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Coupled systems of subdifferential type with integral perturbation and fractional differential equations

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Abstract

This paper is mainly devoted to study a class of first-order differential inclusions governed by time-dependent subdifferential operators involving an integral perturbation. We also handle the associated second-order differential inclusion. Our final topic, accomplished in infinite-dimensional Hilbert spaces, is to develop some variants related to coupled systems by such differential inclusions and fractional differential equations.

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1. introduction

Recently, integro-differential sweeping processes have been investigated in the scientific literature see e.g., [9], [10], [11], [12], [27], see also [16] when integro-differential inclusions for m-accretive operators have been considered. Here, we deal with a new class of First-Order Problems described by subdifferential operators with integral perturbation as follows

$$(FOP)\begin{cases} -\dot{x}(t) \in \partial\varphi(t, x(t)) + \displaystyle\int_0^t f(t, s, x(s))ds \quad \text{a.e. } t \in I := [0, 1], \\ x(0) = x_0 \in \operatorname{dom}\varphi(0, \cdot), \end{cases}$$

where $\partial\varphi(t, \cdot)$ stands for the subdifferential of a proper, lower semi-continuous, convex function $\varphi(t, \cdot)$ from a real separable Hilbert space H into $[0, +\infty]$, whose effective domain is denoted $\operatorname{dom}\varphi(t, \cdot)$ (for each $t \in I$).

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We adopt an assumption expressed in term of the conjugate function of φ namely Peralba's assumption (see (H_2) below). We require the map $f : I \times I \times H \to H$ to be measurable, Lipschitz with respect to its third variable on bounded subsets of H, and satisfying a suitable linear growth condition.
Unlike the aforementioned references [10], [11], [16], where a discretization approach is used there, we proceed by Schauder's fixed point theorem to show the well-posedness result to (FOP).
First-order differential inclusions of subdifferential type have been studied in the pioneering works [6], [13], [30], [32], [43], [44], [45], and then have been extensively developed in the literature, where the main attention was paid to the existence and uniqueness of solutions and various applications; see e.g., [15], [24], [29], [34], [39], [40], [42], with their references.

Then, we address the Second-Order Problem with integral perturbation

$$(SOP)\begin{cases} -\ddot{x}(t) \in \partial\varphi(t, \dot{x}(t)) + \int_0^t f(t, s, x(s), \dot{x}(s))ds \quad \text{a.e. } t \in I, \\ x(0) = x_0,\ \dot{x}(0) = v_0 \in \operatorname{dom}\varphi(0, \cdot). \end{cases}$$

By a reduction to the corresponding first-order differential inclusion and an adaptation of the techniques used in the study of (FOP), we state the existence result to (SOP).
Recent contributions on second-order problems driven by subdifferential operators (under (H_2) below) have been appeared in [35], [38]. For some results regarding second-order evolution problems with subdifferentials or maximal monotone operators, see e.g., [1], [2], [7], [19], [20], [23], [36], [37].

Fractional differential theory and applications constitute an important field of research with serious mathematical achievements discussed by several authors in recent years, see e.g., [3], [4], [5], [8], [17], [18], [21], [31], [33], [41], [46], among others. More recently, coupled systems by evolution problems of subdifferential type and fractional differential equations have been treated in [25] and [34], while related systems subject to maximal monotone operators have been occurred in e.g., [19], [20], [25]. We refer the reader to [16] dealing with fractional-order boundary problems involving differential inclusions governed by m-accretive operators.
We are motivated by the investigation in [16]. So, in the final topic of the paper we aim to establish new results in the new context of coupled fractional differential equations by evolution problems involving subdifferentials with integral perturbations. The first one concerns the fractional differential inclusion with nonlocal boundary conditions

$$\begin{cases} -\dot{x}(t) \in \partial\varphi(t, x(t)) + \displaystyle\int_0^t f(t, s, u(s), x(s))ds \quad \text{a.e. } t \in I, \\ D^\alpha u(t) + \lambda D^{\alpha-1}u(t) = x(t), \quad t \in I, \\ I_{0^+}^\beta u(t)|_{t=0} = 0,\ u(1) = I_{0^+}^\gamma u(1), \\ \quad x(0) = x_0 \in \operatorname{dom}\varphi(0, \cdot), \end{cases}$$

where $\alpha \in]1, 2]$, $\beta \in [0, 2-\alpha]$, $\lambda \geq 0$, $\gamma > 0$, and D^α stands for the Riemann-Liouville fractional derivative of order α.
Our approach consists of combining the existence result to (FOP) and the topological properties of the solution set to the fractional differential equation above (see [18]), in order to apply the fixed point theorem.
In the same spirit, relying on the aforementioned arguments and the structure of the solution set to some differential equations, we succeed in establishing new results regarding other variants of coupled systems.

In what follows, Section 2 provides notation and preliminaries. Section 3 contains the statement of overall assumptions and the main result regarding (FOP). Section 4 is devoted to study (SOP), while Section 5 is concerned with fractional-order equations coupled with (FOP).

2. notation and preliminaries

Throughout the paper, let $I := [0, 1]$ denote an interval of $\mathbb{R}$ and let H be a real separable Hilbert space whose inner product is denoted by $\langle \cdot, \cdot \rangle$ and the associated norm by $\| \cdot \|$. We denote by $\overline{B}_H[x, r]$ the closed ball of center x and radius r on H, and by $\overline{B}_H$ the closed unit ball. Let $\overline{co}(S)$ stand for the closed convex hull of a subset S of H. On the space $\mathcal{C}_H(I)$ of continuous maps $x : I \to H$, we consider the norm of uniform

convergence on I, $\|x\|_\infty = \sup_{t\in I} \|x(t)\|$.
By $L^p_H(I)$ for $p \in [1, +\infty[$ (resp. $p = +\infty$), we denote the space of measurable maps $x : I \to H$ such that $\int_I \|x(t)\|^p dt < +\infty$ (resp. which are essentially bounded) endowed with the usual norm $\|x\|_{L^p_H(I)} = (\int_I \|x(t)\|^p dt)^{\frac{1}{p}}$, $1 \le p < +\infty$ (resp. endowed with the usual essential supremum norm $\|\cdot\|_{L^\infty_H(I)}$).
Denote by $W^{1,2}_H(I)$, the space of absolutely continuous functions from I to H with derivatives in $L^2_H(I)$.
Denote by $W^{2,2}_H(I)$ (resp. $W^{2,1}_H(I)$), the space of absolutely continuous functions $u : I \to H$ with absolutely continuous derivatives w such that $\dot{w} \in L^2_H(I)$ (resp. $\dot{w} \in L^1_H(I)$).
We denote by $\sigma(E, E')$ the weak topology on a topological space E, where E' is the topological dual of E. Recall that the topological dual of $L^1_H(I)$ is $L^\infty_H(I)$.
Let φ be a lower semi-continuous convex function from H into $\mathbb{R} \cup \{+\infty\}$ which is proper in the sense that its effective domain ($\operatorname{dom}\varphi$) defined by

$$\operatorname{dom}\varphi = \{x \in H : \varphi(x) < +\infty\}$$

is non-empty. As usual, its Fenchel conjugate is defined by

$$\varphi^*(v) = \sup_{x\in H} [\langle v, x\rangle - \varphi(x)].$$

The subdifferential $\partial\varphi(x)$ of φ at $x \in \operatorname{dom}\varphi$ is

$$\partial\varphi(x) = \{v \in H : \varphi(y) \ge \langle v, y - x\rangle + \varphi(x)\ \forall y \in \operatorname{dom}\varphi\},$$

and its effective domain is $\operatorname{Dom}\partial\varphi = \{x \in H : \partial\varphi(x) \ne \emptyset\}$. It is well known (see, e.g., [13]) that if φ is a proper lower semi-continuous convex function, then its subdifferential operator $\partial\varphi$ is a maximal monotone operator.

For more details on the properties of maximal monotone operators in Hilbert spaces, we refer to [13]. We also refer to [26] for details concerning convex analysis and measurable set-valued maps.

We recall Schauder's fixed point theorem [28].

Theorem 2.1. *Let C be a non-empty closed bounded convex subset of a Banach space E. Let $f : C \to C$ be a continuous map. If $f(C)$ is relatively compact, then f has a fixed point.*

We need the following lemma (see Lemma A.5 [13]).

Lemma 2.2. *Let $h \in L^1_{\mathbb{R}}(I)$ be such that $h \ge 0$ a.e. on I and let α be a positive real constant. Consider a continuous function $g : I \to \mathbb{R}$ satisfying*

$$\frac{1}{2} g^2(t) \le \frac{1}{2}\alpha^2 + \int_0^t h(s)g(s)ds \quad \text{for all } t \in I.$$

Then, one has

$$|g(t)| \le \alpha + \int_0^t h(s)ds \quad \text{for all } t \in I.$$

We end this section by recalling the Gronwall-like differential inequality proved in [10].

Lemma 2.3. *Let $y : I \to \mathbb{R}$ be a non-negative absolutely continuous function and let $h_1, h_2, g : I \to \mathbb{R}_+$ be non-negative integrable functions. Suppose for some $\varepsilon > 0$*

$$\dot{y}(t) \le g(t) + \varepsilon + h_1(t)y(t) + h_2(t)(y(t))^{\frac{1}{2}} \int_0^t (y(s))^{\frac{1}{2}} ds \text{ a.e. } t \in I.$$

Then, for all $t \in I$, one has

$$\begin{aligned}(y(t))^{\frac{1}{2}} &\le (y(0)+\varepsilon)^{\frac{1}{2}} \exp\left(\int_0^t (h(s)+1)ds\right) + \frac{\varepsilon^{\frac{1}{2}}}{2}\int_0^t \exp\left(\int_s^t (h(r)+1)dr\right)ds \\ &+ 2\left[\left(\int_0^t g(s)ds+\varepsilon\right)^{\frac{1}{2}} - \varepsilon^{\frac{1}{2}}\exp\left(\int_0^t (h(r)+1)dr\right)\right] \\ &+ 2\int_0^t \Big(h(s)+1\Big)\exp\left(\int_s^t (h(r)+1)dr\right)\left(\int_0^s g(r)dr+\varepsilon\right)^{\frac{1}{2}} ds,\end{aligned}$$

where $h(t) = \max\left(\frac{h_1(t)}{2}, \frac{h_2(t)}{2}\right)$ a.e. $t \in I$.

3. standing assumptions and statement of the main result

In the reminder of the paper, we consider the following assumptions.
Let $\varphi : I \times H \to [0, +\infty]$ be a map such that
(H_1) for each $t \in I$, the function $x \mapsto \varphi(t, x)$ is proper, lower semi-continuous, and convex;
(H_2) there exist a ρ-Lipschitz function $k : H \longrightarrow \mathbb{R}_+$ and an absolutely continuous function $a : I \to \mathbb{R}$, with a derivative $\dot{a} \in L^2_{\mathbb{R}_+}(I)$, such that

$$\varphi^*(t, x) \le \varphi^*(s, x) + k(x)|a(t) - a(s)| \text{ for every } (t, s, x) \in I \times I \times H.$$

(H_3) For each $t \in I$, $\operatorname{dom}\varphi(t, \cdot)$ is ball-compact, that is, for all $\xi > 0$, the set $\{x \in \operatorname{dom}\varphi(t, \cdot) : \ \|x\| \le \xi\}$ is compact for all $t \in I$.
(H_4) For each $t \in I$, $\operatorname{dom}\varphi(t, \cdot) \subset X(t) \subset M\overline{B}_H$, where $X : I \rightrightarrows H$ is a measurable set-valued map with convex compact values, and M is a non-negative real constant.
Let $f : I \times I \times H \to H$ be a map such that
(i) $f(\cdot, \cdot, x)$ is measurable on $I \times I$, for all $x \in H$;
(ii) there exists a non-negative function $\alpha(\cdot, \cdot) \in L^2_{\mathbb{R}}(I \times I)$ such that

$$\|f(t, s, x)\| \le \alpha(t, s)(1 + \|x\|) \text{ for all } (t, s, x) \in I \times I \times H;$$

(iii) for every $\eta > 0$, there exists a non-negative function $\beta_\eta(\cdot) \in L^2_{\mathbb{R}}(I)$ such that for all $(t, s) \in I \times I$ and for any $x, y \in \overline{B}_H[0, \eta]$

$$||f(t, s, x) - f(t, s, y)|| \le \beta_\eta(t)||x - y||.$$

Let $f : I \times I \times H \times H \to H$ be a map satisfying
(j) $f(\cdot, \cdot, x, y)$ is measurable on $I \times I$ for all $(x, y) \in H \times H$;
(jj) there exists a non-negative function $\kappa(\cdot, \cdot) \in L^2_{\mathbb{R}}(I \times I)$ such that for all $(t, s) \in I \times I$ and for all $(x, y) \in H \times H$, one has

$$\|f(t, s, x, y)\| \le \kappa(t, s)(1 + \|x\| + \|y\|);$$

(jjj) for every $\eta > 0$, there exists a non-negative function $\delta_\eta(\cdot) \in L^2_{\mathbb{R}}(I)$ such that for all $(t, s) \in I \times I$ and for any $x, y, u, v \in \overline{B}_H[0, \eta]$

$$\|f(t, s, u, x) - f(t, s, v, y)\| \le \delta_\eta(t)(\|u - v\| + \|x - y\|).$$

Let us recall an existence and uniqueness result from [32].

Theorem 3.1. *Let $\varphi : I \times H \to [0, +\infty]$ be a map satisfying (H_1)-(H_2).
Let $x_0 \in \operatorname{dom}\varphi(0, \cdot)$ be fixed. Then, the differential inclusion*

$$\begin{cases} -\dot{x}(t) \in \partial\varphi(t, x(t)) & \text{a.e. } t \in I, \\ \quad x(0) = x_0 \in \operatorname{dom}\varphi(0, \cdot), \end{cases}$$

has a unique absolutely continuous solution $x(\cdot)$ on I such that $x(t) \in \operatorname{dom}\varphi(t, \cdot)$ for all $t \in I$.

Now, denote by $A(t) := \partial\varphi(t,\cdot)$ the maximal monotone operator in H associated with $\partial\varphi(t,\cdot)$, $t \in I$ (φ satisfies conditions (H_1)-(H_2)). Let us consider the operator $\mathcal{A} : L^2_H(I) \rightrightarrows L^2_H(I)$ defined by

$$\mathcal{A}x = \{y \in L^2_H(I) : y(t) \in A(t)x(t) \text{ a.e.}\}.$$

Then, $\mathcal{A}$ is well defined since by Theorem 3.1, the differential inclusion

$$-\dot{x}(t) \in A(t)x(t) = \partial\varphi(t, x(t)) \text{ a.e. } t \in I,\ x(0) = x_0 \in \operatorname{dom}\varphi(0,\cdot),$$

admits a unique absolutely continuous solution.

The operator $\mathcal{A}$ enjoys the following property, see [32] (see also [24] or [34]).

Proposition 3.2. *Assume that for any $t \in I$, $A(t) = \partial\varphi(t,\cdot)$ where φ satisfies conditions (H_1)-(H_2). Then, one has*
$(\mathcal{J})$ the operator $\mathcal{A}$ is maximal monotone.
$(\mathcal{J}\mathcal{J})$ If $(x_n)_n$ and $(y_n)_n$ are two sequences in $L^2_H(I)$ such that $y_n(t) \in A(t)x_n(t)$ a.e. $t \in I$; the sequence $(x_n)_n$ strongly converges to x in $L^2_H(I)$; and $(y_n)_n$ weakly converges to y in $L^2_H(I)$. Then, one has $y(t) \in A(t)x(t)$ a.e. $t \in I$.

We will need the following useful application of Theorem 3.1, regarding an evolution problem with single-valued perturbation depending only on time (see Proposition 3.2 [40]).

Proposition 3.3. *Under the assumptions of Theorem 3.1, if $y \in L^2_H(I)$ and $x_0 \in \operatorname{dom}\varphi(0,\cdot)$, then the perturbed evolution problem*

$$\begin{cases} -\dot{x}(t) \in \partial\varphi(t, x(t)) + y(t) & \text{a.e. } t \in I, \\ x(0) = x_0 \in \operatorname{dom}\varphi(0,\cdot), \end{cases}$$

admits a unique absolutely continuous solution $x(\cdot)$ satisfying

$$\int_0^1 \|\dot{x}(t)\|^2 dt \le \sigma \|y\|^2_{L^2_H(I)} + d, \tag{1}$$

where d and σ are the non-negative real constants defined by

$$d = (k^2(0) + 3(\rho+1)^2) \int_0^1 \dot{a}^2(t) dt + 2[1 + \varphi(0, x_0)], \tag{2}$$

$$\sigma = k^2(0) + 3(\rho+1)^2 + 4. \tag{3}$$

Now, we are able to show the main result of this section concerning the well-posedness of (FOP).

Theorem 3.4. *Let $\varphi : I \times H \to [0, +\infty]$ be a map satisfying (H_1)-(H_2)-(H_3). Let $f : I \times I \times H \to H$ be a map satisfying (i)-(ii)-(iii). Then, for any $x_0 \in \operatorname{dom}\varphi(0,\cdot)$, there is a unique $W^{1,2}_H(I)$-solution $x(\cdot)$ to the First-Order Problem (FOP). Moreover, there exist non-negative real constants L and M depending on $k(\cdot)$, ρ, $\dot{a}(\cdot)$, $\|x_0\|$, $\varphi(0, x_0)$ and $\alpha(\cdot,\cdot)$ such that*

$$\int_0^1 \|\dot{x}(t)\|^2 dt \le L \text{ and } \|x(t)\| \le M \quad \text{for all } t \in I. \tag{4}$$

Proof. **Existence.** Let $x : I \to H$ be the unique absolutely continuous solution to the differential inclusion

$$\begin{cases} -\dot{x}(t) \in \partial\varphi(t, x(t)) & \text{a.e. } t \in I, \\ x(0) = x_0 \in \operatorname{dom}\varphi(0,\cdot), \end{cases}$$

ensured by Theorem 3.1. Let $z : I \to \mathbb{R}_+$ be the unique absolutely continuous solution to the differential equation

$$\dot{z}(t) = \int_0^t \alpha(t,s)(1+z(s))ds \quad \text{a.e. } t \in I \text{ with } z(0) = \sup_{t\in I} \|x(t)\|. \tag{5}$$

Let us define the set $\mathcal{Y}$ by

$$\mathcal{Y} := \{y \in L^2_H(I) : \|y(t)\| \le \dot{z}(t) \text{ a.e. } t \in I\},$$

which is convex $\sigma(L^2_H(I), L^2_H(I))$-compact.
For any $y \in \mathcal{Y}$, denote by x_y the unique absolutely continuous solution to the differential inclusion

$$\begin{cases} -\dot{x}_y(t) \in \partial\varphi(t, x_y(t)) + y(t) \quad \text{a.e. } t \in I, \\ x_y(0) = x_0 \in \operatorname{dom}\varphi(0,\cdot), \end{cases}$$

ensured by Proposition 3.3. Note that for all $t \in I$

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|x_y(t) - x(t)\|^2 &= \langle x_y(t) - x(t), \dot{x}_y(t) - \dot{x}(t)\rangle \\ &\le \langle y(t), x(t) - x_y(t)\rangle \\ &\le \|y(t)\|\|x_y(t) - x(t)\|, \end{aligned}$$

by monotonocity of $\partial\varphi(t,\cdot)$. Integrating then yields

$$\frac{1}{2}\|x_y(t) - x(t)\|^2 \le \int_0^t \|y(s)\|\|x_y(s) - x(s)\|ds.$$

An application of Lemma 2.2 gives

$$\|x_y(t) - x(t)\| \le \int_0^t \|y(s)\|ds,$$

along with (5), it follows

$$\|x_y(t)\| \le z(0) + \int_0^t \dot{z}(s)ds = z(t). \tag{6}$$

Observe by (ii) and (6) that for each $y \in \mathcal{Y}$

$$\begin{aligned} \left\|\int_0^t f(t,s,x_y(s))ds\right\| &\le \int_0^t \|f(t,s,x_y(s))\|ds \\ &\le \int_0^t \alpha(t,s)(1 + \|x_y(s)\|)ds \\ &\le \int_0^t \alpha(t,s)(1+z(s))ds = \dot{z}(t). \end{aligned} \tag{7}$$

Let us define the map Ψ for each $y \in \mathcal{Y}$ by

$$\Psi(y)(t) = \int_0^t f(t,s,x_y(s))ds \ \text{ for all } t \in I.$$

Let us equip $\mathcal{Y}$ by the $\sigma(L^2_H(I), L^2_H(I))$-topology. Recall that $\mathcal{Y}$ is convex $\sigma(L^2_H(I), L^2_H(I))$-compact and since H is separable, then $\mathcal{Y}$ is convex $\sigma(L^2_H(I), L^2_H(I))$-compact metrizable. Estimate (7) shows that $\Psi(y) \in \mathcal{Y}$ so that $\Psi : \mathcal{Y} \to \mathcal{Y}$. Now, let us check that Ψ is continuous. For this purpose, we prove that it is sequentially $\sigma(L^2_H(I), L^2_H(I))$-continuous on $\mathcal{Y}$.
Let $(y_n) \subset \mathcal{Y}$ be a sequence that $\sigma(L^2_H(I), L^2_H(I))$-converges to $y \in \mathcal{Y}$. Then, the absolutely continuous solution x_{y_n} associated with y_n to the evolution problem

$$\begin{cases} -\dot{x}_{y_n}(t) \in \partial\varphi(t, x_{y_n}(t)) + y_n(t) \quad \text{a.e. } t \in I,\ y_n \in \mathcal{Y}, \\ x_{y_n}(0) = x_0 \in \operatorname{dom}\varphi(0,\cdot), \end{cases}$$

satisfies (using (1) with the same constants d and σ given by (2) and (3))

$$\sup_{n\in\mathbb{N}} \int_0^1 \|\dot{x}_{y_n}(t)\|^2 dt \leq \sigma \|\dot{z}\|^2_{L^2_{\mathbb{R}}(I)} + d = L, \tag{8}$$

which yields by the absolute continuity of x_{y_n} and the Cauchy-Schwarz inequality

$$\sup_{n\in\mathbb{N}} \|x_{y_n}(t)\| \leq \|x_0\| + L^{\frac{1}{2}} = M \quad \text{for all } \ t \in I. \tag{9}$$

Since $x_{y_n}(t) \in \operatorname{dom} \varphi(t,\cdot)$, then taking assumption (H_3) into account, one deduces that $(x_{y_n}(t))$ is relatively compact in H, for each $t \in I$. It is clear that $(x_{y_n}(\cdot))$ is equi-continuous. By Ascoli's theorem, there is a map $v \in \mathcal{C}_H(I)$ such that (x_{y_n}) (up to a subsequence that we do not relabel) uniformly converges in $\mathcal{C}_H(I)$ to v with $v(0) = x_0$. Combining this with (8), one concludes that $(\dot{x}_{y_n})$ $\sigma(L^2_H(I), L^2_H(I))$-converges to some $w(\cdot)$ in $L^2_H(I)$ with $w = \dot{v}$ a.e. Hence, Proposition 3.2 gives

$$-\dot{v}(t) \in \partial\varphi(t, v(t)) + y(t) \ \text{ a.e. } t \in I,$$

and by the uniqueness, it follows $x_y = v$. This proves that (x_{y_n}) uniformly converges to x_y in $\mathcal{C}_H(I)$.
Let $g \in L^2_H(I)$. Note by (7) that

$$\left| \left\langle g(t), \int_0^t f(t,s,x_{y_n}(s))ds \right\rangle \right| \leq \dot{z}(t)\|g(t)\|, \tag{10}$$

with the map $t \mapsto \dot{z}(t)\|g(t)\| \in L^1_{\mathbb{R}}(I)$.
From (ii) and (9), remark that for each $n \in \mathbb{N}$

$$\|f(t,s,x_{y_n}(s))\| \leq \alpha(t,s)(1+M) \ \text{ for all } (t,s) \in I \times I.$$

Combining (9) and (iii), there is $\beta_M(\cdot) \in L^2_{\mathbb{R}}(I)$ such that

$$||f(t,s,x_{y_n}(s)) - f(t,s,x_y(s))|| \leq \beta_M(t)||x_{y_n}(s) - x_y(s)|| \ \text{ for all } (t,s) \in I \times I.$$

Noting that (x_{y_n}) uniformly converges to x_y, then, Lebesgue's dominated convergence theorem yields

$$\begin{aligned}
&\| \int_0^t f(t,s,x_{y_n}(s))ds - \int_0^t f(t,s,x_y(s))ds\| \\
&\leq \int_0^t \|f(t,s,x_{y_n}(s)) - f(t,s,x_y(s))\|ds \\
&\leq \int_0^t \beta_M(t)\|x_{y_n}(s) - x_y(s)\|ds \to \ 0 \quad \text{as } n \to \infty.
\end{aligned}$$

This along with (10) entail

$$\lim_{n\to\infty} \int_0^1 \left\langle g(t), \int_0^t f(t,s,x_{y_n}(s))ds \right\rangle dt = \int_0^1 \left\langle g(t), \int_0^t f(t,s,x_y(s))ds \right\rangle dt,$$

using Lebesgue's dominated convergence theorem.
This shows the $\sigma(L^2_H(I), L^2_H(I))$-convergence.
Hence, $\Psi(y_n)$ $\sigma(L^2_H(I), L^2_H(I))$-converges to $\Psi(y)$, that is, $\Psi : \mathcal{Y} \to \mathcal{Y}$ is continuous with respect to the $\sigma(L^2_H(I), L^2_H(I))$-topology. Applying Schauder's fixed point theorem (see Theorem 2.1), Ψ admits a fixed point, $y = \Psi(y)$. This justifies the existence of an absolutely continuous solution to (FOP).
A passage to the limit in (8) and (9) (invoking the preceding modes of convergence), yields the desired

estimates in (4).
Uniqueness. Let $x_1(\cdot)$ and $x_2(\cdot)$ be two solutions to (FOP). Since $\partial\varphi(t,\cdot)$ is monotone, then one has

$$\frac{1}{2}\frac{d}{dt}\|x_2(t)-x_1(t)\|^2 \le \left\langle \int_0^t f(t,s,x_1(s))ds - \int_0^t f(t,s,x_2(s))ds, x_2(t)-x_1(t) \right\rangle. \tag{11}$$

Since $\|x_1(t)\| \le M$ and $\|x_2(t)\| \le M$, for each $t \in I$, along with (iii), there is $\beta_M(\cdot) \in L^2_{\mathbb{R}}(I)$ such that

$$||f(t,s,x_1(s)) - f(t,s,x_2(s))|| \le \beta_M(t)||x_1(s)-x_2(s)|| \text{ for all } (t,s) \in I\times I,$$

so that coming back to (11), it follows

$$\frac{1}{2}\frac{d}{dt}\|x_2(t)-x_1(t)\|^2 \le \beta_M(t)\|x_2(t)-x_1(t)\| \int_0^t \|x_2(s)-x_1(s)\|ds.$$

An application of Lemma 2.3 with $\varepsilon > 0$ arbitrary yields $x_1 = x_2$ and guarantees the uniqueness of the solution to (FOP). $\square$

4. Second-order problem of subdifferential type with integral perturbation

The current section provides a result concerning (SOP).

Theorem 4.1. *Let $\varphi : I \times H \to H$ be a map satisfying (H_1)-(H_2)-(H_4). Let $f : I\times I\times H\times H \to H$ be a map satisfying (j)-(jj)-(jjj). Then, for any $(x_0,v_0) \in H \times \operatorname{dom}\varphi(0,\cdot)$, there is an absolutely continuous solution $(x,v) : I \to H\times H$ to the coupled system*

$$\begin{cases} -\dot{v}(t) \in \partial\varphi(t,v(t)) + \int_0^t f(t,s,x(s),v(s))ds \quad \text{a.e. } t\in I,\\ x(t) = x_0 + \int_0^t v(s)ds \quad \text{a.e. } t \in I,\\ v(0) = v_0 \in \operatorname{dom}\varphi(0,\cdot). \end{cases}$$

In other words, there exists a $W^{2,2}_H(I)$-solution $x(\cdot)$ to the Second-Order Problem (SOP).

Proof. For any continuous map $y : I \to H$, let us define the map f_y by $f_y(t,s,v) := f(t,s,y(s),v)$, for each $(t,s,v) \in I\times I\times H$. It is clear by (j) that $f_y(\cdot,\cdot,v)$ is measurable on $I\times I$. Moreover, by (jj) for all $(t,s,v) \in I\times I\times H$, one has

$$\|f_y(t,s,v)\| \le \kappa(t,s)(1+\|y(s)\|+\|v\|),$$

so that there exists $\kappa_1(\cdot,\cdot) \in L^2_{\mathbb{R}}(I\times I)$ such that

$$\|f_y(t,s,v)\| \le \kappa_1(t,s)(1+\|v\|).$$

Besides, by (jjj) for some $\eta > 0$, there exists $\delta_\eta(\cdot) \in L^2_{\mathbb{R}}(I)$ such that for all $(t,s) \in I\times I$ and for any $u, v, y(s) \in \overline{B}_H[0,\eta]$, one has

$$\|f_y(t,s,u) - f_y(t,s,v)\| = \|f(t,s,y(s),u) - f(t,s,y(s),v)\| \le \delta_\eta(t)\|u-v\|.$$

Hence, by Theorem 3.4 there is a unique absolutely continuous solution v_y to the evolution problem

$$(P_y)\begin{cases} -\dot{v}_y(t) \in \partial\varphi(t,v_y(t)) + \int_0^t f_y(t,s,v_y(s))ds \quad \text{a.e. } t\in I,\\ v_y(0) = v_0 \in \operatorname{dom}\varphi(0,\cdot), \end{cases}$$

with $\int_0^1 \|\dot{v}_y(t)\|^2 dt \le L$ for some non-negative real (appropriate) constant L (see Theorem 3.4) and $\|v_y(t)\| \le M$ for all $t \in I$, by (H_4).
Now, consider the closed convex subset $\mathcal{Y}$ in the Banach space $\mathcal{C}_H(I)$ by

$$\mathcal{Y} := \{x_g : I \to H : x_g(t) = x_0 + \int_0^t g(s)ds,\ g \in S^1_{M\overline{B}_H}\},$$

where $S^1_{M\overline{B}_H}$ denotes the set of all integrable selections of the convex weakly compact set-valued map $t \mapsto M\overline{B}_H$.
Next, define the map Λ on $\mathcal{Y}$ by

$$\Lambda(y)(t) = x_0 + \int_0^t v_y(s)ds, \text{ for all } t \in I \text{ and } y \in \mathcal{Y},$$

where v_y denotes the unique absolutely continuous solution to (P_y), and remark that $\Lambda(y) \in \mathcal{Y}$.
Since $v_y(t) \in \operatorname{dom}\varphi(t,\cdot)$, for each $t \in I$, then (H_4) entails that $v_y(t) \in X(t)$. Recall that the set-valued map X is a measurable integrably bounded with convex compact values (see (H_4)). Thus, for any $y \in \mathcal{Y}$, one obtains

$$\Lambda(y)(t) \in x_0 + \int_0^t \overline{co}(X(s))ds.$$

Note that $s \mapsto \overline{co}(X(s))$ is an integrably bounded set-valued map with convex compact values, then, the second member is convex compact-valued [14] so that $\Lambda(\mathcal{Y})(t)$ is relatively compact in H for each $t \in I$. Since moreover the set $\Lambda(\mathcal{Y})$ is equi-continuous, then $\Lambda(\mathcal{Y})$ is relatively compact in $\mathcal{C}_H(I)$.
Now, it remains to check that $\Lambda : \mathcal{Y} \to \mathcal{Y}$ is continuous.
Let $(y_n)_n \subset \mathcal{Y}$ be a sequence that uniformly converges to y in $\mathcal{Y}$. Then, for each n the absolutely continuous solution v_{y_n} associated with y_n to the evolution problem

$$\begin{cases} -\dot{v}_{y_n}(t) \in \partial\varphi(t, v_{y_n}(t)) + \int_0^t f(t,s,y_n(s),v_{y_n}(s))ds \quad \text{a.e. } t \in I, \\ v_{y_n}(0) = v_0 \in \operatorname{dom}\varphi(0,\cdot), \end{cases}$$

satisfies $\int_0^1 \|\dot{v}_{y_n}(t)\|^2 dt \le L$ and $\|v_{y_n}(t)\| \le M$, $t \in I$.
Since $v_{y_n}(t) \in \operatorname{dom}\varphi(t,\cdot)$, then, taking assumption (H_4) into account, one deduces that $(v_{y_n}(t))$ is relatively compact in H, for each $t \in I$. It is clear that $(v_{y_n}(\cdot))$ is equi-continuous. By Ascoli's theorem, there is a map $v \in \mathcal{C}_H(I)$ such that (v_{y_n}) (up to a subsequence that we do not relabel) uniformly converges in $\mathcal{C}_H(I)$ to v with $v(0) = v_0$. Since moreover $\sup\limits_n \int_0^1 \|\dot{v}_{y_n}(t)\|^2 dt \le L$, one concludes that $(\dot{v}_{y_n})$ $\sigma(L^2_H(I), L^2_H(I))$-converges to some $w(\cdot)$ in $L^2_H(I)$ with $w = \dot{v}$ a.e.
From (jj), remark that

$$\|f(t,s,y_n(s),v_{y_n}(s))\| \le \kappa(t,s)(1 + \|x_0\| + 2M) \text{ for all } (t,s) \in I \times I.$$

Let $g \in L^2_H(I)$. Then, the latter inequality yields

$$\left| \left\langle g(t), \int_0^t f(t,s,y_n(s),v_{y_n}(s))ds \right\rangle \right| \le (1 + \|x_0\| + 2M)\|g(t)\| \int_0^t \kappa(t,s)ds, \tag{12}$$

with the map $t \mapsto (1 + \|x_0\| + 2M)\|g(t)\| \int_0^t \kappa(t,s)ds \in L^1_{\mathbb{R}}(I)$.
Put $M_1 = M + ||x_0||$. Then, in view of (jjj), there is $\delta_{M_1}(\cdot) \in L^2_{\mathbb{R}}(I)$ such that

$$\begin{aligned} &||f(t,s,y_n(s),v_{y_n}(s)) - f(t,s,y(s),v(s))|| \\ &\le \delta_{M_1}(t)(||y_n(s) - y(s)|| + ||v_{y_n}(s) - v(s)||) \end{aligned}$$

for all $(t,s) \in I \times I$.
Noting that (v_{y_n}) (resp. (y_n)) uniformly converges to v (resp. y), then, Lebesgue's dominated convergence theorem yields

$$\begin{aligned} &\| \int_0^t f(t,s,y_n(s),v_{y_n}(s))ds - \int_0^t f(t,s,y(s),v(s))ds \| \\ &\le \int_0^t \|f(t,s,y_n(s),v_{y_n}(s)) - f(t,s,y(s),v(s))\|ds \\ &\le \int_0^t \delta_{M_1}(t)(||y_n(s) - y(s)|| + ||v_{y_n}(s) - v(s)||)ds \to 0 \text{ as } n \to \infty. \end{aligned}$$

This along with (12) entail

$$\lim_{n\to\infty}\int_0^1 \left\langle g(t), \int_0^t f(t,s,y_n(s),v_{y_n}(s))ds\right\rangle dt$$
$$= \int_0^1 \left\langle g(t), \int_0^t f(t,s,y(s),v(s))ds\right\rangle dt,$$

using Lebesgue's dominated convergence theorem. This justifies the $\sigma(L^2_H(I), L^2_H(I))$-convergence.
Hence, Proposition 3.2 yields

$$-\dot{v}(t) \in \partial\varphi(t,v(t)) + \int_0^t f(t,s,y(s),v(s))ds \quad \text{a.e. } t\in I,$$

and by uniqueness, it results $v_y = v$. This proves that (v_{y_n}) uniformly converges to v_y in $\mathcal{C}_H(I)$.
Coming back to the map Λ, for all $t\in I$, one has

$$\Lambda(y_n)(t) - \Lambda(y)(t) = \int_0^t v_{y_n}(s)ds - \int_0^t v_y(s)ds.$$

As $\|v_{y_n}(\cdot) - v_y(\cdot)\|_\infty \to 0$, and $\|v_{y_n}(\cdot) - v_y(\cdot)\|_\infty \leq 2M$, one concludes

$$\sup_{t\in I}\|\Lambda(y_n)(t) - \Lambda(y)(t)\| \leq \|v_{y_n}(\cdot) - v_y(\cdot)\|_\infty \to 0 \text{ as } n\to\infty.$$

Thus, $\Lambda : \mathcal{Y} \to \mathcal{Y}$ is continuous. Applying Schauder's fixed point theorem (see Theorem 2.1), the map Λ admits a fixed point, $y = \Lambda(y)$ with

$$\begin{cases} -\dot{v}_y(t) \in \partial\varphi(t,v_y(t)) + \int_0^t f(t,s,y(s),v_y(s))ds \quad \text{a.e. } t\in I,\\ y(t) = \Lambda(y)(t) = x_0 + \int_0^t v_y(s)ds, \quad t\in I,\\ v_y(0) = v_0 \in \operatorname{dom}\varphi(0,\cdot). \end{cases}$$

This justifies the existence of a $W^{2,2}_H(I)$-solution to (SOP). □

5. Coupled systems with subdifferentials and fractional differential equations

5.1. *Riemann-Liouville fractional derivative coupled with subdifferentials*

Before going further, let us recall some useful definitions and properties taken from [31], [41].

Definition 5.1. *Let $f : I \to H$. The fractional Bochner integral of order $\alpha > 0$ of the function f is defined by*

$$I^\alpha_{a^+} f(t) := \int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau)d\tau, \ \ t > a.$$

We recall the following lemma from [33].

Lemma 5.2. *Let $f \in L^1_H(I)$. One has*
if $0 < \alpha < 1$, then $I^\alpha f$ exists a.e. on I and one has $I^\alpha f \in L^1_H(I)$.
If $\alpha \geq 1$, then $I^\alpha f \in \mathcal{C}_H(I)$.

Definition 5.3. *(Riemann-Liouville fractional derivative) Let $f \in L^1_H(I)$. Define the Riemann-Liouville fractional derivative of order $\alpha > 0$ of f by*

$$D^\alpha f(t) := D^\alpha_{0^+} f(t) = \frac{d^n}{dt^n} I^{n-\alpha}_{0^+} f(t) = \frac{d^n}{dt^n}\int_0^t \frac{(t-s)^{n-\alpha-1}}{\Gamma(n-\alpha)} f(s)ds,$$

where $n = [\alpha] + 1$.

Denote by

$$W^{\alpha,1}_H(I) := \{u \in \mathcal{C}_H(I) : \ D^{\alpha-1}u \in \mathcal{C}_H(I), \ D^\alpha u \in L^1_H(I)\}.$$

5.1.1. Coupled systems with nonlocal boundary conditions

We need to define the Green function and its properties taken from [18].

Lemma 5.4. *Let $\alpha \in]1,2]$, $\beta \in [0, 2-\alpha]$, $\lambda \geq 0$ and $\gamma > 0$. Let $G_1 : I \times I \to \mathbb{R}$ be a function defined by*

$$G_1(t,s) = \phi(s) I_{0^+}^{\alpha-1}(\exp(-\lambda t)) + \begin{cases} \exp(\lambda s) I_{s^+}^{\alpha-1}(\exp(-\lambda t)), & 0 \leq s \leq t \leq 1, \\ 0 & 0 \leq t \leq s \leq 1, \end{cases}$$

where

$$\phi(s) = \frac{\exp(\lambda s)}{\mu_0}\left[\left(I_{s^+}^{\alpha-1+\gamma}(\exp(-\lambda t))\right)(1) - \left(I_{s^+}^{\alpha-1}(\exp(-\lambda t))\right)(1)\right]$$

with

$$\mu_0 = \left(I_{0^+}^{\alpha-1}(\exp(-\lambda t))\right)(1) - \left(I_{0^+}^{\alpha-1+\gamma}(\exp(-\lambda t))\right)(1).$$

Then,
(A) the following estimate holds true

$$|G_1(t,s)| \leq \frac{1}{\Gamma(\alpha)}\left(\frac{1+\Gamma(\gamma+1)}{|\mu_0|\Gamma(\alpha)\Gamma(\gamma+1)} + 1\right) = M_{G_1}.$$

(B) If $u \in W_H^{\alpha,1}(I)$ satisfies

$$\begin{cases} D^\alpha u(t) + \lambda D^{\alpha-1} u(t) = f(t), \quad t \in I, \ f \in L_H^1(I), \\ I_{0^+}^{\beta} u(t)|_{t=0} = 0, \ u(1) = I_{0^+}^{\gamma} u(1), \end{cases}$$

then, one has

$$u(t) = \int_0^1 G_1(t,s)(D^\alpha u(s) + \lambda D^{\alpha-1} u(s))ds \ \text{ for all } t \in I.$$

(C) Let $f \in L_H^1(I)$ and $u_f : I \to H$ be the function defined by

$$u_f(t) := \int_0^1 G_1(t,s) f(s) ds, \quad t \in I.$$

Then, one has

$$I_{0^+}^{\beta} u_f(t)|_{t=0} = 0, \ \text{ and } \ u_f(1) = (I_{0^+}^{\gamma} u_f)(1).$$

Moreover, $u_f \in W_H^{\alpha,1}(I)$ and one has

$$(D^\alpha u_f)(t) + \lambda(D^{\alpha-1} u_f)(t) = f(t) \quad \text{for all } t \in I.$$

The solution set is characterized as follows (see [18]).

Theorem 5.5. *Let $X : I \rightrightarrows H$ be a measurable set-valued map with convex compact values such that $X(t) \subset S\overline{B}_H$ for all $t \in I$, where S is a non-negative real constant and S_X^1 denotes the set of all integrable selections of X. Then, the $W_H^{\alpha,1}(I)$-solutions set of*

$$\begin{cases} D^\alpha u(t) + \lambda D^{\alpha-1} u(t) = f(t), \ f \in S_X^1, \quad t \in I, \\ I_{0^+}^{\beta} u(t)|_{t=0} = 0, \ u(1) = I_{0^+}^{\gamma} u(1), \end{cases}$$

is a convex compact subset in $\mathcal{C}_H(I)$.

Now, we are ready to prove a new result concerning a coupled system with nonlocal boundary conditions.

Theorem 5.6. *Let $\varphi : I \times H \to [0, +\infty]$ be a map satisfying (H_1)-(H_2)-(H_4). Let $f : I \times I \times H \times H \to H$ be a map satisfying (j)-(jj)-(jjj). Then, there is a $W_H^{\alpha,1}(I)$ map $u : I \to H$ and an absolutely continuous map $x : I \to H$ satisfying*

$$\begin{cases} -\dot{x}(t) \in \partial\varphi(t, x(t)) + \displaystyle\int_0^t f(t, s, u(s), x(s))ds \quad \text{a.e. } t \in I, \\ D^\alpha u(t) + \lambda D^{\alpha-1}u(t) = x(t), \ \ t \in I, \\ I_{0^+}^\beta u(t)|_{t=0} = 0, \ u(1) = I_{0^+}^\gamma u(1), \\ \quad x(0) = x_0 \in \operatorname{dom}\varphi(0, \cdot). \end{cases}$$

Proof. Consider the set $\mathcal{Y}$ defined by

$$\mathcal{Y} := \{u_f : I \to H : u_f(t) = \int_0^1 G_1(t, s)f(s)ds, \ f \in S^1_{M\overline{B}_H}, \ t \in I\}.$$

By Theorem 4.1 [18], the set $\mathcal{Y}$ is closed convex bounded and equi-Lipschitz in $\mathcal{C}_H(I)$.
For any $y \in \mathcal{Y}$, let us define the map f_y by $f_y(t, s, v) := f(t, s, y(s), v)$, for each $(t, s, v) \in I \times I \times H$, which satisfies (i)-(ii)-(iii). Hence, by Theorem 3.4 there is a unique absolutely continuous solution x_y to the evolution problem

$$(P_y) \begin{cases} -\dot{x}_y(t) \in \partial\varphi(t, x_y(t)) + \int_0^t f_y(t, s, x_y(s))ds \quad \text{a.e. } t \in I, \\ \quad x_y(0) = x_0 \in \operatorname{dom}\varphi(0, \cdot), \end{cases}$$

with $\int_0^1 \|\dot{x}_y(t)\|^2 dt \leq L$ for some non-negative real constant L (see Theorem 3.4). Note by assumption (H_4), that since $x_y(t) \in \operatorname{dom}\varphi(t, \cdot)$ then, for all $t \in I$: $\|x_y(t)\| \leq M$.
Next, define the map Λ on $\mathcal{Y}$ by

$$\Lambda(y)(t) = \int_0^1 G_1(t, s)x_y(s)ds, \ \ t \in I,$$

where x_y denotes the unique absolutely continuous solution to (P_y) for each $y \in \mathcal{Y}$, and remark that $\Lambda(y) \in \mathcal{Y}$. Since $x_y(t) \in \operatorname{dom}\varphi(t, \cdot)$, for each $t \in I$, then (H_4) entails that $x_y(t) \in X(t)$, where $X(t)$ is convex compact. Thus, for any $y \in \mathcal{Y}$, one obtains $\Lambda(y) \in \mathcal{Z}$, where

$$\mathcal{Z} := \{u_f : I \to H : u_f(t) = \int_0^1 G_1(t, s)f(s)ds, \ f \in S^1_X, t \in I\},$$

such that $\mathcal{Z}$ is convex compact in $\mathcal{C}_H(I)$ by Theorem 5.5, with
$\Lambda(\mathcal{Y}) \subset \mathcal{Z} \subset \mathcal{Y}$. This proves that $\Lambda(\mathcal{Y})$ is relatively compact. It is sufficient to show that Λ is continuous on $\mathcal{Y}$.
Let $(y_n) \subset \mathcal{Y}$ be a sequence that uniformly converges to $y \in \mathcal{Y}$ and prove that the sequence of solutions x_{y_n} associated with y_n to the evolution problem

$$\begin{cases} -\dot{x}_{y_n}(t) \in \partial\varphi(t, x_{y_n}(t)) + \int_0^t f_{y_n}(t, s, x_{y_n}(s))ds \quad \text{a.e. } t \in I, \\ \quad x_{y_n}(0) = x_0 \in \operatorname{dom}\varphi(0, \cdot), \end{cases}$$

uniformly converges to x_y solution to (P_y).
Taking assumption (H_4) into account, one deduces that $(x_{y_n}(t))$ is relatively compact in H, for each $t \in I$. It is clear that $(x_{y_n}(\cdot))$ is equi-continuous. By Ascoli's theorem, there is a map $v \in \mathcal{C}_H(I)$ such that (x_{y_n}) (up to a subsequence that we do not relabel) uniformly converges in $\mathcal{C}_H(I)$ to v with $v(0) = x_0$. Since moreover $\sup_n \int_0^1 \|\dot{x}_{y_n}(t)\|^2 dt \leq L$, one concludes that $(\dot{x}_{y_n})$ $\sigma(L^2_H(I), L^2_H(I))$-converges to some $w(\cdot)$ in $L^2_H(I)$ with $w = \dot{v}$ a.e.
From (jj) and Lemma 5.4 (A), remark that

$$\|f(t, s, y_n(s), x_{y_n}(s))\| \leq \kappa(t, s)(1 + MM_{G_1} + M) \ \text{ for all } (t, s) \in I \times I.$$

Let $g \in L^2_H(I)$. Then, the latter inequality yields

$$\left| \left\langle g(t), \int_0^t f(t,s,y_n(s),x_{y_n}(s))ds \right\rangle \right| \le (1 + MM_{G_1} + M)\|g(t)\| \int_0^t \kappa(t,s)ds, \tag{13}$$

with the map $t \mapsto (1 + MM_{G_1} + M)\|g(t)\| \int_0^t \kappa(t,s)ds \in L^1_{\mathbb{R}}(I)$.
Put $M_1 = \max(M, MM_{G_1})$. Then, in view of (jjj), there is $\delta_{M_1}(\cdot) \in L^2_{\mathbb{R}}(I)$ such that

$$\begin{aligned} &||f(t,s,y_n(s),x_{y_n}(s)) - f(t,s,y(s),v(s))|| \\ &\le \delta_{M_1}(t)(||y_n(s) - y(s)|| + ||x_{y_n}(s) - v(s)||) \end{aligned}$$

for all $(t,s) \in I \times I$.
Then, noting that (x_{y_n}) (resp. (y_n)) uniformly converges to v (resp. y), Lebesgue's dominated convergence theorem yields

$$\begin{aligned} &\| \int_0^t f(t,s,y_n(s),x_{y_n}(s))ds - \int_0^t f(t,s,y(s),v(s))ds\| \\ &\le \int_0^t \|f(t,s,y_n(s),x_{y_n}(s)) - f(t,s,y(s),v(s))\|ds \\ &\le \delta_{M_1}(t) \int_0^t (||y_n(s) - y(s)|| + ||x_{y_n}(s) - v(s)||)ds \to 0 \text{ as } n \to \infty. \end{aligned}$$

This along with (13) entail

$$\begin{aligned} &\lim_{n\to\infty} \int_0^1 \left\langle g(t), \int_0^t f(t,s,y_n(s),x_{y_n}(s))ds \right\rangle dt \\ &= \int_0^1 \left\langle g(t), \int_0^t f(t,s,y(s),v(s))ds \right\rangle dt, \end{aligned}$$

using Lebesgue's dominated convergence theorem. This justifies the
$\sigma(L^2_H(I), L^2_H(I))$-convergence.
Hence, Proposition 3.2 yields

$$-\dot{v}(t) \in \partial\varphi(t, v(t)) + \int_0^t f(t,s,y(s),v(s))ds \quad \text{a.e. } t \in I,$$

and by uniqueness, it results $x_y = v$.
Coming back to the map Λ, with the help of Lemma 5.4 (A), one has for all $t \in I$

$$\begin{aligned} \|\Lambda(y_n)(t) - \Lambda(y)(t)\| &= \| \int_0^1 G_1(t,s)x_{y_n}(s)ds - \int_0^1 G_1(t,s)x_y(s)ds\| \\ &\le M_{G_1} \int_0^1 \|x_{y_n}(s) - x_y(s)\|ds \end{aligned}$$

Since $\|x_{y_n}(\cdot) - x_y(\cdot)\|_\infty \to 0$ and (x_{y_n}) is uniformly bounded, it follows

$$\sup_{t\in I} \|\Lambda(y_n)(t) - \Lambda(y)(t)\| \le M_{G_1}\|x_{y_n}(\cdot) - x_y(\cdot)\|_\infty \to 0 \text{ as } n \to \infty.$$

Thus, $\Lambda : \mathcal{Y} \to \mathcal{Y}$ is continuous. Applying Schauder's fixed point theorem (see Theorem 2.1), the map Λ admits a fixed point, $y = \Lambda(y)$ with

$$y(t) = \Lambda(y)(t) = \int_0^1 G_1(t,s)x_y(s)ds, \quad t \in I,$$

and x_y is a solution to (P_y).
In other words, there exist a $W_H^{\alpha,1}(I)$ map $y : I \to H$ and an absolutely continuous map $x_y : I \to H$ such that

$$\begin{cases} -\dot{x}_y(t) \in \partial\varphi(t, x_y(t)) + \displaystyle\int_0^t f(t, s, y(s), x_y(s))ds \quad \text{a.e. } t \in I, \\ D^\alpha y(t) + \lambda D^{\alpha-1} y(t) = x_y(t), \quad t \in I, \\ I_{0+}^\beta y(t)|_{t=0} = 0, \ y(1) = I_{0+}^\gamma y(1), \\ \quad x_y(0) = x_0 \in \operatorname{dom}\varphi(0, \cdot). \end{cases}$$

□

5.1.2. Coupled systems with integral boundary conditions

In order to prove our main theorem, let us recall some useful results from [21].

Lemma 5.7. *Let $\alpha \in]1, 2]$, $b \in H$ and $f \in L_H^1(I)$. Then, the map $u_f : I \to H$ defined by*

$$u_f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds + \frac{b}{\Gamma(\alpha)} t^{\alpha-1}, \ t \in I$$

is the unique $W_H^{\alpha,1}(I)$-solution to the problem

$$\begin{cases} D^\alpha u(t) = f(t), \quad t \in I, \\ u(0) = 0, \ D^\alpha u(0) = b, \\ D^{\alpha-1} u(t) = \int_0^t f(s)ds + b, \end{cases}$$

Lemma 5.8. *Let $b \in H$. Let $X : I \rightrightarrows H$ be a measurable and integrably bounded set-valued map with convex compact values. Then, the $W_H^{\alpha,1}(I)$-solution set to the fractional differential inclusion*

$$\begin{cases} D^\alpha u(t) \in X(t), \quad t \in I, \\ u(0) = 0, \ D^\alpha u(0) = b, \end{cases}$$

is convex, bounded, equi-continuous, compact in $\mathcal{C}_H(I)$. Moreover, the $W_H^{\alpha,1}(I)$-solution set is characterized by

$$\mathcal{Y} := \{u_f : I \to H, \ u_f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds + \frac{b}{\Gamma(\alpha)} t^{\alpha-1}, \ f \in S_X^1, \ t \in I\}.$$

Now, we are able to prove a new result concerning a coupled system with integral boundary conditions.

Theorem 5.9. *Let $b \in H$. Let $\varphi : I \times H \to [0, +\infty]$ be a map satisfying (H_1)-(H_2)-(H_4). Let $f : I \times I \times H \times H \to H$ be a map satisfying (j)-(jj)-(jjj). Then, there is a $W_H^{\alpha,1}(I)$ map $u : I \to H$ and an absolutely continuous map $x : I \to H$ satisfying*

$$\begin{cases} -\dot{x}(t) \in \partial\varphi(t, x(t)) + \displaystyle\int_0^t f(t, s, u(s), x(s))ds \quad \text{a.e. } t \in I, \\ D^\alpha u(t) = x(t), \quad t \in I, \\ u(0) = 0, \ D^\alpha u(0) = b, \\ D^{\alpha-1} u(t) = \int_0^t x(s)ds + b, \\ \quad x(0) = x_0 \in \operatorname{dom}\varphi(0, \cdot). \end{cases}$$

Proof. Consider the set $\mathcal{Y}$ defined by

$$\mathcal{Y} := \{u_f : I \to H, \ u_f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds + \frac{b}{\Gamma(\alpha)} t^{\alpha-1}, f \in S_{M\overline{B}_H}^1, \ t \in I\}.$$

By Lemma 4.5 [21], the set $\mathcal{Y}$ is closed convex bounded and equi-Lipschitz in $\mathcal{C}_H(I)$.
For any $y \in \mathcal{Y}$, let us define the map f_y by $f_y(t,s,v) := f(t,s,y(s),v)$, for each $(t,s,v) \in I \times I \times H$, which satisfies (i)-(ii)-(iii). Hence, by Theorem 3.4 there is a unique absolutely continuous solution x_y to the evolution problem

$$(P_y)\begin{cases} -\dot{x}_y(t) \in \partial\varphi(t, x_y(t)) + \int_0^t f_y(t,s,x_y(s))ds \quad \text{a.e. } t \in I, \\ \quad x_y(0) = x_0 \in \operatorname{dom}\varphi(0,\cdot), \end{cases}$$

with $\int_0^1 \|\dot{x}_y(t)\|^2 dt \le L$ for some non-negative real constant L (see Theorem 3.4). Note by assumption (H_4), that since $x_y(t) \in \operatorname{dom}\varphi(t,\cdot)$ then, for all $t \in I$: $\|x_y(t)\| \le M$.
Next, define the map Λ on $\mathcal{Y}$ by

$$\Lambda(y)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} x_y(s)ds + \frac{b}{\Gamma(\alpha)}t^{\alpha-1},$$

where x_y denotes the unique absolutely continuous solution to (P_y) for each $y \in \mathcal{Y}$, and remark that $\Lambda(y) \in \mathcal{Y}$. Since $x_y(t) \in \operatorname{dom}\varphi(t,\cdot)$, for each $t \in I$, then (H_4) entails that $x_y(t) \in X(t)$, where $X(t)$ is convex compact. Thus, for any $y \in \mathcal{Y}$, one obtains $\Lambda(y) \in \mathcal{Z}$, where

$$\mathcal{Z} := \{u_f : I \to H : u_f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)ds + \frac{b}{\Gamma(\alpha)}t^{\alpha-1},\ f \in S_X^1, t \in I\},$$

such that $\mathcal{Z}$ is convex compact in $\mathcal{C}_H(I)$ by Lemma 5.8, with $\Lambda(\mathcal{Y}) \subset \mathcal{Z} \subset \mathcal{Y}$. This proves that $\Lambda(\mathcal{Y})$ is relatively compact in $\mathcal{C}_H(I)$. It is sufficient to show that Λ is continuous on $\mathcal{Y}$.
Let $(y_n) \subset \mathcal{Y}$ be a sequence that uniformly converges to $y \in \mathcal{Y}$ and prove that the sequence of solutions x_{y_n} associated with y_n to the evolution problem

$$\begin{cases} -\dot{x}_{y_n}(t) \in \partial\varphi(t, x_{y_n}(t)) + \int_0^t f_{y_n}(t,s,x_{y_n}(s))ds \quad \text{a.e. } t \in I, \\ \quad x_{y_n}(0) = x_0 \in \operatorname{dom}\varphi(0,\cdot), \end{cases}$$

uniformly converges to x_y solution to (P_y) (using the same arguments used in the proof of Theorem 5.6). Thus it is easy to deduce that $\Lambda : \mathcal{Y} \to \mathcal{Y}$ is continuous. Applying Schauder's fixed point theorem (see Theorem 2.1), the map Λ admits a fixed point, $y = \Lambda(y)$ with

$$y(t) = \Lambda(y)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} x_y(s)ds + \frac{b}{\Gamma(\alpha)}t^{\alpha-1},\ t \in I,$$

and x_y is a solution to (P_y).
In other words, there exist a $W_H^{\alpha,1}(I)$ map $y : I \to H$ and an absolutely continuous map $x_y : I \to H$ such that

$$\begin{cases} -\dot{x}_y(t) \in \partial\varphi(t, x(t)) + \displaystyle\int_0^t f(t,s,y(s),x_y(s))ds \quad \text{a.e. } t \in I, \\ D^\alpha y(t) = x_y(t), \quad t \in I, \\ y(0) = 0,\ D^\alpha y(0) = b, \\ D^{\alpha-1}y(t) = \int_0^t x_y(s)ds + b, \\ \quad x_y(0) = x_0 \in \operatorname{dom}\varphi(0,\cdot). \end{cases}$$

□

5.2. Second-order differential equation coupled with subdifferentials

We need to define the Green function and its properties taken from [22].

Lemma 5.10. *Let* $0 < \eta_1 < \eta_2 < \cdots < \eta_m - 2 < 1$, $\gamma > 0$, $m > 3$ *be an integer number and* $\alpha_i \in \mathbb{R}$ $(i = 1, \cdots, m-2)$ *satisfy the condition*

$$\sum_{i=1}^{m-2} \alpha_i - 1 + \exp(-\gamma) - \sum_{i=1}^{m-2} \alpha_i \exp(-\gamma\eta_i) \neq 0.$$

Let $G_2 : I \times I \to \mathbb{R}$ *be the function defined by*

$$G_2(t,s) = \begin{cases} \frac{1}{\gamma}(1 - \exp(-\gamma(t-s))) + \frac{A}{\gamma}(1 - \exp(-\gamma t))\psi(s) & 0 \leq s \leq t \leq 1 \\ \frac{A}{\gamma}(1 - \exp(-\gamma t))\psi(s) & t < s \leq 1, \end{cases}$$

such that

$$\psi(s) = \begin{cases} 1 - \exp(-\gamma(1-s)) - \sum_{i=1}^{m-2} \alpha_i(1 - \exp(-\gamma(\eta_i - s))), & 0 \leq s < \eta_1 \\ 1 - \exp(-\gamma(1-s)) - \sum_{i=2}^{m-2} \alpha_i(1 - \exp(-\gamma(\eta_i - s))), & \eta_1 \leq s \leq \eta_2 \\ \ldots\ldots. & \\ 1 - \exp(-\gamma(1-s)), & \eta_{m-2} \leq s \leq 1, \end{cases}$$

and

$$A = \left(\sum_{i=1}^{m-2} \alpha_i - 1 + \exp(-\gamma) - \sum_{i=1}^{m-2} \alpha_i \exp(-\gamma\eta_i) \right)^{-1}.$$

Then, the following hold true
(A) for each $(t,s) \in I \times I$,

$$|G_2(t,s)| \leq M_{G_2},$$

with

$$M_{G_2} := \max\{\gamma^{-1}, 1\} \left[1 + |A|\big(1 + \sum_{i=1}^{m-2} |\alpha_i|\big) \right].$$

(B) If $u \in W_H^{2,1}(I)$ with $u(0) = c$ and $u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i)$, then

$$u(t) = e_c(t) + \int_0^1 G_2(t,s)(\ddot{u}(s) + \gamma\dot{u}(s))ds, \quad \forall t \in I,$$

where

$$e_c(t) = c + A(1 - \sum_{i=1}^{m-2} \alpha_i)(1 - \exp(-\gamma t))c.$$

(C) Let $f \in L_H^1(I)$ and let $u_f : I \to H$ be the function defined by

$$u_f(t) = e_c(t) + \int_0^1 G_2(t,s)f(s)ds \quad \forall t \in I.$$

Then, one has

$$u_f(0) = c \quad u_f(1) = \sum_{i=1}^{m-2} \alpha_i u_f(\eta_i).$$

(D) If $f \in L_H^1(I)$, then $\dot{u}_f$ is scalarly derivable, and its weak derivative $\ddot{u}_f$ satisfies

$$\ddot{u}_f(t) + \gamma\dot{u}_f(t) = f(t) \quad \text{a.e. } t \in I.$$

Proposition 5.11. *Let $f \in L^1_H(I)$. Then, the m-points boundary problem*

$$\begin{cases} \ddot{u}(t) + \gamma \dot{u}(t) = f(t), \quad t \in I, \\ u(0) = c, \ u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i), \end{cases}$$

has a unique solution $u_f \in W^{2,1}_H(I)$, with

$$u_f(t) = e_c(t) + \int_0^1 G_2(t,s) f(s) ds, \ t \in I.$$

Proposition 5.12. *Let $X : I \rightrightarrows H$ be a measurable and integrably bounded set-valued map with convex compact values. Then, the $W^{2,1}_H(I)$-solutions set of*

$$\begin{cases} \ddot{u}(t) + \gamma \dot{u}(t) = f(t), \quad t \in I, \ f \in S^1_X, \\ u(0) = c, \ u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i), \end{cases}$$

is bounded, convex, equi-continuous and compact in $\mathcal{C}_H(I)$.

Now, we are able to prove a new result concerning a coupled system with m-points boundary conditions.

Theorem 5.13. *Let $\varphi : I \times H \to [0, +\infty]$ be a map satisfying (H_1)-(H_2)-(H_4). Let $f : I \times I \times H \times H \to H$ be a map satisfying (j)-(jj)-(jjj). Then, there is a $W^{2,1}_H(I)$ map $u : I \to H$ and an absolutely continuous map $x : I \to H$ satisfying*

$$\begin{cases} -\dot{x}(t) \in \partial\varphi(t, x(t)) + \displaystyle\int_0^t f(t, s, u(s), x(s)) ds \quad \text{a.e. } t \in I, \\ \ddot{u}(t) + \gamma \dot{u}(t) = x(t), \quad t \in I, \\ u(0) = c, \ u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i), \\ \quad x(0) = x_0 \in \operatorname{dom} \varphi(0, \cdot). \end{cases}$$

Proof. Consider the set $\mathcal{Y}$ defined by

$$\mathcal{Y} := \{u_f : I \to H, \ u_f(t) = e_c(t) + \int_0^1 G_2(t,s) f(s) ds, \ t \in I, \ f \in S^1_X\},$$

where X is the measurable and integrably bounded set-valued map with convex compact values given in (H_4). By Proposition 5.12, the set $\mathcal{Y}$ is convex, compact, bounded and equi-continuous in $\mathcal{C}_H(I)$. Next, define the map Λ on $\mathcal{Y}$ by

$$\Lambda(y)(t) = e_c(t) + \int_0^1 G_2(t,s) x_y(s) ds, \ t \in I,$$

where x_y is the unique absolutely continuous solution to the evolution problem

$$(P_y) \begin{cases} -\dot{x}_y(t) \in \partial\varphi(t, x_y(t)) + \displaystyle\int_0^t f(t, s, y(s), x_y(s)) ds \quad \text{a.e. } t \in I, \\ \quad x_y(0) = x_0 \in \operatorname{dom} \varphi(0, \cdot). \end{cases}$$

Arguing as in the proof of Theorem 5.6, one proves that $\Lambda : \mathcal{Y} \to \mathcal{Y}$ is continuous. Applying Schauder's fixed point theorem (see Theorem 2.1), the map Λ admits a fixed point, $y = \Lambda(y)$ with

$$y(t) = \Lambda(y)(t) = e_c(t) + \int_0^1 G_2(t,s) x_y(s) ds, \ t \in I,$$

and x_y is a solution to (P_y).
In other words, there is a $W_H^{2,1}(I)$ map $y : I \to H$ and an absolutely continuous map $x_y : I \to H$ such that

$$\begin{cases} -\dot{x}_y(t) \in \partial\varphi(t, x_y(t)) + \displaystyle\int_0^t f(t, s, y(s), x_y(s))ds \quad \text{a.e. } t \in I, \\ \ddot{y}(t) + \gamma\dot{y}(t) = x_y(t), \quad t \in I, \\ y(0) = c, \ y(1) = \sum_{i=1}^{m-2} \alpha_i y(\eta_i), \\ \quad x_y(0) = x_0 \in \operatorname{dom}\varphi(0, \cdot). \end{cases}$$

□

Concluding remarks. In this work, we have proved the well-posedness of a new differential inclusion of subdifferential type with integral perturbation, using Schauder's fixed point theorem. Then, we have obtained with success and established theorems related to coupled systems involving fractional differential equations. These results could provide some insight into the control theory analysis of fractional-order systems and can also be enlarged to the case of other fractional systems.

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