$$

for $\frac{1}{\alpha} + \frac{1}{\theta} = 1$. We take $\theta = 2(1-\sigma)$, to obtain $\frac{\alpha}{1-\sigma} = \frac{2}{1-2\sigma} \leq q^-$ by (17). Therefore, (30) becomes

$$\left|\int_{\Omega} uu_t dx\right|^{\frac{1}{1-\sigma}} \leq C\left(\|u_t\|^2 + \|u\|_{q^-}^{s}\right),$$

where $\frac{2}{1-2\sigma} \leq q^-$. By using (11), we get

$$\left|\int_{\Omega} uu_t dx\right|^{\frac{1}{1-\sigma}} \leq C\left(\|u_t\|^2 + \|u\|_{q^-}^{q^-} + F(t)\right).$$

Thus,

$$\begin{aligned}\Phi^{\frac{1}{1-\sigma}}(t) &= \left[F^{1-\sigma}(t) + \varepsilon\int_{\Omega} uu_t dx\right]^{\frac{1}{1-\sigma}} \\ &\leq 2^{\frac{\sigma}{1-\sigma}}\left(F(t) + \varepsilon^{\frac{1}{1-\sigma}}\left|\int_{\Omega} uu_t dx\right|^{\frac{1}{1-\sigma}}\right) \\ &\leq C\left(\|u_t\|^2 + \|u\|_{q^-}^{q^-} + F(t)\right) \\ &\leq C\left(F(t) + \|u_t\|^2 + \|\nabla u\|_p^p + \|\nabla u\|_p^{2p} + \|u\|_{q^-}^{q^-}\right) \end{aligned} \tag{31}$$

where

$$(a+b)^p \leq 2^{p-1}(a^p + b^p)$$

is used.

In summary, our aim here is to obtain an inequality between the derivative of the Φ function and its numerical power. By combining of (29) and (31), we get

$$\Phi'(t) \geq \mu\Phi^{\frac{1}{1-\sigma}}(t). \tag{32}$$

where $\mu > 0$.

Integrating the inequality (32) over $(0,t)$ yields

$$\Phi^{\frac{\sigma}{1-\sigma}}(t) \geq \frac{1}{\Psi^{-\frac{\sigma}{1-\sigma}}(0) - \frac{\mu\sigma t}{1-\sigma}}.$$

This shows that solution blows up in a finite time T^*, with

$$T^* \leq \frac{1-\sigma}{\mu\sigma\Phi^{\frac{\sigma}{1-\sigma}}(0)}.$$

Hence, we finish the proof.

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