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Zipper Fractal Functions with Variable Scalings

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Abstract

Zipper fractal interpolation function (ZFIF) is a generalization of fractal interpolation function through an improved version of iterated function system by using a binary parameter called a signature. The signature allows the horizontal scalings to be negative. ZFIFs have a complex geometric structure, and they can be non-differentiable on a dense subset of an interval I. In this paper, we construct k-times continuously differentiable ZFIFs with variable scaling functions on I. Some properties like the positivity, monotonicity, and convexity of a zipper fractal function and the one-sided approximation for a continuous function by a zipper fractal function are studied. The existence of Schauder basis of zipper fractal functions for the space of k-times continuously differentiable functions and the space of p-integrable functions for $p \in [1, \infty)$ are studied. We introduce the zipper versions of full Müntz theorem for continuous function and p-integrable functions on I for $p \in [1, \infty)$.

Keywords: Fractals zipper smooth fractal function topological isomorphism Schauder basis linear operator.
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1. Introduction

To describe irregular and sophisticated objects in nature and various scientific experiments, Mandelbrot [15] coined the word fractals due to presence of self-similar patterns in these objects. The fractal geometry tools are supplement to all existing tools in Euclidean geometry. Fractals have been used to describe most of natural objects mountains, clouds, trees, lightening, etc, apart from their applications in bio-enginnering, financial series, image compression, pattern recognition, computer graphics, physics, chemistry, antennas,

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and architecture. A common way to construct fractals by using the theory of iterated function system (IFS) was introduced by Hutchinson [13]. Then, Barnsley [3] introduced the concept of recursive interpolants or fractal interpolation functions (FIFs) using construction of suiatble IFS from a given interpolation data set. Fractal interpolation addresses roughness on different scales or some degree of self-similarity of a data generating function, whereas traditional interpolants are generally smooth or peicewise smooth in nature. Since fractal function are not differentiable in general, Barnsley and Harrington [4] introduced the concept of fractal splines by studying the calculus of fractal functions. They showed the fact that the indefinite integral of a continuous FIF is also a FIF with different IFS. Using this, they constructed C^k FIFs with the property that the k-th derivative of the fractal interpolation function can be nowhere differentiable. Therefore, fractal interpolation techniques can produce smooth and non-smooth interpolants. The theoritical developement of FIFs with constant scaling can be seen in the references [8, 17–21]. The constant scaling in fractal functions gives a strict self-referentail structures to its graph. In [34], Wang and Yu proposed fractal interpolation functions with variable scaling that are suitable to approximate data egenrating function with lesser self-similarity. In [22], Navascués et al. constructed a k-times continuously differentiable fractal interpolation function with variable scaling and showed the existence of a fractal basis for the space of k-times continuously differentiable functions on the domain of interpolation.

In [2], Aseev et al. constructed various types of fractals using the concept of zipper in graph theoritical setting. In [30–32], Tetenov and his group examined many attractive topological and structural properties of zippers related to dendrites and self-similar continua. Recently, Chand et al. in [9], developed a univariate interpolation theory using the zipper. They constructed affine zipper fractal interpolation functions using a suitable affine zipper. They also approximated the solution of the Volterra integral equation using the affine ZFIFs. In the construction of fractal functions, a zipper IFS presented more flexibility than an IFS. This article will extend the concept of the fractal functions using zipper IFS, which relates its theory to functional analysis, approximation theory, operator theory, etc.

This paper is organized as follows: We define zipper IFS and construct zipper fractal interpolation functions with variable scaling functions in Section 2. In Section 3, we construct k-times continuously differentiable ZFIF with variable scaling functions for a given data and for a prescribed k-times continuously differentiable function f. In Section 4, we find sufficient conditions on the scaling functions and base function so that the zipper α-fractal functions become copositive, comonotone, or coconvex corresponding to continuous function f. Similarly, we find sufficient conditions on the scaling functions and base function so that the zipper α-fractal function corresponding to f lies above or below f. We provide some numerical examples of zipper α-fractal functions to feature these properties. We also approximate a continuous function with zipper fractal polynomial on I. In Section 5, we define the zipper fractal operator from $C^k(I)$ to itself, where $C^k(I)$, $k \in \mathbb{N} \cup \{0\}$, is the Banach space of real-valued functions having k continuous derivatives defined on a real compact interval I. That operator forms an isomorphism for some prescribed conditions on scaling functions. Then, we demonstrate that the space $C^k(I)$ has Schauder basis of k-times continuously differentiable zipper fractal functions on I. In Section 6, using the density of $C^k(I)$ in $L^p(I)$ for $p \in [1, \infty)$, where $L^p(I)$ is the Banach space of all measurable functions $\Psi : I \to \mathbb{R}$ such that $\int_I |\Psi|^p dx < \infty$, we extend the zipper fractal operator from $L^p(I)$ to itself and using some conditions on scaling functions, this extension forms an isomorphism. Then, we demonstrate that the space $L^p(I)$ for $p \in [1, \infty)$ has Schauder basis of zipper fractal functions. In Section 7, we define the zipper fractal Müntz space and prove zipper fractal versions of the full Müntz theorem for $C([0, 1])$ and $L^p([0, 1])$ $(1 \le p < \infty)$.

2. Zipper Fractal Interpolation Function (ZFIF)

In this section, first we define zipper IFS, then we construct a zipper IFS for a given interpolation data, which has a unique attractor. That attractor is a graph of a continuous function, and we call that continuous function a zipper fractal interpolation function. In the end, we find a recurrence relation for the zipper fractal interpolation function.

Definition 2.1. For a binary vector $\epsilon := (\epsilon_1, \epsilon_2, \dots, \epsilon_{N-1}) \in \{0,1\}^{N-1}$ called signature, let W_i, $i \in \mathbb{N}_{N-1} := \{1, 2, \dots, N-1\}$, be non-surjective maps on a complete metric space $\{X, d\}$. Then the system $\Upsilon := \{X; W_i, i \in \mathbb{N}_{N-1}\}$ is called a zipper IFS with vertices $\{v_1, v_2, \dots, v_N\}$, if W_i satisfies $W_i(v_1) = v_{i+\epsilon_i}$ and $W_i(v_N) = v_{i+1-\epsilon_i}$, for all $i = 1, 2, \dots, N-1$. Any compact set $A \subset X$ satisfying the self-referential equation

$$A = \bigcup_{i=1}^{N-1} W_i(A),$$

is called the attractor, self-referential set, or fractal corresponding to the zipper IFS Υ.

The definition of a zipper IFS meets with the definition of an IFS with vertices $\{v_1, v_2, \dots, v_N\}$, if we choose signature vector ϵ such that $\epsilon_i = 0$ for all $i \in \mathbb{N}_{N-1}$. Now, we review the construction of zipper fractal interpolation functions based on the zipper as a suitable IFS. The details can be seen in [1, 3, 9].

Let a set of interpolation points $\{(x_i, y_i) \in I \times \mathbb{R} : i \in \mathbb{N}_N (N > 2)\}$ be given, where $x_1 < x_2 < \dots < x_N$ is a partition of closed interval $I := [x_1, x_N]$, $y_i \in [c, d] \subset \mathbb{R}, \forall i \in \mathbb{N}_N$. Set $I_i := [x_i, x_{i+1}]$ and $K := I \times [c, d]$. Let $L_i : I \to I_i$, $i = 1, 2, \dots, N-1$ be contractive homeomorphisms such that

$$L_i(x_1) = x_{i+\epsilon_i}, \quad L_i(x_N) = x_{i+1-\epsilon_i}. \tag{1}$$

If $L_i(x) = a_i x + b_i$ and $\epsilon_i = 1$, then the horizontal scaling a_i can be negative. Furthermore, let for $i \in \mathbb{N}_{N-1}$, $F_i : K \to \mathbb{R}$ is a function of the form

$$F_i(x, y) = \alpha_i(x) y + q_i(x),$$

where α_i's and q_i's are continuous functions on I such that $\|\alpha_i\|_\infty := \{|\alpha_i(x)| : x \in I\} < 1$ and following holds:

$$F_i(x_1, y_1) = y_{i+\epsilon_i}, \quad F_i(x_N, y_N) = y_{i+1-\epsilon_i}, \quad i \in \mathbb{N}_{N-1}. \tag{2}$$

These F_i's for $i \in \mathbb{N}_{N-1}$, contracted either the graph of a function or its flipped version from I to I_i.
Now define mappings $W_i : K \to I_i \times \mathbb{R}$, $i = 1, 2, \dots, N-1$ by

$$W_i(x, y) = (L_i(x), F_i(x, y)), \quad \forall (x, y) \in K.$$

Then the system $\Upsilon = \{K; W_i, i = 1, 2, \dots, N-1\}$ is a zipper IFS with vertices $\{(x_i, y_i) : i \in \mathbb{N}_N\}$, and signature $\epsilon = \{\epsilon_1, \epsilon_2, \dots, \epsilon_{N-1}\}$. The existence of ZFIF has been proved recently in [1] :

Theorem 2.2. *For zipper IFS $\Upsilon = \{K; W_i, i = 1, 2, \dots, N-1\}$, where W_i's are as defined above, the following holds.*

(i) There exists a unique compact set $G \subset K$ such that $G = \bigcup_{i=1}^{N-1} W_i(G)$.

(ii) G is the graph of a continuous function $f_\epsilon^\alpha : I \to \mathbb{R}$ which interpolates the data $\{(x_i, y_i) : i = 1, 2, \dots, N\}$, i.e., $G = \{(x, f_\epsilon^\alpha(x) : x \in I\}$ and for $i = 1, 2, \dots, N$, $f_\epsilon^\alpha(x_i) = y_i$.

Since the existence and uniqueness of zipper fractal interpolation function f_ϵ^α are given by Theorem 2.2, now we obtain a recursive formula for f_ϵ^α.
Let $\epsilon \in \{0,1\}^{N-1}$ be fixed. Suppose $\widetilde{C}(I) := \{g \in C(I) \mid g(x_1) = y_1, \ g(x_N) = y_N\}$. Then $\widetilde{C}(I)$ is closed subspace of $C(I)$ and $\widetilde{C}(I)$ is complete with respect to uniform norm. Now define the Read-Bajraktarević operator $T : \widetilde{C}(I) \to \widetilde{C}(I)$ as

$$(Tg^*)(x) = \alpha_i(L_i^{-1}(x)) g^*(L_i^{-1}(x)) + q_i(L_i^{-1}(x)), \quad x \in I.$$

Note that T is contraction on $(\widetilde{C}(I), \|.\|_\infty)$. By Banach fixed point theorem, T has a unique fixed point f_ϵ^α that satisfies

$$f_\epsilon^\alpha(L_i(x)) = \alpha_i(x) f_\epsilon^\alpha(x) + q_i(x), \quad i \in \mathbb{N}_{N-1}.$$

We call this interpolating function f_ϵ^α as a zipper fractal interpolation function (ZFIF) corresponding to the given data $\{(x_i, y_i) : i = 1, 2, \dots, N\}$, $\alpha := (\alpha_1, \alpha_2, \dots, \alpha_{N-1})$ and the signature $\epsilon = (\epsilon_1, \epsilon_2, \dots, \epsilon_{N-1}) \in \{0,1\}^{N-1}$. For a prescribed function $f \in C(I)$, if we choose $q_i(x) := f(L_i(x)) - \alpha_i(x)b(x)$ for $i \in \mathbb{N}_{N-1}$ and $y_i = f(x_i)$ for $i \in \mathbb{N}_N$, where b is called a base function which satisfies $f(x_1) = b(x_1)$ and $f(x_N) = b(x_N)$, then the corresponding ZFIF f_α^ϵ is called a zipper α-fractal function. It satisfies

$$f_\epsilon^\alpha(L_i(x)) = f(L_i(x)) + \alpha_i(x)(f_\epsilon^\alpha(x) - b(x)), \ \ i \in \mathbb{N}_{N-1}.$$

3. Smooth and Shape Preserving ZFIFs

In this section, we will try to find a k-times continuously differentiable ZFIF with variable scaling functions for a given data. After that, we can easily find a k-times continuously differentiable zipper α-fractal function with variable scaling functions corresponding to a given function $f \in C^k(I)$, where the norm on $C^k(I)$ is defined as $\|f\|_k := \max\{\|f\|_\infty, \|f^{(1)}\|_\infty, \dots, \|f^{(k)}\|_\infty\}$. Then, we will give some conditions on scaling functions or base function so that the proposed zipper α-fractal function satisfies positivity, monotonicity, or convexity, whenever the original has the same characteristic, and also we will give a one-sided approximation of a given function by a zipper α-fractal function or zipper fractal polynomial.

Theorem 3.1. *For $x_1 < x_2 < \cdots < x_N$, let $\{(x_i, y_i) : i \in \mathbb{N}_N\}$ be a given set of interpolation data. Apart from $y_{1,0} = y_1$ and $y_{N,0} = y_N$, Suppose $y_{1,p}$ and $y_{N,p}$, $p \in \mathbb{N}_k^0 := \{0, 1, 2, \dots, k\}$, are arbitrarily chosen real numbers. For $i \in \mathbb{N}_{N-1}$, let $L_i(x) = a_i x + b_i$ and $F_i(x, y) = \alpha_i(x)y + q_i(x)$ satisfy (1) and (2) respectively. Let $\epsilon = \{\epsilon_1, \epsilon_2, \dots, \epsilon_{N-1}\} \in \{0,1\}^{N-1}$ and for $i \in \mathbb{N}_{N-1}$, assume that there exist k-times continuously differentiable functions α_i and q_i on I such that $\|\alpha_i\|_k < |\frac{a_i}{2}|^k$ and for $p \in \mathbb{N}_k$,*

$$\begin{cases} \frac{B_i + q_i^{(p)}(x_N)}{a_i^p} = \frac{A_{i+1} + q_{i+1}^{(p)}(x_1)}{a_{i+1}^p} & \text{if } \epsilon_i = 0, \epsilon_{i+1} = 0, \\ \frac{B_i + q_i^{(p)}(x_N)}{a_i^p} = \frac{B_{i+1} + q_{i+1}^{(p)}(x_N)}{a_{i+1}^p} & \text{if } \epsilon_i = 0, \epsilon_{i+1} = 1, \\ \frac{A_i + q_i^{(p)}(x_1)}{a_i^p} = \frac{A_{i+1} + q_{i+1}^{(p)}(x_1)}{a_{i+1}^p} & \text{if } \epsilon_i = 1, \epsilon_{i+1} = 0, \\ \frac{A_i + q_i^{(p)}(x_1)}{a_i^p} = \frac{B_{i+1} + q_{i+1}^{(p)}(x_N)}{a_{i+1}^p} & \text{if } \epsilon_i = 1, \epsilon_{i+1} = 1, \ \ i = 1, 2, \dots, N-1, \end{cases} \tag{3}$$

$$\begin{aligned} y_{1,p} &= \begin{cases} \frac{q_1^{(p)}(x_1) + \sum_{j=0}^{p} \binom{p}{j} y_{1,j} \alpha_1^{(p-j)}(x_1)}{a_1^p} & \text{if } \epsilon_1 = 0, \\ \frac{q_1^{(p)}(x_N) + \sum_{j=0}^{p} \binom{p}{j} y_{N,j} \alpha_1^{(p-j)}(x_N)}{a_1^p} & \text{if } \epsilon_1 = 1, \end{cases} \\ y_{N,p} &= \begin{cases} \frac{q_{N-1}^{(p)}(x_N) + \sum_{j=0}^{p} \binom{p}{j} y_{N,j} \alpha_{N-1}^{(p-j)}(x_N)}{a_{N-1}^p} & \text{if } \epsilon_{N-1} = 0, \\ \frac{q_{N-1}^{(p)}(x_1) + \sum_{j=0}^{p} \binom{p}{j} y_{1,j} \alpha_{N-1}^{(p-j)}(x_1)}{a_{N-1}^p} & \text{if } \epsilon_{N-1} = 1, \end{cases} \end{aligned} \tag{4}$$

where $A_i := \sum_{j=0}^{p} \binom{p}{j} y_{1,j} \alpha_i^{(p-j)}(x_1)$ and $B_i := \sum_{j=0}^{p} \binom{p}{j} y_{N,j} \alpha_i^{(p-j)}(x_N)$ for $i \in \mathbb{N}_{N-1}$. Then $\{(L_i(x), F_i(x, y)) : i \in \mathbb{N}_{N-1}\}$ clinch a k-times continuously differentiable ZFIF f_ϵ^α and, for $p \in \mathbb{N}_k$, it satisfies

$$f_\epsilon^{\alpha(p)}(L_i(x)) = a_i^{-p}\Big[\sum_{j=0}^{p} \binom{p}{j} f_\epsilon^{\alpha(j)}(x) \alpha_i^{(p-j)}(x) + q_i^{(p)}(x)\Big], \ \ x \in I, \ i \in \mathbb{N}_{N-1}.$$

Proof. Let $\mathbb{D}^k(I) := \{g \in C^k(I) : g^{(p)}(x_1) = y_{1,p},\ g^{(p)}(x_N) = y_{N,p},\ p \in \mathbb{N}_k^0\}$. Since $\mathbb{D}^k(I)$ is a closed subset of the complete metric space $(C^k(I), \|.\|_k)$, the space $(\mathbb{D}^k(I), \|.\|_k)$ is also a complete metric space. Now define $T : \mathbb{D}^k(I) \to \mathbb{D}^k(I)$ by

$$(Tg)(x) = \alpha_i(L_i^{-1}(x))g(L_i^{-1}(x)) + q_i(L_i^{-1}(x)), \ \ x \in I_i, \ i \in \mathbb{N}_{N-1}. \tag{5}$$

Now Tg is k-times continuously differentiable on each open subinterval (x_i, x_{i+1}), $i \in \mathbb{N}_{N-1}$, as the functions g, α_i and q_i are in $C^k(I)$. Since $L_i : I \to [x_i, x_{i+1}]$ for $i \in \mathbb{N}_{N-1}$ satisfying $L_i(x_1) = x_{i+\epsilon_i}$ and $L_i(x_N) = x_{i+1-\epsilon_i}$, therefore

$$x_{i+1} = \begin{cases} L_i(x_N), & \text{if } \epsilon_i = 0, \\ L_i(x_1), & \text{if } \epsilon_i = 1. \end{cases}$$

Similarly,

$$x_{i+1} = \begin{cases} L_{i+1}(x_1), & \text{if } \epsilon_{i+1} = 0, \\ L_{i+1}(x_N), & \text{if } \epsilon_{i+1} = 1. \end{cases}$$

This implies,

$$(Tg)^{(p)}(x_{i+1}^-)a_i^p = \begin{cases} \sum_{j=0}^p \binom{p}{j} g^{(j)}(x_N)\alpha_i^{(p-j)}(x_N) + q_i^{(p)}(x_N), & \text{if } \epsilon_i = 0, \\ \sum_{j=0}^p \binom{p}{j} g^{(j)}(x_1)\alpha_i^{(p-j)}(x_1) + q_i^{(p)}(x_1), & \text{if } \epsilon_i = 1, \end{cases}$$

$$(Tg)^{(p)}(x_{i+1}^+)a_{i+1}^p = \begin{cases} \sum_{j=0}^p \binom{p}{j} g^{(j)}(x_1)\alpha_{i+1}^{(p-j)}(x_1) + q_{i+1}^{(p)}(x_1), & \text{if } \epsilon_{i+1} = 0, \\ \sum_{j=0}^p \binom{p}{j} g^{(j)}(x_N)\alpha_{i+1}^{(p-j)}(x_N) + q_{i+1}^{(p)}(x_N), & \text{if } \epsilon_{i+1} = 1. \end{cases}$$

Utilising the conditions prescribed in (3) and (4), we can easily obtain that

$$\begin{aligned} &(Tg)^{(p)}(x_{i+1}^+) = (Tg)^{(p)}(x_{i+1}^-), \ \ i = 1, 2, \ldots, N-2, \ \ p \in \mathbb{N}_k^0, \\ &(Tg)^{(p)}(x_1) = y_{1,p}, \ \ (Tg)^{(p)}(x_N) = y_{N,p}, \ \ p \in \mathbb{N}_k^0. \end{aligned}$$

Thus, T is well defined and $Tg \in \mathbb{D}^k(I)$. For g, g^* in $\mathbb{D}^k(I)$, $p \in \mathbb{N}_k^0$ and for all $x \in I_i$, we have

$$|(Tg)^{(p)}(x) - (Tg^*)^{(p)}(x)| = |a_i|^{-p}\Big[\Big|\sum_{j=0}^{p}\binom{p}{j}\alpha_i^{(p-j)}(L_i^{-1}(x))(g-g^*)^{(j)}(L_i^{-1}(x))\Big|\Big],$$

therefore,

$$|(Tg)^{(p)}(x) - (Tg^*)^{(p)}(x)| \le |a_i|^{-p}\|\alpha_i\|_p\|g-g^*\|_p\sum_{j=0}^{p}\binom{p}{j}. \tag{6}$$

For $p \in \mathbb{N}_k^0$, easily we can get, $|a_i|^{-p}\|\alpha_i\|_p \le |a_i|^{-k}\|\alpha_i\|_k$, for all $i \in \mathbb{N}_{N-1}$. So (6) produces

$$\|(Tg)^{(p)} - (Tg^*)^{(p)}\|_\infty \le \max\{(2/|a_i|)^k\|\alpha_i\|_k : i \in \mathbb{N}_{N-1}\}\|g-g^*\|_k < s\|g-g^*\|_k,$$

where $s := \max\{(2/|a_i|)^k\|\alpha_i\|_k : i \in \mathbb{N}_{N-1}\} < 1$.

$$\implies \|Tg - Tg^*\|_k < s\|g-g^*\|_k.$$

i.e., T is a contraction map on $\mathbb{D}^k(I)$. Hence by Banach fixed point theorem, T has a unique fixed point, say $f_\alpha^\epsilon \in \mathbb{D}^k(I) \subset C^k(I)$. Further, for $p \in \mathbb{N}_k$, by successive differentiating equation (5) p-times, our ZFIF f_ϵ^α satisfies

$$f_\epsilon^{\alpha(p)}(L_i(x)) = a_i^{-p}\Big[\sum_{j=0}^{p}\binom{p}{j}f_\epsilon^{\alpha(j)}(x)\alpha_i^{(p-j)}(x) + q_i^{(p)}(x)\Big], \ \ x \in I, i \in \mathbb{N}_{N-1}.$$

This completes our proof. □

Remark 3.1.1. (*i*) *Theorem 3.2 in [22] is a particular case of Theorem 3.1 (take all $\epsilon_i = 0$).*
(*ii*) *For all $i \in \mathbb{N}_{N-1}$, if we choose $\alpha_i(x) = \alpha_i$ for all $x \in I$ with $|\alpha_i| < a_i^k$ and $\epsilon_i = 0$, then Theorem 3.1 includes the Barnsley-Harrington theorem in [4] as a special case.*

By choosing $q_i(x) = f(L_i(x)) - \alpha_i(x)b(x)$ in Theorem 3.1, we can easily get the following result:

Corollary 3.2. For a given function $f \in C^k(I)$ and a signature $\epsilon \in \{0,1\}^{N-1}$, if k-times continuously differentiable scaling functions α_i $(i \in \mathbb{N}_{N-1})$ and b are chosen such that $\|\alpha_i\|_k < |\frac{a_i}{2}|^k$ and

$$b^{(p)}(x_1) = f^{(p)}(x_1), \qquad b^{(p)}(x_N) = f^{(p)}(x_N), \;\; p \in \mathbb{N}_k^0,$$

then the corresponding zipper α-fractal function f_ϵ^α belongs to $C^k(I)$. Also, for $p \in \mathbb{N}_k^0$ and $i \in \mathbb{N}_k$, $(f_\epsilon^\alpha)^{(p)}$ satisfies $(f_\epsilon^\alpha)^{(p)}(x_i) = f^{(p)}(x_i)$ and

$$(f_\epsilon^\alpha)^{(p)}(L_i(x)) = f^{(p)}(L_i(x)) + a_i^{-p}\Big[\sum_{j=0}^{p}\binom{p}{j}\alpha_i^{(p-j)}(x)(f_\epsilon^\alpha - b)^{(j)}(x)\Big]. \tag{7}$$

Remark 3.2.1. (i) *Theorem* 3.2 *in [33] is a particular case of Corollary 3.2 (take all* $\epsilon_i = 0$*).*
(ii) *If we fix the non-zero variable scaling functions* α_i *for* $i \in \mathbb{N}_{N-1}$ *and base function* b *satisfy conditions given in Corollary 3.2, then we have* 2^{N-1} k*-times continuously differentiable zipper* α*-fractal functions based on the signature* ϵ *for a prescribed function* $f \in C^k(I)$ *such that* $f_\epsilon^{\alpha(p)}(x_i) = f^{(p)}(x_i)$*, for* $p \in \mathbb{N}_k^0$ *and* $i \in \mathbb{N}_k$*.*

From Corollary 3.2, we can observe that zipper α-fractal function f_ϵ^α corresponding to $f \in C^k(I)$ depends on the choice of variable scaling functions and the base function b. In this next theorem for $f \in C(I)$ such that $f(x) \geq 0$ on I, we give constraints on scaling functions such that the corresponding zipper α-fractal function f_ϵ^α is non-negative for any choice of base function b except $b \in C(I)$, $b(x_1) = f(x_1)$ and $b(x_N) = f(x_N)$.

Theorem 3.3. *Let* $f \in C(I)$ *such that* $f(x) \geq 0$ *for all* $x \in I$*,* $\Delta := \{x_1, x_2, \ldots, x_N\}$ *be a partition of* I *and* $\epsilon = \{\epsilon_1, \epsilon_2, \ldots, \epsilon_{N-1}\} \in \{0,1\}^{N-1}$*. Let* $L_i(x) = a_i x + b_i$ *and* $F_i(x,y) = \alpha_i(x)y + q_i(x)$ *are satisfying (1) and (2) respectively, where* $q_i(x) = f(L_i(x)) - \alpha_i(x)b(x)$*,* $b(x_1) = f(x_1)$*,* $b(x_N) = f(x_N)$*, and* $\|\alpha_i\|_\infty = \max\{|\alpha_i(x)| : x \in I\} < 1$*. So if* $\forall i \in \mathbb{N}_{N-1}$ *and* $\forall x \in I$*, variable scaling functions* $\alpha_i \in C(I)$ *are chosen such that* $0 \leq \alpha_i(x) \leq \frac{m_i}{M^*}$*, where* $m_i = \min\{f(x) : x \in I_i\}$ *and* $M^* = \max\{b(x) : x \in I\}$*, then zipper* α*-fractal function* $f_\epsilon^\alpha(x)$ *corresponding to* f *satisfies* $f_\epsilon^\alpha(x) \geq 0$ *for all* $x \in I$*.*

Proof. For $k = 0$, (7) implies that zipper α-fractal function f_ϵ^α corresponding to f satisfies the functional equation

$$f_\epsilon^\alpha(L_i(x)) = f(L_i(x)) + \alpha_i(x)(f_\epsilon^\alpha(x) - b(x)) \;\; x \in I, \; i \in \mathbb{N}_{N-1} \tag{8}$$

and $f_\epsilon^\alpha(x_i) = f(x_i)$, i.e., $f_\epsilon^\alpha(x_i) \geq 0$ for all $i = 1, 2, \ldots, N$. Now I is the attractor of the zipper $\{I; L_i, i = 1, 2, \ldots, N-1\}$ and f_ϵ^α is constructed using an iterative scheme, so it is easy to see that proving $f_\epsilon^\alpha(x) \geq 0$ for all $x \in I$ enough to prove that $f_\epsilon^\alpha \geq 0$ holds on I obtained at $n+1$-th iteration whenever $f_\epsilon^\alpha \geq 0$ is satisfied for the points on I at n-th iteration, i.e., enough to prove $f_\epsilon^\alpha(L_i(x)) \geq 0$ for all $i \in \mathbb{N}_{N-1}$ whenever $f_\epsilon^\alpha(x) \geq 0$. Now let $f_\epsilon^\alpha(x) \geq 0$. Then the equation (8) can be rewritten as,

$$f_\epsilon^\alpha(L_i(x)) = f(L_i(x)) + \alpha_i(x)f_\epsilon^\alpha(x) - \alpha_i(x)b(x).$$

Now for all $x \in I$ and $i \in \mathbb{N}_{N-1}$, if $\alpha_i(x) \geq 0$, then $\alpha_i(x)f_\epsilon^\alpha(x) \geq 0$. Therefore,

$$f_\epsilon^\alpha(L_i(x)) \geq f(L_i(x)) - \alpha_i(x)b(x),$$

and $f(L_i(x)) - \alpha_i(x)b(x) \geq 0$, when $\alpha_i(x) \leq \frac{m_i}{M^*}$. This completes the proof. □

Remark 3.3.1. *Similarly we can also prove that if* $f(x) \leq 0$ *for all* $x \in I$*, then corresponding zipper* α*-fractal function* f_ϵ^α *is also non-positive on* I*, by choosing variable scaling functions such that* $0 \leq \alpha_i(x) \leq \frac{m_i^*}{M_*}$*,* $\|\alpha_i\|_\infty < 1$ *for all* $i \in \mathbb{N}_{N-1}$*, where* $m_i^* = \max\{f(x) : x \in I_i\}$ *and* $M_* = \min\{b(x) : x \in I\}$*.*

Now we will show that for the non-negative variable scaling functions if base function b satisfies some prescribed constraints, then zipper α-fractal function f_ϵ^α corresponding to f satisfies $f_\epsilon^\alpha(x) \geq f(x)$ for all $x \in I$. Consequently, if $f(x) \geq 0$ on I, then f_ϵ^α also satisfies $f_\epsilon^\alpha(x) \geq 0$, $\forall x \in I$.

Theorem 3.4. *Let f be a continuous function on I, $\Delta = \{x_1, x_2, \dots, x_N\}$ be a partition of I and $\epsilon = \{\epsilon_1, \epsilon_2, \dots, \epsilon_{N-1}\} \in \{0,1\}^{N-1}$. Let $L_i(x) = a_i x + b_i$ and $F_i(x,y) = \alpha_i(x)y + f(L_i(x)) - \alpha_i(x)b(x)$ be satisfying (1) and (2) respectively, where the base function b is any continuous function on I satisfying $b(x_1) = f(x_1)$ and $b(x_N) = f(x_N)$. So, if scaling functions $\alpha_i \in C(I)$, $i \in \mathbb{N}_{N-1}$ are chosen such that $\alpha_i(x) \geq 0$ and $\|\alpha_i\|_\infty := \max\{|\alpha_i(x)| : x \in I\} < 1$ and the base function is elected as $b(x) \leq f(x)$ for all $x \in I$, then zipper α-fractal function f_ϵ^α corresponding to f satisfies $f_\epsilon^\alpha(x) \geq f(x)$, $\forall x \in I$.*

Proof. Using the same argument as in Theorem 3.3, for proving $f_\epsilon^\alpha(x) \geq f(x)$ for all $x \in I$, enough to prove $(f_\epsilon^\alpha - f)(L_i(x)) \geq 0$ for all $i \in \mathbb{N}_{N-1}$ whenever $(f_\epsilon^\alpha - f)(x) \geq 0$. We can reshape (8) as

$$\begin{aligned}(f_\epsilon^\alpha - f)(L_i(x)) &= \alpha_i(x)(f_\epsilon^\alpha - b)(x)\\ &= \alpha_i(x)(f_\epsilon^\alpha - f)(x) + \alpha_i(x)(f - b)(x).\end{aligned} \tag{9}$$

So, if $\alpha_i(x) \geq 0$ and $f(x) \geq b(x)$ for all $x \in I$, then $(f_\epsilon^\alpha - f)(x) \geq 0$ implies $(f_\epsilon^\alpha - f)(L_i(x)) \geq 0$, which completes the proof. □

Remark 3.4.1. (i) *Similarly one can also prove that for variable scaling functions satisfying $\alpha_i(x) \geq 0$ and $\|\alpha_i\|_\infty := \max\{|\alpha_i(x)| : x \in I\} < 1$ and for base function satisfying $b(x) \geq f(x)$ for all $x \in I$, zipper α-fractal function f_ϵ^α corresponding to f satisfies $f_\epsilon^\alpha(x) \leq f(x)$ for all $x \in I$.*
(ii) *If we fix the non-zero variable scaling functions α_i for $i \in \mathbb{N}_{N-1}$ and base function b satisfying conditions given in Theorem 3.4, then we have 2^{N-1} different zipper α-fractal functions corresponding to $f \in C(I)$ based on the signature ϵ such that $f_\epsilon^\alpha(x) \geq f(x)$ for all $x \in I$.*

In the next theorem, we will show a one-sided approximation of a continuous function by a zipper fractal polynomial on I.

Theorem 3.5. *For a continuous function g on I and arbitrary $\delta > 0$, there exists a zipper fractal polynomial P_ϵ^α satisfying $P_\epsilon^\alpha(x) \geq g(x)$ for all $x \in I$ and $\|P_\epsilon^\alpha - g\|_\infty < \delta$.*

Proof. It is known from Weierstrass theorem that for a continuous function g on I, there exists a polynomial $\widetilde{P}$ such that $\|g - \widetilde{P}\|_\infty < \frac{\delta}{4}$. Therefore, we have

$$\widetilde{P}(x) - \frac{\delta}{4} \leq g(x) \leq \widetilde{P}(x) + \frac{\delta}{4}, \ \ \forall x \in I.$$

Now let $P(x) = \widetilde{P}(x) + \frac{\delta}{4}$. So the last inequality changes into

$$P(x) - \frac{\delta}{2} \leq g(x) \leq P(x) \leq P(x) + \frac{\delta}{2}, \ \ \forall x \in I.$$

Therefore, for $g \in C(I)$, we have a polynomial P such that $P(x) \geq g(x)$ for all $x \in I$ and $\|g - P\|_\infty < \frac{\delta}{2}$. Now, for $\Delta = \{x_1, x_2, \dots, x_N\} \subset I$ and $\epsilon = (\epsilon_1, \epsilon_2, \dots, \epsilon_{N-1}) \in \{0,1\}^{N-1}$, choose a base function b such that $b(x_1) = P(x_1)$, $b(x_N) = P(x_N)$, and $b(x) \leq P(x)$, $\forall x \in I$. Then select non-negative variable scaling functions such that for all $i \in \mathbb{N}_{N-1}$, $\alpha_i \in C(I)$ and $\|\alpha\|_\infty < \frac{\delta}{\delta + 2\|P-b\|_\infty}$, where $\|\alpha\|_\infty := \max\{\|\alpha_i\|_\infty : i \in \mathbb{N}_{N-1}\}$. Utilizing Theorem 3.4, zipper fractal polynomial P_ϵ^α corresponding to P satisfies $P_\epsilon^\alpha(x) \geq P(x) \geq g(x)$ for all $x \in I$. Now consider for $i \in \mathbb{N}_{N-1}$ and $x \in I$,

$$(P_\epsilon^\alpha - P)(L_i(x)) = \alpha_i(x)(P_\epsilon^\alpha - b)(x) = \alpha_i(x)(P_\epsilon^\alpha - P + P - b)(x).$$

Taking modulus and using a triangle inequality, we obtain

$$\|P_\epsilon^\alpha - P\|_\infty \leq \frac{\|\alpha\|_\infty}{1 - \|\alpha\|_\infty}\|P - b\|_\infty < \frac{\delta}{2}.$$

Then, it is concluded that

$$\|P_\epsilon^\alpha - g\|_\infty \leq \|P_\epsilon^\alpha - P\|_\infty + \|P - g\|_\infty < \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

□

Now, we will find conditions on variable scaling functions so that the first derivative of zipper α-fractal function f_ϵ^α corresponding to f satisfies $r \le (f_\epsilon^\alpha)^{(1)} \le R$ whenever $r \le f^{(1)} \le R$. Then, it helps to construct monotonicity preserving zipper α-fractal function f_ϵ^α whenever germ function f is monotone (take $r = 0$). For this, let us define some notation first:
$z_k := \min_{x\in I}\{b^{(k)}(x) : i \in \mathbb{N}_{N-1}\}$; $Z_k := \max_{x\in I}\{b^{(k)}(x) : i \in \mathbb{N}_{N-1}\}$; $r_i := \min_{x\in I_i} f^{(1)}(x)$, $R_i := \max_{x\in I_i} f^{(1)}(x)$; $m := \min_{x\in I} f_\epsilon^\alpha(x)$; $M := \max_{x\in I} f_\epsilon^\alpha(x)$.
By putting $k = 1$ in Corollary 3.2, our zipper α-fractal function f_ϵ^α corresponding to f satisfies

$$(f_\epsilon^\alpha)^{(1)}(L_i(x)) = f^{(1)}(L_i(x)) + \frac{\alpha_i'(x)}{a_i}(f_\epsilon^\alpha - b)(x) + \frac{\alpha_i(x)}{a_i}(f_\epsilon^\alpha - b)^{(1)}(x). \tag{10}$$

Let $r \le f^{(1)} \le R$. To prove $r \le (f_\epsilon^\alpha)^{(1)} \le R$, it is enough to verify $r \le (f_\epsilon^\alpha)^{(1)}(L_i(x)) \le R$ for all $i \in \mathbb{N}_{N-1}$ and $x \in I$. For this, we will choose monotonic scaling functions $\alpha_i \in C^1(I)$ satisfying $\|\alpha_i\|_1 < \frac{|a_i|}{2}$, where $\|\alpha_i\|_1 = \max\{\|\alpha_i\|_\infty, \|\alpha_i'\|_\infty\}$. Then, we need to verify

$$r \le f^{(1)}(L_i(x)) + \frac{\alpha_i'(x)}{a_i}(f_\epsilon^\alpha - b)(x) + \frac{\alpha_i(x)}{a_i}(f_\epsilon^\alpha - b)^{(1)}(x) \le R. \tag{11}$$

Case 1: Let $0 \le (-1)^{\epsilon_i}\alpha_i(x) \le (-1)^{\epsilon_i}\frac{a_i}{2}$ and $(-1)^{\epsilon_i}\alpha_i'(x) \ge 0$ for all $x \in I$. Hence $\frac{\alpha_i(x)}{a_i} \ge 0$ and $\frac{\alpha_i'(x)}{a_i} \ge 0$. Therefore, (11) is true whenever

$$r\left(1 - \frac{\alpha_i(x)}{a_i}\right) \le f^{(1)}(L_i(x)) + \frac{\alpha_i'(x)}{a_i}(f_\epsilon^\alpha - b)(x) - \frac{\alpha_i(x)}{a_i}b^{(1)}(x) \le R\left(1 - \frac{\alpha_i(x)}{a_i}\right). \tag{12}$$

It is easy to observe that

$$f^{(1)}(L_i(x)) + \frac{\alpha_i'(x)}{a_i}(f_\epsilon^\alpha - b)(x) - \frac{\alpha_i(x)}{a_i}b^{(1)}(x) \ge r_i + \frac{\alpha_i'(x)}{a_i}(m - Z_0) - \frac{\alpha_i(x)}{a_i}Z_1. \tag{13}$$

In view of (13), for the validity of the first inequality of (12), we need a condition for which

$$\begin{aligned}
&r_i + \frac{\alpha_i'(x)}{a_i}(m - Z_0) - \frac{\alpha_i(x)}{a_i}Z_1 \ge r\left(1 - \frac{\alpha_i(x)}{a_i}\right),\\
&\text{i.e.,}\ \ r_i + \frac{\alpha_i'(x)}{a_i}(m - Z_0) + \frac{\alpha_i(x)}{a_i}(r - Z_1) \ge r,\\
&\text{i.e.,}\ \ \frac{\alpha_i'(x)}{a_i}(Z_0 - m) + \frac{\alpha_i(x)}{a_i}(Z_1 - r) \le (r_i - r),\\
&\text{i.e.,}\ \ \frac{(-1)^{\epsilon_i}\alpha_i'(x)}{(-1)_i^{\epsilon}a_i}(Z_0 - m) + \frac{(-1)^{\epsilon_i}\alpha_i(x)}{(-1)^{\epsilon_i}a_i}(Z_1 - r) \le (r_i - r).
\end{aligned}$$

We know $\|\alpha_i\|_1 = \max\{\|\alpha_i\|_\infty, \|\alpha_i'\|_\infty\}$. The above inequality is true if

$$\|\alpha_i\|_1(Z_0 - m + Z_1 - r) \le (-1)^{\epsilon_i}a_i(r_i - r).$$

Therefore, we choose

$$\|\alpha_i\|_1 \le \frac{(-1)^{\epsilon_i}a_i(r_i - r)}{Z_0 - m + Z_1 - r},$$

for validity of the left inequality of (12). Similarly, if we choose

$$\|\alpha_i\|_1 \le \frac{(-1)^{\epsilon_i}a_i(R - R_i)}{M - z_0 + R - z_1},$$

then the right inequality of (12) is true.
Case 2: Let $(-1)^{\epsilon_i}\frac{-a_i}{2} \leq (-1)^{\epsilon_i}\alpha_i \leq 0$ and $(-1)^{\epsilon_i}\alpha_i'(x) \geq 0$ for all $x \in I$. Then we have $\frac{\alpha_i(x)}{a_i} \leq 0$ and $\frac{\alpha_i'(x)}{a_i} \geq 0$ for all $x \in I$ as $(-1)^{\epsilon_i} = \pm 1$. Then, the first inequality of (11) satisfies when

$$r - R\frac{\alpha_i(x)}{a_i} \leq f^{(1)}(L_i(x)) + \frac{\alpha_i'(x)}{a_i}(f_\epsilon^\alpha - b)(x) - \frac{\alpha_i(x)}{a_i}b^{(1)}(x),$$
$$\text{i.e., } r \leq f^{(1)}(L_i(x)) + \frac{\alpha_i'(x)}{a_i}(f_\epsilon^\alpha - b)(x) + \frac{\alpha_i(x)}{a_i}(R - b^{(1)}(x)) \ \ \forall x \in I.$$

The sufficient condition for validity of the above inequality is

$$r \leq r_i + \frac{\alpha_i'(x)}{a_i}(m - Z_0) + \frac{\alpha_i(x)}{a_i}(R - z_1),$$
$$\text{i.e., } r - r_i \leq \frac{\alpha_i'(x)}{a_i}(m - Z_0) + \frac{\alpha_i(x)}{a_i}(R - z_1),$$
$$\text{i.e., } r_i - r \geq \frac{\alpha_i'(x)}{a_i}(Z_0 - m) - \frac{\alpha_i(x)}{a_i}(R - z_1),$$
$$\text{i.e., } r_i - r \geq \frac{(-1)^{\epsilon_i}\alpha_i'(x)}{(-1)^{\epsilon_i}a_i}(Z_0 - m) - \frac{(-1)^{\epsilon_i}\alpha_i(x)}{(-1)^{\epsilon_i}a_i}(R - z_1),$$
$$\text{i.e., } r_i - r \geq \frac{\|\alpha_i\|_1}{(-1)^{\epsilon_i}a_i}(Z_0 - m) + \frac{\|\alpha_i\|_1}{(-1)^{\epsilon_i}a_i}(R - z_1),$$
$$\text{i.e., } (-1)^{\epsilon_i}a_i(r - r_i) \geq \|\alpha_i\|_1(Z_0 - m + R - z_1).$$

Therefore, if we choose

$$\|\alpha_i\|_1 \leq \frac{(-1)^{\epsilon_i}a_i(r_i - r)}{Z_0 - m + R - z_1},$$

then the left inequality of (11) is true in this case. Similarly, other inequality of (11) is true when

$$\|\alpha_i\|_1 \leq \frac{(-1)^{\epsilon_i}a_i(R - R_i)}{M - z_0 + Z_1 - r}.$$

In a similarly analysis, for Case 3: $0 \leq (-1)^{\epsilon_i}\alpha_i(x) \leq (-1)^{\epsilon_i}\frac{a_i}{2}$ and $(-1)^{\epsilon_i}\alpha_i'(x) \leq 0$, (11) is true when

$$\|\alpha_i\|_1 \leq \min\left\{\frac{(-1)^{\epsilon_i}a_i(r_i - r)}{M - z_0 + Z_1 - r}, \frac{(-1)^{\epsilon_i}a_i(R - R_i)}{Z_0 - m + R - z_1}\right\},$$

and for case 4: $(-1)^{\epsilon_i}\frac{-a_i}{2} \leq (-1)^{\epsilon_i}\alpha_i(x) \leq 0$ and $(-1)^{\epsilon_i}\alpha_i'(x) \leq 0$, (11) is true when

$$\|\alpha_i\|_1 \leq \min\left\{\frac{(-1)^{\epsilon_i}a_i(r_i - r)}{M - z_0 + R - z_1}, \frac{(-1)^{\epsilon_i}a_i(R - R_i)}{Z_0 - m + Z_1 - r}\right\}.$$

Now we will summarize these above discussions in the following theorem.

Theorem 3.6. *Let $f \in C^1(I)$ satisfying $r \leq f^{(1)}(x) \leq R$ for all $x \in I$. For a partition $\Delta = \{x_1, x_2, \dots, x_N\}$ of I with increasing abscissae, let $\epsilon \in \{0,1\}^{N-1}$ and the base function $b \in C^1(I)$ satisfying the conditions $b^{(p)}(x_1) = f^{(p)}(x_1)$ and $b^{(p)}(x_N) = f^{(p)}(x_N)$ for $i \in \mathbb{N}_{N-1}$, $p = 0, 1$. Then the corresponding zipper α-fractal function f_ϵ^α obeys $r \leq (f_\epsilon^\alpha)^{(1)}(x) \leq R$ for all $x \in I$, if the variable scaling functions $\alpha_i \in C^1(I)$ are monotone and satisfy $\|\alpha_i\|_1 < \frac{a_i}{2}$, and*

$$\|\alpha_i\|_1 \leq \min\left\{\frac{e_i}{\mho_{i1}}, \frac{e_i}{\mho_{i2}}, \frac{e_i}{\mho_{i3}}, \frac{e_i}{\mho_{i4}}, \frac{E_i}{\mho_{i1}}, \frac{E_i}{\mho_{i2}}, \frac{E_i}{\mho_{i3}}, \frac{E_i}{\mho_{i4}}\right\}, \tag{14}$$

where $e_i = (-1)^{\epsilon_i}a_i(r_i - r)$, $E_i = (-1)^{\epsilon_i}a_i(R - R_i)$,
$\mho_{i1} = Z_0 - m + Z_1 - r$, $\mho_{i2} = Z_0 - m + R - z_1$,
$\mho_{i3} = M - z_0 + Z_1 - r$, $\mho_{i4} = M - z_0 + R - z_1$.

Remark 3.6.1. *(i) Theorem 3.6 includes the Theorem 5 of [14] as a special case by taking all $\epsilon_i = 0$ for $i \in \mathbb{N}_{N-1}$.*
(ii) If $r = 0$, then we can construct a class of 2^{N-1} monotonically increasing zipper α-fractal functions corresponding to monotonically increasing function f.

Similarly, for a given convex function f, we can restrict the variable scaling functions so that the proposed class of zipper α-fractal functions corresponding to f becomes convex. Substituting $\|\alpha_i\|_2$ in place of $\|\alpha_i\|_{C^2}$ for any $\epsilon \in \{0,1\}^{N-1}$, the restrictions on variable scaling functions are the same as given in Theorem 7 of [14]. But our advantage is that we can find 2^{N-1} different convex zipper α-fractal functions including the convex α-fractal function given in [14], based on ϵ for a fixed set of scaling functions.

4. Examples of Shape Preserving ZFIFs

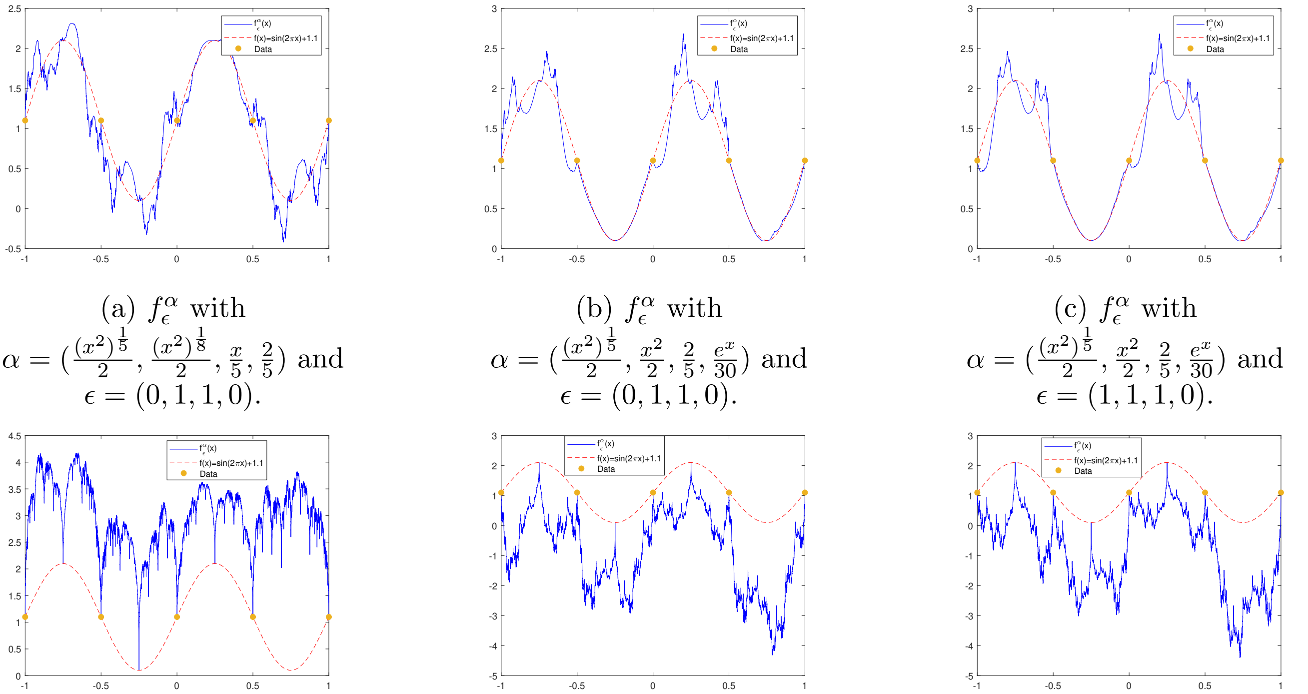
In this section, we present some numerical examples to support the theory in Section 3. If we do not restrict the scaling functions or the base function as prescribed in the last section, then the proposed zipper α-fractal functions may not preserve the desired property. The sufficient conditions are verified through suitable examples. We also plot graphs with different signatures to show that for the same set of scaling functions, we get the different zipper α-fractal functions.

Example 4.1. *In this example, we want to check the positivity preserving property and one-sided approximation property of the proposed zipper α-fractal function. Consider a non-negative germ function $f(x) = sin(2\pi x) + 1.1$ on $I = [-1, 1]$ with partition $\Delta = \{-1, -1/2, 0, 1/2, 1\}$. The following parameters are used to plot the graphs for Figs. 1(a)-(f):*

Fig. 1	$b(x)$	α	ϵ
(a)	1.1	$(\frac{(x^2)^{\frac{1}{5}}}{2}, \frac{(x^2)^{\frac{1}{8}}}{2}, \frac{x}{5}, \frac{2}{5})$	$(0,1,1,0)$
(b)	1.1	$(\frac{(x^2)^{\frac{1}{5}}}{2}, \frac{x^2}{2}, \frac{2}{5}, \frac{e^x}{30})$	$(0,1,1,0)$
(c)	1.1	$(\frac{(x^2)^{\frac{1}{5}}}{2}, \frac{x^2}{2}, \frac{2}{5}, \frac{e^x}{30})$	$(1,1,1,0)$
(d)	$5x^2 - 3.9$	$(\frac{(x^2)^{\frac{1}{5}}}{2}, \frac{(x^2)^{\frac{1}{8}}}{2}, \frac{(x^2)^{\frac{1}{7}}}{3}, \frac{1}{2})$	$(0,1,1,0)$
(e)	$6.1 - 5x^2$	$(\frac{(x^2)^{\frac{1}{5}}}{2}, \frac{(x^2)^{\frac{1}{8}}}{2}, \frac{(x^2)^{\frac{1}{7}}}{3}, \frac{1}{2})$	$(1,1,0,1)$
(f)	$6.1 - 5x^2$	$(\frac{(x^2)^{\frac{1}{5}}}{2}, \frac{(x^2)^{\frac{1}{8}}}{2}, \frac{(x^2)^{\frac{1}{7}}}{3}, \frac{1}{2})$	$(0,0,0,0)$

In Fig. 1(a), we do not restrict scaling functions or a base function as given in Theorem 3.3 or Theorem 3.4 and we can see that the corresponding zipper α-fractal function is neither positive nor lying completely above or below f. But when we restrict the scaling functions as prescribed in Theorem 3.3, the corresponding zipper α-fractal functions become positive see Figs. 1(b)-(c). We have used $\epsilon_1 = 1$ for Fig. 1(b) and $\epsilon_1 = 0$ for Fig. 1(c), whereas all other parameters are the same for both the figures. Notice between Figs. 1(b) and (c) over I_1 as they are almost the flipped versions of each other. In this way, for the fixed non-zero scalings, we can get 2^{N-1} different zipper α-fractal functions based on different signatures.
In Fig. 1(d), we have used the base function $b(x) = 5x^2 - 3.9$ which satisfies $b(x) \le f(x)$ for all $x \in I$. All the scaling functions are non-negative, so by Theorem 3.4, the corresponding zipper α-fractal function lies completely above f on I. Similarly, for Figs. 1(e)-(f), we have used non-negative scaling functions and the base function $b(x) = 6.1 - 5x^2$ that satisfies $b(x) \ge f(x)$ for all $x \in I$ to get the zipper fractal approximants from below for f. For Fig. 1(f), we have chosen $\epsilon_i = 0$ for each $i \in \mathbb{N}_4$, so the proposed zipper α-fractal function reduces to classical α-fractal function approximant from below.

Example 4.2. *In this example, we want to check the range restriction by the first derivative of the proposed zipper α-fractal functions. Let us consider $f(x) = \frac{x}{10} + sin(\pi x)$ on $I = [\frac{-1}{2}, \frac{1}{2}]$ with partition*



(a) f_ϵ^α with $\alpha = (\frac{(x^2)^{\frac{1}{5}}}{2}, \frac{(x^2)^{\frac{1}{8}}}{2}, \frac{x}{5}, \frac{2}{5})$ and $\epsilon = (0, 1, 1, 0)$.

(b) f_ϵ^α with $\alpha = (\frac{(x^2)^{\frac{1}{5}}}{2}, \frac{x^2}{2}, \frac{2}{5}, \frac{e^x}{30})$ and $\epsilon = (0, 1, 1, 0)$.

(c) f_ϵ^α with $\alpha = (\frac{(x^2)^{\frac{1}{5}}}{2}, \frac{x^2}{2}, \frac{2}{5}, \frac{e^x}{30})$ and $\epsilon = (1, 1, 1, 0)$.

(d) f_ϵ^α with $\alpha = (\frac{(x^2)^{\frac{1}{5}}}{2}, \frac{(x^2)^{\frac{1}{8}}}{2}, \frac{(x^2)^{\frac{1}{7}}}{3}, \frac{1}{2})$ and $\epsilon = (0, 1, 1, 0)$.

(e) f_ϵ^α with $\alpha = (\frac{(x^2)^{\frac{1}{5}}}{2}, \frac{(x^2)^{\frac{1}{8}}}{2}, \frac{(x^2)^{\frac{1}{7}}}{3}, \frac{1}{2})$ and $\epsilon = (1, 1, 0, 1)$.

(f) f_ϵ^α with $\alpha = (\frac{(x^2)^{\frac{1}{5}}}{2}, \frac{(x^2)^{\frac{1}{8}}}{2}, \frac{(x^2)^{\frac{1}{7}}}{3}, \frac{1}{2})$ and $\epsilon = (0, 0, 0, 0)$.

Figure 1: Verification of positivity and one-sided approximation by zipper α-fractal functions.

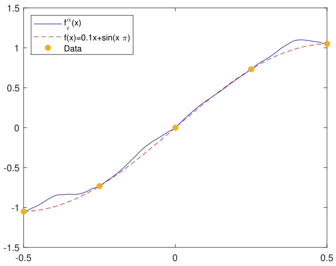
$\Delta = \{\frac{-1}{2}, \frac{-1}{4}, 0, \frac{1}{4}, \frac{1}{2}\}$. *For the fixed base function*

$$b(x) = \begin{cases} \frac{x}{10} - 1 & \text{if } x \in [\frac{-1}{2}, 0], \\ -31x^3 + 24x^2 + 0.1x - 1 & \text{if } x \in [0, \frac{1}{2}], \end{cases}$$

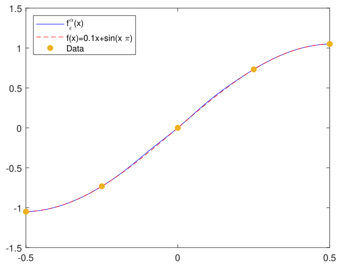
scaling function and signature vectors $\{(\frac{3}{25}, \frac{e^x}{16}, \frac{x}{9}, \frac{(\frac{x+1}{2})^{\frac{1}{10}}}{10}), (1, 0, 1, 0)\}$, $\{(\frac{-x^3}{300}, \frac{e^x}{75}, \frac{e^{-x}}{80}, \frac{x}{400}), (1, 0, 1, 0)\}$, *and* $\{(\frac{-x^3}{300}, \frac{e^x}{75}, \frac{e^{-x}}{80}, \frac{x}{400}), (1, 1, 0, 0)\}$ *are used for plotting Figs. 2(a)-(c) respectively. Figs 2(d)-(f) are the graphs of the derivative of zipper* α*-fractal functions plotted in Figs 2(a)-(c) respectively. Note that the given function* f *satisfies* $r = 0 \leq f^{(1)}(x) \leq R = 4$ *for all* $x \in I$*. If we restrict the scaling functions as prescribed in Theorem 3.6, then zipper* α*-fractal function* f_ϵ^α *corresponding to* f *satisfies* $0 \leq (f_\epsilon^\alpha)^{(1)} \leq 4$*, and consequently, we get monotonically increasing zipper* α*-fractal function corresponding to monotonically increasing function* f*. For Fig. 2(a) we do not restrict the scaling functions as prescribed in Theorem 3.6 and we can see in Fig. 2(d) that corresponding zipper* α*-fractal function* f_ϵ^α *does not satisfy the condition* $0 \leq (f_\epsilon^\alpha)^{(1)}(x) \leq 4$ *on* I *and hence* f_ϵ^α *is not a monotone function on* I*. When the scaling functions are elected as per the prescription of 3.6, the corresponding zipper* α*-fractal functions in Figs. 2(e)-(f) satisfy* $0 \leq (f_\epsilon^\alpha)^{(1)}(x) \leq 4$ *for all* $x \in I$*. We have used different signatures, but all other parameters are the same for Figs. 2(b)-(c) to show the effect of signature. We may not have seen the differences in Figs. 2(b) and 2(c) as the magnitude of scaling functions are nearly close to zero, but the difference in shape can be seen from their derivatives in Figs. 2(e)-(f).*

5. Schauder basis consists of zipper fractal functions for $C^k(I)$

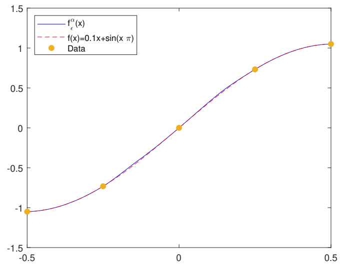
First let us recall the definition of Schauder basis:



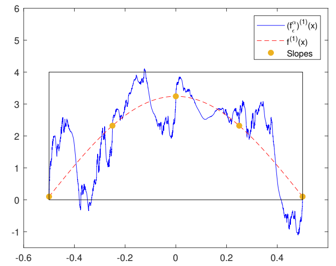
(a) f_ϵ^α with $\alpha = (\frac{3}{25}, \frac{e^x}{16}, \frac{x}{9}, \frac{(\frac{x+1}{2})^{\frac{1}{10}}}{10})$ and $\varepsilon = (1, 0, 1, 0)$.



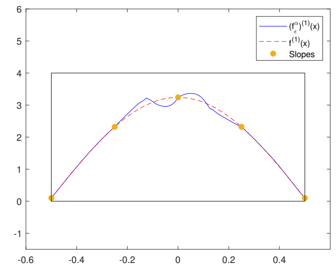
(b) f_ϵ^α with $\alpha = (\frac{-x^3}{300}, \frac{e^x}{75}, \frac{e^{-x}}{80}, \frac{x}{400})$ and $\epsilon = (1, 0, 1, 0)$.



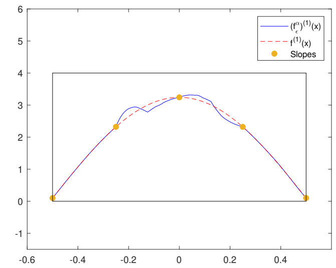
(c) f_ϵ^α with $\alpha = (\frac{-x^3}{300}, \frac{e^x}{75}, \frac{e^{-x}}{80}, \frac{x}{400})$ and $\epsilon = (1, 1, 0, 0)$.



(d) $f_\epsilon^{\alpha(1)}$ with $\alpha = (\frac{3}{25}, \frac{e^x}{16}, \frac{x}{9}, \frac{(\frac{x+1}{2})^{\frac{1}{10}}}{10})$ and $\varepsilon = (1, 0, 1, 0)$.



(e) $f_\epsilon^{\alpha(1)}$ with $\alpha = (\frac{-x^3}{300}, \frac{e^x}{75}, \frac{e^{-x}}{80}, \frac{x}{400})$ and $\epsilon = (1, 0, 1, 0)$.



(f) $f_\epsilon^{\alpha(1)}$ with $\alpha = (\frac{-x^3}{300}, \frac{e^x}{75}, \frac{e^{-x}}{80}, \frac{x}{400})$ and $\epsilon = (1, 1, 0, 0)$.

Figure 2: Verification of monotonicity by zipper α-fractal functions.

Definition 5.1. A sequence $\{u_n\}$ in an infinite dimensional Banach space X is said be a Schauder basis of X if for every $\zeta \in X$, there exists a unique sequence $\{\beta_n(\zeta)\}$ of scalars such that

$$\zeta = \sum_{n=1}^{\infty} \beta_n(\zeta) u_n,$$

where β_n is a linear functional on X, $n \in \mathbb{N}$. β_n is called the n^{th} coefficient functional corresponding to the Schauder basis $\{u_1, u_2, \dots\}$.

Our aim in this section is to show that space $C^k(I)$ has a Schauder basis consisting of zipper fractal functions. Corollary 3.2 stated that for a given function $f \in C^k(I)$, we can choose scaling functions and base function in such a way that the corresponding zipper fractal function is also a member of the space $C^k(I)$ and satisfies (7). From here, we assume that $b = Lf$, where L is a bounded linear operator on $C^k(I)$ such that for $p \in \mathbb{N}_k^0$, $Lf^{(p)}(x_1) = f^{(p)}(x_1)$ and $Lf^{(p)}(x_N) = f^{(p)}(x_N)$. Therefore, (7) changes into

$$(f_\epsilon^\alpha)^{(p)}(L_i(x)) = f^{(p)}(L_i(x)) + a_i^{-p} \left(\sum_{j=0}^{p} \binom{p}{j} \alpha_i^{(p-j)}(x) \big((f_\epsilon^\alpha)^{(p)} - Lf^{(p)} \big)(x) \big) \right). \tag{15}$$

So the corresponding zipper fractal function f_ϵ^α now depends on the partition of I, scaling functions, signature, and L. So for the fixed parameters Δ, α, ϵ, and L, let us define an operator

$$\Omega_{\Delta,\epsilon}^{L,\alpha} : C^k(I) \to C^k(I), \ \ \Omega_{\Delta,\epsilon}^{L,\alpha}(f) = f_\epsilon^\alpha,$$

where $\Delta = \{x_1, x_2, \dots, x_N\} \subset I$, $\alpha(x) = (\alpha_i(x), \alpha_2(x), \dots, \alpha_{N-1}(x))$, $\epsilon = (\epsilon_1, \epsilon_2, \dots, \epsilon_{N-1})$, and the linear operator L are satisfying the conditions prescribed in Corollary 3.2. For $f \in C^k(I)$, $\Omega_{\Delta,\epsilon}^{L,\alpha}(f) \in C^k(I)$ i.e., the operator $\Omega^{L,\alpha}_{\Delta,\epsilon}$ is well defined and $\Omega^{L,\alpha}_{\Delta,\epsilon}(f)$ is a zipper fractal function corresponding to f. We call this operator $\Omega^{L,\alpha}_{\Delta,\epsilon}$ as a zipper fractal operator. For fixed L and Δ, we denote $\Omega^{L,\alpha}_{\Delta,\epsilon} = \Omega^{\alpha}_{\epsilon}$. The operator norm for a bounded linear operator $S : (X_1, \|.\|_{X_1}) \to (X_2, \|.\|_{X_2})$ is defined as

$$\|S\| := \sup\{\|Sv\|_{X_2} : v \in X_1,\ \|v\|_{X_1} \le 1\}.$$

If $u_n = v_n|_{[0,1]}$, then $\{u_0, u_1, u_2, \dots\}$ is a Schauder basis for $C([0,1])$, where

$$v_0(t) = t,\ \ v_1(t) = 1 - t,\ \ t \in \mathbb{R},$$
$$v_2(t) = \begin{cases} 2t, & \text{if } 0 \le t < 1/2 \\ 2 - 2t, & \text{if } 1/2 < t \le 1 \\ 0, & \text{otherwise,} \end{cases}$$

and $v_{2^n+j}(t) = v_2(2^n t - j + 1)$ for $n = 1, 2, \dots$ and $j = 1, 2, \dots, 2^n$. Each u_n is a non-negative piecewise linear continuous function, known as a saw-tooth function.

Cheney in [10] provided Schauder basis of polygonal functions for the space $C(I)$ endowed with supremum norm. Schonefeld in [25] constructed Schauder basis for $C^k(I)$ corresponding to any Schauder basis of $C(I)$. Therefore, Schauder basis for $C^k(I)$ exists. If we prove that the operator $\Omega^{\alpha}_{\epsilon}$ is an isomorphism from $C^k(I)$ to $C^k(I)$, then our aim for this section is completed. Thus, we will show that for some prescribed conditions, the zipper fractal operator is an isomorphism from $C^k(I)$ to $C^k(I)$. For the main theorem of this section, we will prove some related results first.

Proposition 5.1. *Zipper fractal operator $\Omega^{\alpha}_{\epsilon} : C^k(I) \to C^k(I)$ is linear and bounded .*

Proof. Let $f, g \in C^k(I)$ and $a, b \in \mathbb{R}$. Then linearty of L implies

$$(af + bg)^{\alpha}_{\epsilon}(L_i(x)) = (af + bg)(L_i(x)) + \alpha_i(x)\Big((af + bg)^{\alpha}_{\epsilon}(x) - (aLf + bLg)(x)\Big),$$

for all $i \in \mathbb{N}_{N-1}$. From the above equation we find that $af^{\alpha}_{\epsilon} + bg^{\alpha}_{\epsilon}$ is the fixed point of Read-Bajraktarević operator $T^{\alpha}_{\epsilon} : C^k(I) \to C^k(I)$, where

$$(T^{\alpha}_{\epsilon} h)(L_i(x)) := (af + bg)(L_i(x)) + \alpha_i(x)\Big(h(x) - (aLf + bLg)(x)\Big).$$

The uniqueness of the fixed point shows that $(af + bg)^{\alpha}_{\epsilon} = af^{\alpha}_{\epsilon} + bg^{\alpha}_{\epsilon}$. That is, for any $a, b \in \mathbb{R}$ and any $f, g \in C^k(I)$,

$$\Omega^{\alpha}_{\epsilon}(af + bg) = a\Omega^{\alpha}_{\epsilon}(f) + b\Omega^{\alpha}_{\epsilon}(g).$$

Thus $\Omega^{\alpha}_{\epsilon}$ is linear. From (15), we obtain

$$\begin{aligned} \|f^{\alpha}_{\epsilon} - f\|_k &\le \max\{(2/|a_i|)^k \|\alpha_i\|_k : i \in \mathbb{N}_{N-1}\}\|f^{\alpha}_{\epsilon} - Lf\|_k \\ &\le \frac{s}{1-s}\|f - Lf\|_k, \end{aligned} \tag{16}$$

where $s = \max\{(2/|a_i|)^k \|\alpha_i\|_k : i \in \mathbb{N}_{N-1}\}$. Hence,

$$\|f^{\alpha}_{\epsilon}\|_k - \|f\|_k \le \frac{s}{1-s}\|I_d - L\|\|f\|_k,$$

and we have

$$\|\Omega^{\alpha}_{\epsilon}(f)\|_k = \|f^{\alpha}_{\epsilon}\|_k \le \big(1 + \frac{s\|I_d - L\|}{1-s}\big)\|f\|_k. \tag{17}$$

Clearly, $\|\Omega^{\alpha}_{\epsilon}\| \le \big(1 + \frac{s\|I_d - L\|}{1-s}\big)$ and $\Omega^{\alpha}_{\epsilon}$ is bounded. □

Proposition 5.2. *Zipper fractal operator Ω_ϵ^α is injective and bounded below, when $\|\alpha_i\|_k < \big(\frac{|a_i|}{2}\big)^k \|L\|^{-1}$ for all $i \in \mathbb{N}_{N-1}$. In particular, the range of Ω_ϵ^α denoted by $\mathcal{R}(\Omega_\epsilon^\alpha)$ is a closed subspace of $C^k(I)$.*

Proof. For any $f \in C^k(I)$, let $\Omega_\epsilon^\alpha(f) = f_\epsilon^\alpha = 0$. Then from (16) we get,

$$\|f\|_k \leq s\|L\|\|f\|_k, \tag{18}$$

where $s = \max\{(2/|a_i|)^k \|\alpha_i\|_k : i \in \mathbb{N}_{N-1}\} < \|L\|^{-1}$. Since $s\|L\| < 1$, this gives $f = 0$, hence Ω_ϵ^α is injective. Also from (16) we get

$$\|f\|_k - \|f_\epsilon^\alpha\|_k \leq \|f - f_\epsilon^\alpha\|_k \leq s\|f_\epsilon^\alpha - Lf\|_k \leq s(\|f_\epsilon^\alpha\|_k + \|L\|\|f\|_k),$$

and this implies

$$\|f\|_k \leq \frac{1+s}{1 - s\|L\|}\|f_\epsilon^\alpha\|_k, \tag{19}$$

i.e.,

$$\frac{1 - s\|L\|}{1+s}\|f\|_k \leq \|\Omega_\epsilon^\alpha(f)\|_k, \ \ \forall f \in C^k(I). \tag{20}$$

Hence, the operator Ω_ϵ^α is bounded below.
Now let $g \in \overline{\mathcal{R}(\Omega_\epsilon^\alpha)}$, where $\mathcal{R}(\Omega_\epsilon^\alpha) := \{q \in C^k(I) : q = \Omega_\epsilon^\alpha(h) \text{ for some } h \in C^k(I)\}$. Then, there exist a sequence $\{f_{\epsilon,n}^\alpha\}$ in $\mathcal{R}(\Omega_\epsilon^\alpha) \subset C^k(I)$ such that $f_{\epsilon,n}^\alpha \xrightarrow{\|\cdot\|_k} g$ as $n \to \infty$. Since for each $n \in \mathbb{N}$, $f_{\epsilon,n}^\alpha \in \mathcal{R}(\Omega_\epsilon^\alpha)$, there exist $f_n \in C^k(I)$ such that $\Omega_\epsilon^\alpha(f_n) = f_{\epsilon,n}^\alpha$, $n \in \mathbb{N}$. From (20) we get,

$$\|f_n - f_m\|_k \leq \frac{1+s}{1 - s\|L\|}\|f_{\epsilon,n}^\alpha - f_{\epsilon,m}^\alpha\|_k.$$

Since $\{f_{\epsilon,n}^\alpha\}$ is a Cauchy sequence, the sequence $\{f_n\}$ is also a Cauchy sequence in the Banach space $C^k(I)$ and hence convergent, say, $f_n \xrightarrow{\|\cdot\|_k} f$ in $C^k(I)$. Boundedness of the map Ω_ϵ^α implies that $\Omega_\epsilon^\alpha(f_n) \xrightarrow{\|\cdot\|_k} \Omega_\epsilon^\alpha(f)$ and hence $g = \Omega_\epsilon^\alpha(f)$. Thus, $g \in \mathcal{R}(\Omega_\epsilon^\alpha)$, i.e., $\mathcal{R}(\Omega_\epsilon^\alpha)$ is closed in $C^k(I)$. This completes our proof. □

Remark 5.1.1. *Using (17) and (20), we get*

$$\frac{1 - s\|L\|}{1+s}\|f\|_k \leq \|\Omega_\epsilon^\alpha(f)\| \leq \big(1 + \frac{s\|I_d - L\|}{1-s}\big)\|f\|_k, \ \ \forall f \in C^k(I). \tag{21}$$

Now let us recall a result on bounded linear operators, which help us to proceed further.

Lemma 5.2. *([35]) If S is a bounded linear operator from a Banach space into itself such that $\|S\| < 1$, then $I - S$ has a bounded inverse and the Neumann series $\sum_{j=0}^{\infty} S^j$ converges in operator norm to $(I - S)^{-1}$.*

Now we will prove the main theorem of this section.

Theorem 5.3. *Let $\|\alpha_i\|_k < \big(\frac{|a_i|}{2}\big)^k (1 + \|I_d - L\|)^{-1}$ for all $i \in \mathbb{N}_{N-1}$, where I_d is the identity operator on $C^k(I)$. Then, Ω_ϵ^α is an isomorphism (linear, bijective and bicontinuous map). In particular, the space C^k admits a Schauder basis, consisting of C^k-continuous zipper fractal functions.*

Proof. Consider,

$$\|L\| - \|I_d - L\| \leq \|I_d - L + L\| = \|I_d\| = 1,$$

i.e.,

$$\|L\| \leq 1 + \|I_d - L\|.$$

Hence,

$$\|\alpha_i\|_k < \left(\frac{|a_i|}{2}\right)^k (1 + \|I_d - L\|)^{-1} \leq \left(\frac{|a_i|}{2}\right)^k \|L\|^{-1}.$$

From Proposition 5.1 and Proposition 5.2, we know that Ω_ϵ^α is a bounded, linear and injective operator. Now from (16),

$$\|f_\epsilon^\alpha - f\|_k \leq \frac{s}{1-s}\|f - Lf\|_k,$$

where $s < (1+\|I_d - L\|)^{-1}$ by given condition on the scaling functions. Now the inequality $s < (1+\|I_d-L\|)^{-1}$

$$\begin{aligned}
&\implies s(1+\|I_d - L\|) < 1,\\
&\implies s + s\|I_d - L\|) < 1,\\
&\implies \|I_d - L\| < \frac{1-s}{s},\\
&\implies \frac{s}{1-s}\|I_d - L\| < 1.
\end{aligned}$$

Thus,

$$\|\Omega_\epsilon^\alpha - I_d\| \leq \frac{s}{1-s}\|I_d - L\| < 1.$$

From Lemma 5.2, Ω_ϵ^α has a bounded inverse, therefore $\Omega_\epsilon^\alpha : C^k(I) \to C^k(I)$ is an isomorphism. Thus, Ω_ϵ^α preserves the basis, equivalently if $\{g_n\}$ is Schauder basis for $C^k(I)$, then the set of zipper fractal functions $\{g_{\epsilon,n}^\alpha\}$ is also Schauder basis for $C^k(I)$, where $g_{\epsilon,n}^\alpha = \Omega_\epsilon^\alpha(g_n)$. □

Theorem 5.4. *Let $\|\alpha_i\|_k < \left(\frac{|a_i|}{2}\right)^k(1+\|I_d - L\|)^{-1}$ for all $i \in \mathbb{N}_{N-1}$. If $\{g_n\}$ be a Schauder basis for $C^k(I)$ with associated coefficient functionals $\{\beta_n\}$, then $\{g_{\epsilon,n}^\alpha\}$ is the corresponding Schauder basis of zipper fractal functions with associated coefficient functionals $\{\beta_n \circ (\Omega_\epsilon^\alpha)^{-1}\}$, where $g_{\epsilon,n}^\alpha = \Omega_\epsilon^\alpha(g_n)$.*

Proof. From Theorem 5.3, we know that given assumptions on variable scaling functions implies Ω_ϵ^α is an isomorphism on $C^k(I)$. So if $\{g_n\}$ is Schauder basis for $C^k(I)$, then, $\{g_{\epsilon,n}^\alpha\}$ is the corresponding Schauder basis of zipper fractal functions for $C^k(I)$.
For $g \in C^k(I)$, we can write

$$g = \sum_{n=1}^{\infty} \beta_n(g) g_n.$$

Also, for $g \in C^k(I)$, we know $(\Omega_\epsilon^\alpha)^{-1}(g) \in C^k(I)$. Thus,

$$(\Omega_\epsilon^\alpha)^{-1}(g) = \sum_{n=1}^{\infty} \beta_n((\Omega_\epsilon^\alpha)^{-1}(g)) g_n.$$

By continuity of Ω_ϵ^α,

$$g = \sum_{n=1}^{\infty} \beta_n((\Omega_\epsilon^\alpha)^{-1}(g)) g_{\epsilon,n}^\alpha.$$

This completes our proof. □

6. Schauder basis consists of zipper fractal functions for $L^p(I)$

It can happen that a space X with some norm may not be a Banach space, but it is a subspace of a Banach space. We know that $C^k(I)$ with respect to norm $\|.\|_p$ $\left(\text{for } g \in L^p(I),\ \|g\|_p := (\int_I |g|^p dx)^{1/p}\right)$, $1 \leq p < \infty$ is not a Banach space, but it is a dense subspace of the Banach space $L^p(I)$. So for the extension of a linear and bounded operator, we have the following result:

Lemma 6.1. *If X_2 is Banach and X_1 is dense in X', then a linear and bounded operator $S : X_1 \to X_2$ can be extended to X' preserving the norm of S.*

From functional analysis we know that Schauder basis for $L^p(I)$ for $p \in [1,\infty)$ exists. It is known that Haar system $\{u_1, u_2, u_3, \dots\}$ with

$$u_1(t) = 1$$
$$u_2(t) = \begin{cases} 1, & \text{if } 0 \le t \le 1/2 \\ -1, & \text{if } 1/2 < t \le 1, \end{cases}$$

and for $n = 1, 2, \dots,\ j = 1, 2, \dots, 2^n$,

$$u_{2^n+j}(t) = \begin{cases} 2^{n/p}, & \text{if } (2j-2)/2^{n+1} \le t \le (2j-1)/2^{n+1}, \\ -2^{n/p}, & \text{if } (2j-1)/2^{n+1} < t \le 2j/2^{n+1}, \\ 0, & \text{otherwise,} \end{cases}$$

is a Schauder basis for $L^p([0,1])$ for $1 \le p < \infty$. The set of Legendre polynomials forms an orthonormal basis for $L^2(I)$. For $p \in (1,\infty)$, the sequence of trigonometric functions $\{1, cos(nx), sin(nx) : n \in \mathbb{N}\}$ is a basis for $L^p([0,2\pi])$.
So in this section, we will extend the operator $\Omega^\alpha_\epsilon : C^k(I) \to L^p(I)$ from $C^k(I)$ to $L^p(I)$ for $p \in [1,\infty)$ and we will show that for some prescribed conditions, the extension of Ω^α_ϵ is an isomorphism from the space $L^p(I)$ to itself. In particular, we will show that for $p \in [1,\infty)$, the space $L^p(I)$ admits a Schauder basis of zipper fractal functions. Aiming to this task, first we will prove some auxiliary results.
By Lemma 6.1, the density of $C^k(I)$ in $L^p(I)$ for $p \in [1,\infty)$ allows us to extend the operators $\Omega^\alpha_\epsilon : C^k(I) \to L^p(I)$ and $L : C^k(I) \to L^p(I)$ from $C^k(I)$ to $L^p(I)$, preserving the operator norm. Due to density of $C^k(I)$ in $L^p(I)$, if $g \in L^p(I)$, then there exist a sequence $\{g_n\} \subset C^k(I)$ such that $\lim_{n\to\infty} g_n = g$ (with respect to L^p-norm). Therefore, the extensions $\overline{\Omega^\alpha_\epsilon} : L^p(I) \to L^p(I)$ and $\overline{L} : L^p(I) \to L^p(I)$ are defined as if $g \in L^p(I)$, then $\overline{\Omega^\alpha_\epsilon}(g) = \lim_{n\to\infty} \Omega^\alpha_\epsilon(g_n) = \lim_{n\to\infty} g^\alpha_{\epsilon,n}$ i.e., $\overline{g^\alpha_\epsilon} = \overline{\Omega^\alpha_\epsilon}(g)$ is the limit of a sequence of zipper fractal functions $\{g^\alpha_{\epsilon,n}\} \subset C^k(I)$, and similarly $\overline{L}(g) = \lim_{n\to\infty} L(g_n)$. The function $\overline{g^\alpha_\epsilon}$ will be the zipper α-fractal function of $g \in L^p(I)$. These extensions preserve the norm, so we have $\|\overline{\Omega^\alpha_\epsilon}\| = \|\Omega^\alpha_\epsilon\|$ and $\|\overline{L}\| = \|L\|$.

Proposition 6.1. *For $f \in C^k(I)$, $p \in [1,\infty)$,*

$$\|f^\alpha_\epsilon - f\|_p \le \frac{\|\alpha\|_\infty}{1 - \|\alpha\|_\infty} \|f - Lf\|_p.$$

Proof. Zipper α-fractal function $\Omega^\alpha_\epsilon(f) = f^\alpha_\epsilon$ satisfies

$$f^\alpha_\epsilon(x) = f(x) + \alpha_i(L_i^{-1}(x))(f^\alpha_\epsilon(L_i^{-1}(x)) - Lf(L_i^{-1}(x))) \qquad x \in I_i, i \in \mathbb{N}_{N-1}.$$

So, we have

$$\|f^\alpha_\epsilon - f\|_p^p = \sum_{i=1}^{N-1} \int_{I_i} |\alpha_i(L_i^{-1}(x))|^p |(f^\alpha_\epsilon - Lf)(L_i^{-1}(x))|^p dx.$$

By putting $t = L_i^{-1}(x)$, we get

$$\begin{aligned} \|f^\alpha_\epsilon - f\|_p^p &= \sum_{i=1}^{N-1} \int_I (-1)^{\epsilon_i} a_i |\alpha_i(t)|^p |(f^\alpha_\epsilon - Lf)(t)|^p dt \\ &\le \sum_{i=1}^{N-1} (-1)^{\epsilon_i} a_i \|\alpha\|_\infty^p \|f^\alpha_\epsilon - Lf\|_p^p. \end{aligned}$$

Since $\sum_{i=1}^{N-1} (-1)^{\epsilon_i} a_i = \sum_{i=1}^{N-1} (-1)^{\epsilon_i} \frac{x_{i+1-\epsilon_i} - x_{i+\epsilon_i}}{x_N - x_1} = 1$, we have

$$\|f^\alpha_\epsilon - f\|_p^p \le \sum_{i=1}^{N-1} (-1)^{\epsilon_i} a_i \|\alpha\|_\infty^p \|f^\alpha_\epsilon - Lf\|_p^p = \|\alpha\|_\infty^p \|f^\alpha_\epsilon - Lf\|_p^p,$$

which implies

$$\|f_\epsilon^\alpha - f\|_p \le \|\alpha\|_\infty \|f_\epsilon^\alpha - Lf\|_p. \tag{22}$$

Using $\|f_\epsilon^\alpha - Lf\|_p \le \|f_\epsilon^\alpha - f\|_p + \|f - Lf\|_p$, we have the desired result:

$$\|f_\epsilon^\alpha - f\|_p \le \frac{\|\alpha\|_\infty}{1-\|\alpha\|_\infty}\|f - Lf\|_p. \tag{23}$$

□

Remark 6.1.1. *Using Proposition 6.1, we get*

$$\|\overline{\Omega_\epsilon^\alpha}\| = \|\Omega_\epsilon^\alpha\| \le 1 + \frac{\|\alpha\|_\infty \|I_d - L\|}{1-\|\alpha\|_\infty}. \tag{24}$$

Proposition 6.2. *For any $f \in L^p(I)$, $p \in [1,\infty)$,*

$$\|\overline{\Omega_\epsilon^\alpha}(f) - f\|_p \le \frac{\|\alpha\|_\infty}{1-\|\alpha\|_\infty}\|f - \overline{L}(f)\|_p. \tag{25}$$

Proof. Let $f \in L^p(I)$. Then $\overline{\Omega_\epsilon^\alpha}(f) = \lim_{n\to\infty} \Omega_\epsilon^\alpha(f_n)$ and $\overline{L}(f) = \lim_{n\to\infty} L(f_n)$, where $\{f_n\} \subset C^k(I)$ is a sequence such that $f = \lim_{n\to\infty} f_n$ with respect to L^p-norm. Since $f_n \in C^k(I)$, we have $\overline{\Omega_\epsilon^\alpha}(f_n) = \Omega_\epsilon^\alpha(f_n)$, $\overline{L}(f_n) = L(f_n)$. The continuity of the norm gives

$$\|\overline{\Omega_\epsilon^\alpha}(f) - f\|_p = \lim_{n\to\infty} \|\overline{\Omega_\epsilon^\alpha}(f_n) - f_n\|_p.$$

Using (22),

$$\begin{aligned}
\|\overline{\Omega_\epsilon^\alpha}(f) - f\|_p &= \lim_{n\to\infty} \|\Omega_\epsilon^\alpha(f_n) - f_n\|_p,\\
&\le \|\alpha\|_\infty \lim_{n\to\infty} \|\Omega_\epsilon^\alpha(f_n) - L(f_n)\|_p,\\
&= \|\alpha\|_\infty \lim_{n\to\infty} \|\overline{\Omega_\epsilon^\alpha}(f_n) - \overline{L}(f_n)\|_p,\\
&= \|\alpha\|_\infty \|\overline{\Omega_\epsilon^\alpha}(f) - \overline{L}(f)\|_p,\\
&\le \|\alpha\|_\infty \|\overline{\Omega_\epsilon^\alpha}(f) - f + f - \overline{L}(f)\|_p,\\
&\le \|\alpha\|_\infty \|\overline{\Omega_\epsilon^\alpha}(f) - f\|_p + \|\alpha\|_\infty \|f - \overline{L}(f)\|_p.
\end{aligned}$$

From above we easily get the inequality (25). □

Proposition 6.3. *If $\|\alpha\|_\infty < \|L\|^{-1}$, then $\overline{\Omega_\epsilon^\alpha}$ is injective and bounded below. In particular, the range of $\overline{\Omega_\epsilon^\alpha}$ is closed in $L^p(I)$.*

Proof of the above proposition is similar to the proof of Proposition 5.2. We know that $\|L\| = \|\overline{L}\|$, so similar to (20), we can easily obtain

$$\|f\|_p \le \frac{1+\|\alpha\|_\infty}{1-\|L\|\|\alpha\|_\infty}\|\overline{\Omega_\epsilon^\alpha}(f)\|_p, \ \ \forall f \in L^p(I). \tag{26}$$

Now we will prove the main theorem of this section.

Theorem 6.2. *Let $\|\alpha\|_\infty < (1 + \|I_d - L\|)^{-1}$. For $p \in [1,\infty)$, if $\{g_n\}$ is a Schauder basis of $L^p(I)$, then $\{\overline{\Omega_\epsilon^\alpha}(g_n)\}$ is a Schauder basis of $L^p(I)$.*

Proof. Since $\overline{L}$ is extension of operator L, the extension of $I_d - L$ (here, $I_d : C^k(I) \to L^p(I)$ such that $I_d(f) = f$ for all $f \in C^k(I)$) agrees with $\overline{I_d} - \overline{L}$ (here, $\overline{I_d} : L^p(I) \to L^p(I)$ such that $\overline{I_d}(f) = f$ for all $f \in L^p(I)$) and $\|I_d - L\| = \|\overline{I_d} - \overline{L}\|$. Now the inequality $\|\alpha\|_\infty < (1 + \|I_d - L\|)^{-1}$ gives that

$$\begin{aligned} & \|\alpha\|_\infty(1 + \|I_d - L\|) < 1, \\ \implies & \|\alpha\|_\infty + \|\alpha\|_\infty\|I_d - L\| < 1, \\ \implies & \|I_d - L\| < \frac{1 - \|\alpha\|_\infty}{\|\alpha\|_\infty}, \\ \implies & \frac{\|\alpha\|_\infty}{1 - \|\alpha\|_\infty}\|I_d - L\| < 1, \\ \implies & \frac{\|\alpha\|_\infty}{1 - \|\alpha\|_\infty}\|\overline{I_d} - \overline{L}\| < 1. \end{aligned}$$

Using (25) and previous inequality, we obtain the L_p operator norm

$$\|\overline{I_d} - \overline{\Omega^\alpha_\epsilon}\| < 1.$$

Therefore, Lemma 5.2 implies that $\overline{\Omega^\alpha_\epsilon} : L_p \to L_p$ is an isomorphism and hence preserves the basis. This completes our proof. □

7. Zipper fractal versions of full Müntz theorems

In this section, we introduce zipper fractal Müntz space and some of its properties using the fixed interval $I = [0, 1]$ and zipper fractal operator $\Omega^\alpha_\epsilon : C(I) \to C(I) \subset L^p(I)$ for $p \in [1, \infty)$.
Let $\Lambda := \{\lambda_k\}_{k=1}^\infty$ be a sequence of distinct real numbers with $0 = \lambda_1 < \lambda_2 < \dots$. Bernstein [5] conjectured that

$$\sum_{k=2}^{\infty} \frac{1}{\lambda_k} = \infty,$$

it is a necessary and sufficient condition for the linear space $\Pi(\Lambda) := \text{span}\{x^{\lambda_k} : k = 1, 2, \dots\}$ to become dense in $C([0, 1])$. This beautiful conjecture that connects the density of a particular subset of a functional space to the divergence of a particular harmonic series was proved by Müntz in [16] and the linear space $\Pi(\Lambda)$ over $\mathbb{R}$ is called the Müntz space associated with Λ. Since then, many generalizations of this theorem have been presented.

Theorem 7.1. *(**Full Müntz Theorem for** $C([0, 1])$) Let Λ be such that $\lambda_k > 0$ for all k, except $\lambda_1 = 0$. Then $\Pi(\Lambda)$ is dense in $C([0, 1])$ if and only if*

$$\sum_{k=1}^{\infty} \frac{\lambda_k}{\lambda_k^2 + 1} = \infty.$$

The above theorem was proved by Siegel in [27] and also proved differently by Borwein and Erdélyi [6, 7] using techniques given by Szász [28]. Researchers then changed the space from $C(I)$ to $L^p(I)$ and provided the necessary and sufficient condition for the density of Müntz space associated with Λ in $L^p(I)$. The following theorem was proved for $p = 1$ by Borwein and Erdélyi and for $1 < p < \infty$ by Operstein [23].

Theorem 7.2. *(**Full Müntz Theorem for** $L^p([0, 1])$) Let $p \in [1, \infty)$ and Λ be such that $\lambda_k > -1/p$ for all k. Then $\Pi(\Lambda)$ is dense in $L^p([0, 1])$ if and only if*

$$\sum_{k=1}^{\infty} \frac{\lambda_k + \frac{1}{p}}{(\lambda_k + \frac{1}{p})^2 + 1} = \infty.$$

With $x_1 = 0$ and $x_N = 1$, let $\Delta = \{x_1, x_2, \dots, x_N\}$ be a partition of $I = [0,1]$ and $L : C(I) \to C(I)$ is a linear and bounded operator satisfying $Lf(x_1) = f(x_1)$ and $Lf(x_N) = f(x_N)$. If $\frac{-1}{p} < \lambda_k < 0$, then $x^{\lambda_k} \in L^p(I)$. Define

$$(x^{\lambda_k})^{\alpha}_{\epsilon} = \begin{cases} \Omega^{\alpha}_{\epsilon}(x^{\lambda_k}) & \text{if } \lambda_k \geq 0, \\ \overline{\Omega^{\alpha}_{\epsilon}}(x^{\lambda_k}) & \text{if } \frac{-1}{p} < \lambda_k < 0, \end{cases} \tag{27}$$

where $\alpha(x) = (\alpha_1(x), \alpha_2(x), \dots, \alpha_{N-1}(x))$ is a continuous function such that $\|\alpha\|_\infty := \max\{\|\alpha_i\|_\infty : i \in \mathbb{N}_{N-1}\} < 1$, and $\epsilon = (\epsilon_1, \epsilon_2, \dots, \epsilon_{N-1}) \in \Gamma := \{0,1\}^{N-1}$.
For the sequence of continuous functions $\alpha^j(x) = (\alpha^j_1(x), \alpha^j_2(x), \dots, \alpha^j_{N-1}(x))$ (not zero on all I) such that $\|\alpha^j\|_\infty \to 0$ as $j \to \infty$, let $\xi = \{(x^{\lambda_k})^{\alpha^j}_{\epsilon} : k, j \in \mathbb{N}, \epsilon \in \Gamma\}$, then the zipper fractal Müntz space associated with Λ is defined as $\Pi^{\alpha}_{\epsilon}(\Lambda) := \text{span}(\xi)$.

Theorem 7.3. *Let Δ be a partition of $I = [0,1]$, L be a linear and bounded operator on $C(I)$ with respect to the uniform norm, and $\{\alpha^j\}$ be a sequence of non-zero variable scaling functions such that $\|\alpha^j\|_\infty \to 0$ as $j \to \infty$. If the set span$\{g_k : k \in \mathbb{N}\}$ is dense in $C(I)$, then the set span$\{(g_k)^{\alpha^j}_{\epsilon} : k, j \in \mathbb{N}, \epsilon \in \Gamma\}$ is also dense in $C(I)$.*

Proof. Let $g \in C(I)$ and $\delta > 0$ be given. Since the set span$\{g_k : k \in \mathbb{N}\}$ is dense in $C(I)$, there exists $f \in \text{span}\{g_k : k \in \mathbb{N}\}$ such that

$$\|f - g\|_\infty < \frac{\delta}{2}. \tag{28}$$

Since $\|\alpha^j\|_\infty \to 0$ as $j \to 0$, we can find α^{j_*} $(j_* \in \mathbb{N})$ such that

$$\frac{(1 + \|L\|)\|\alpha^{j_*}\|_\infty}{1 - \|\alpha^{j_*}\|_\infty}\|f\|_\infty < \frac{\delta}{2}. \tag{29}$$

Now we have $f^{\alpha^{j_*}}_{\epsilon} \in \text{span}\{(g_k)^{\alpha^j}_{\epsilon} : k, j \in \mathbb{N}, \epsilon \in \Gamma\}$, and it satisfies

$$f^{\alpha^{j_*}}_{\epsilon}(L_i(x)) = f(L_i(x)) + \alpha^{j_*}_i(x)(f^{\alpha^{j_*}}_{\epsilon}(x) - Lf(x)).$$

Therefore,

$$\|f^{\alpha^{j_*}}_{\epsilon} - f\|_\infty \leq \frac{\|\alpha^{j_*}\|_\infty}{1 - \|\alpha^{j_*}\|_\infty}\|f - Lf\|_\infty \leq \frac{(1 + \|L\|)\|\alpha^{j_*}\|_\infty}{1 - \|\alpha^{j_*}\|_\infty}\|f\|_\infty. \tag{30}$$

Then, using (28)-(30), we have

$$\|f^{\alpha^{j_*}}_{\epsilon} - g\|_\infty \leq \|f - g\|_\infty + \|f^{\alpha^{j_*}}_{\epsilon} - f\|_\infty < \frac{\delta}{2} + \frac{\delta}{2} < \delta. \tag{31}$$

Hence, span$\{(g_k)^{\alpha^j}_{\epsilon} : k, j \in \mathbb{N}, \epsilon \in \Gamma\}$ is dense in $C(I)$. ☐

In the similar way we can easily proved the following theorem.

Theorem 7.4. *Let Δ be a partition of $I = [0,1]$, L be a linear and bounded operator with respect to the L^p norm, and $\{\alpha^j\}$ be a sequence of non-zero variable scaling functions such that $\|\alpha^j\|_\infty \to 0$ as $j \to \infty$. If the set span$\{g_k : k \in \mathbb{N}\}$ is dense in $L^p(I)$ for $p \in [1, \infty)$, then the set span$\{(g_k)^{\alpha^j}_{\epsilon} : k, j \in \mathbb{N}, \epsilon \in \Gamma\}$ is also dense in $L^p(I)$.*

Using Theorems 7.1 and 7.3, we have following result.

Theorem 7.5. *(**Zipper Fractal version of Full Müntz Theorem for** $C([0,1])$) Let Λ be such that $\lambda_k > 0$ for all k, except $\lambda_1 = 0$. Further let Δ be a partition of $I = [0,1]$, L be a linear and bounded operator on $C(I)$ with respect to the uniform norm, and $\{\alpha^j\}$ be a sequence of non-zero variable scaling functions such that $\|\alpha^j\|_\infty \to 0$ as $j \to \infty$. Then the zipper fractal Müntz space $\Pi^{\alpha}_{\epsilon}(\Lambda) = \text{span}(\xi)$ associated with Λ is dense in $C(I)$, if*

$$\sum_{k=1}^{\infty} \frac{\lambda_k}{\lambda_k^2 + 1} = \infty.$$

Similarly, using Theorems 7.2 and 7.4 we have following result:

Theorem 7.6. *(Zipper Fractal version of Full Müntz Theorem for* $L^p([0,1])$*) Let* $p \in [1,\infty)$ *and* Λ *be such that* $\lambda_k > -1/p$ *for all* k*. Further, let* Δ *be a partition of* $I = [0,1]$*,* L *be a linear and bounded operator with respect to the* L^p *norm, and* $\{\alpha^j\}$ *be a sequence of non-zero variable scaling functions such that* $\|\alpha^j\|_\infty \to 0$ *as* $j \to \infty$*. Then, the zipper fractal Müntz space* $\Pi^\alpha_\epsilon(\Lambda) = span(\xi)$ *is dense in* $L^p([0,1])$*, if*

$$\sum_{k=1}^{\infty} \frac{\lambda_k + \frac{1}{p}}{(\lambda_k + \frac{1}{p})^2 + 1} = \infty.$$

8. Conclusions

In the present work, we have developed the theory of zipper fractal interpolation functions with variable scaling. We have constructed the differentiable zipper fractal interpolation functions and zipper α-fractal functions with variable scaling. Using some constraints on scaling functions or a base function, we have constructed positive or monotone zipper α-fractal functions for given positive or monotone function respectively. We have found sufficient conditions on zipper IFS parameters for a one-sided approximation of a given continuous function by zipper α-fractal functions. We have shown the existence of Schauder basis consists of zipper fractal functions for the spaces $C^k(I)$ and $L^p(I)$, where $p \in [1,\infty)$. In the end, we have introduced zipper fractal Müntz space and proved zipper fractal versions of the full Müntz theorem for $C([0,1])$ and $L^p([0,1])$ where $p \in [1,\infty)$.

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