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On Generalizations of Hölder's and Minkowski's Inequalities

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Abstract
We present the generalizations of Hölder's inequality and Minkowski's inequality along with the generalizations of Aczél's, Popoviciu's, Lyapunov's and Bellman's inequalities. Some applications for the metric spaces, normed spaces, Banach spaces, sequence spaces and integral inequalities are further specified. It is shown that $(\mathbb{R}^n, d)$ and $(l_p, d_{m,p})$ are complete metric spaces and $(\mathbb{R}^n, \|x\|_m)$ and $\left(l_p, \|x\|_{m,p}\right)$ are $\frac{1}{m}-$Banach spaces. Also, it is deduced that $\left(b^{r,s}_{p,1}, \|x\|_{r,s,m}\right)$ is a $\frac{1}{m}-$normed space.

Keywords: *Aczél's inequality; Bellman's inequality; Hölder's inequality; Lyapunov's inequality; Minkowski's inequality; Popoviciu's inequality.*

AMS Subject Classification (2020): *Primary:26D15; Secondary:26A51; 26D10; 26A42; 46B20; 46B25.*

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1. Introduction

We shall use $\mathbb{N}$ to denote the set of positive integers, $\mathbb{C}$ for the set of complex numbers, $\mathbb{R}$ for the set of real numbers and $\mathbb{R}^n$ for the set of all ordered n-tuples $x = (x_1, x_2, \ldots, x_n)$ of real numbers x_i.

In [1], the following extensions of the inequalities of Hölder and Minkowski are given respectively:
If $x_{i,j} > 0 \;\; for\; i = 1, 2, \ldots, n\; and\; j = 1, 2, \ldots, m,\; and\; if\; p_j > 0\; with\; \sum_{j=1}^{m} \frac{1}{p_j} = 1$, then

$$\sum_{i=1}^{n} \prod_{j=1}^{m} x_{ij} \leq \prod_{j=1}^{m} \left(\sum_{i=1}^{n} x_{ij}^{p_j} \right)^{1/p_j}, \tag{1.1}$$

the sign of equality holding if and only if the m sets $(x_{i1}^{p_1}), (x_{i2}^{p_2}), \ldots, (x_{im}^{p_m})$ are proportional, that is, if and only if there are numbers $\lambda_i,\; not\; all\; 0$, such that $\sum_{j=i}^{m} \lambda_j x_{ij}^{p_j} = 0 \;\; for\; i = 1, 2, \ldots, n.$

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If $x_{i,j} \geq 0 \;\; for \; i = 1, 2, \ldots, n \; and \; j = 1, 2, \ldots, k, \; and \; if \; p > 1$, then

$$\left(\sum_{i=1}^{n} \left(\sum_{j=1}^{k} x_{ij} \right)^{p} \right)^{1/p} \leq \sum_{j=1}^{k} \left(\sum_{i=1}^{n} x_{ij}^{p} \right)^{1/p}. \tag{1.2}$$

The inequality is reversed for $p < 1 \; (p \neq 0)$. $(For \; p < 0, \;\; we \; assume \; that \; x_{i,j} > 0)$. In each case, the sign of equality holds if and only if the k sets $(x_{i1}), (x_{i2}), \ldots, (x_{ik})$ are proportional.

Similarly, the integral form of the Hölder inequality is

$$\int_{a}^{b} \left(\prod_{j=1}^{m} f_{j}(x) \right) dx \leq \prod_{j=1}^{m} \left(\int_{a}^{b} f_{j}^{p_{j}}(x) \, dx \right)^{1/p_{j}}, \tag{1.3}$$

where $f_j(x) > 0 \;\; (j = 1, 2, \ldots, m), \; x \in [a, b], \; -\infty < a < b < +\infty, \; p_j > 0, \; \sum_{j=1}^{m} \frac{1}{p_j} = 1$ and $f_j \in L^{p_j}[a, b]$.

Furthermore, the integral form of the Minkowski inequality is

$$\left(\int_{a}^{b} \left(\sum_{j=1}^{k} f_{j}(x) \right)^{p} dx \right)^{1/p} \leq \sum_{j=1}^{k} \left(\int_{a}^{b} f_{j}^{p}(x) dx \right)^{1/p},$$

where $f_j(x) > 0 \;\; (j = 1, 2, \ldots, k), \; x \in [a, b], \; -\infty < a < b < +\infty, \; p > 0$ and $f_j \in L^{p}[a, b]$.

A normed linear space is called complete if every Cauchy sequence in the space converges, that is, if for each Cauchy sequence (f_n) in the space there is an element f in the space such that $f_n \to f$. A complete normed linear space is called a Banach space. [2](p. 115).

For $1 \leq p < \infty$, we denote by l_p the space of all sequences $x = (x_n)_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} |x_n|^p < \infty$. The space l_p is a Banach space by the norm

$$\|x\| = (\sum_{n=1}^{\infty} |x_n|^p)^{1/p},$$

which is given by Yosida in [3] (p. 55).

In [4, 5], the sequence space $b_p^{r,s}$ is given by

$$b_p^{r,s} = \left\{ x = (x_k) \in w : \; \sum_{n} \left| \frac{1}{(s+r)^n} \sum_{k=0}^{n} \binom{n}{k} s^{n-k} r^k x_k \right|^p < \infty \right\}.$$

Where $1 \leq p < \infty, \;$ r and s are nonzero real numbers with $r + s \neq 0$. The binomial matrix $B^{r,s} = (b_{nk}^{r,s})$ is defined as follows:

$$b_{nk}^{r,s} = \begin{cases} \frac{1}{(s+r)^n} \binom{n}{k} s^{n-k} r^k & , 0 \leq k \leq n \\ 0 & , k > n \end{cases},$$

for all $k, n \in N$. For $sr > 0$, one can easily check that the following properties hold for the binomial matrix $B^{r,s} = (b_{nk}^{r,s})$:

(i) $\|B^{r,s}\| < \infty, \quad (ii) \; \lim_{n\to\infty} b_{nk}^{r,s} = 0 \quad (each \; k \in N), \quad (iii) \lim_{n\to\infty} \sum_k b_{nk}^{r,s} = 1$.

Thus, the binomial matrix is regular whenever $sr > 0$.

Young's inequality asserts that,

$$\frac{a^p}{p} + \frac{b^q}{q} \geq ab, \quad for \; all \; a, b \geq 0,$$

whenever $p, q \in (1, \infty) \;\; and \; \frac{1}{p} + \frac{1}{q} = 1$; the equality holds if and only if $a^p = b^q$.

W.H. Young actually proved a much more general inequality which yields the aforementioned one for $f(x) = x^{p-1}$:

Theorem 1.1 (Young's inequality). *Suppose that $f : [0,\infty) \to [0,\infty)$ is an increasing continuous function such that $f(0) = 0$ and $\lim_{x\to\infty} f(x) = \infty$. Then*

$$ab \leq \int_0^a f(x)dx + \int_0^b f^{-1}(x)\,dx,$$

$for\ all\ a, b \geq 0,\ and\ equality\ occurs\ if\ and\ only\ if\ b = f(a)$. [6] (p. 15).

In the next section, we consider the following form of Young's inequality, for $s \geq 1$

$$\frac{a^{sp}}{sp} + \frac{b^{sq}}{sq} \geq ab,$$

where $a, b \geq 0,\ \ sp, sq \in (1,\infty)\ \ and\ \frac{1}{sp} + \frac{1}{sq} = 1.$

In [7], H. Agahi et al. gave some generalizations of Hölder's and Minkowski's inequalities for the pseudo-integral. In [8], C.J. Zhao and W.S. Cheung gave an improvement of Minkowski's inequality. In [9], X. Zhou established some functional generalizations and refinements of Aczél's inequality and of Bellman's inequality. In [10], S.I. Butt et al. gave refinements of the discrete Hölder's and Minkowski's inequalities for finite and infinite sequences by using cyclic refinements of the discrete Jensen's inequality. In [11], S. Rashid et al. established Minkowski and reverse Hölder inequalities by employing weighted $AB\mathscr{A}\mathscr{B}-$fractional integral. In [12], S. Rashid et al. gave new fractional behavior of Minkowski inequality and several other related generalizations in the frame of the newly proposed fractional operators. In [13], S. Rashid et al. presented the major consequences of the certain novel versions of reverse Minkowski and related Hölder-type inequalities via discrete $\hbar\hbar$-proportional fractional sums. In [14], S. Rashid et al. gave the certain novel versions of reverse Minkowski and related Hölder-type inequalities via discrete-fractional operators having $\hbar\hbar$-discrete generalized Mittag-Leffler kernels. In [15], S. Rafeeq et al. presented the explicit bounds for three generalized delay dynamic Gronwall–Bellman type integral inequalities on time scales, which are the unification of continuous and discrete results. In [16], Z. Zong et al. investigated the n-dimensional $(n \geq 1)$ Jensen inequality, Hölder inequality, and Minkowski inequality for dynamically consistent nonlinear evaluations in $L^1\left(\Omega, F, (F_t)_{t\geq 0}, P\right)$. Furthermore, they gave four equivalent conditions on the n-dimensional Jensen inequality for g-evaluations induced by backward stochastic differential equations with non-uniform Lipschitz coefficients in $L^p\left(\Omega, F, (F_t)_{0\leq t\leq T}, P\right)$ $(1 < p \leq 2)$. Finally, they gave a sufficient condition on g that satisfies the non-uniform Lipschitz condition under which Hölder's inequality and Minkowski's inequality for the corresponding g-evaluation hold true.

Hölder's inequality, power-mean inequality and Jensen's inequality are used to obtain Hermite-Hadamard type inequalities and Ostrowski's type inequalities for different kinds of convexity which are used in the fields of integral inequalities, approximation theory, special means theory, optimization theory, information theory and numerical analysis. Furthermore, both the Hölder inequality and the Minkowski inequality play an important role in many areas of pure and applied mathematics. These inequalities have been used in several areas of mathematics, especially in functional analysis and generalized in various directions.

The main aim of this paper is to give generalizations of Hölder's, Minkowski's, Aczél's, Popoviciu's, Lyapunov's and Bellman's inequalities.

For several recent results concerning Hölder's inequality, Minkowski's inequality, Hermite-Hadamard type inequalities and Banach spaces, we refer to [1, 6, 7, 10, 16–40]. For Aczél's, Popoviciu's, Bellman's inequalities and the related results, we refer to [3, 9, 36].

2. Main results

First, we give a generalization of Hölder's inequality.

Theorem 2.1. *If $a_k,\ b_k \geq 0$ for $k = 1, 2, \ldots, n$ and $\frac{1}{sp} + \frac{1}{sq} = 1$ with $p > 1,\ \ s \geq 1$, then*

$$\sum_{k=1}^{n} (a_k b_k)^{1/s} \leq \left(\sum_{k=1}^{n} a_k^p\right)^{1/sp} \left(\sum_{k=1}^{n} b_k^q\right)^{1/sq}, \tag{2.1}$$

with equality holding if and only if $\alpha a_k^p = \beta b_k^q$ for $k = 1, 2, \ldots n$, where α and β are real nonnegative constants such that $\alpha^2 + \beta^2 > 0$.

Proof. If $\sum_{k=1}^{n} a_k^p = 0 \ or \ \sum_{k=1}^{n} b_k^q = 0$, then equality holds in (2.1). Let $\sum_{k=1}^{n} a_k^p > 0 \ and \ \sum_{k=1}^{n} b_k^q > 0$. Substituting

$$a = a_\nu^{1/s}\left(\sum_{k=1}^{n} a_k^p\right)^{-1/sp}, \ b = b_\nu^{1/s}\left(\sum_{k=1}^{n} b_k^q\right)^{-1/sq}, \tag{2.2}$$

into the inequality

$$\frac{a^{sp}}{sp} + \frac{b^{sq}}{sq} \geq ab, \tag{2.3}$$

we get

$$\frac{a_\nu^p}{sp}\left(\sum_{k=1}^{n} a_k^p\right)^{-1} + \frac{b_\nu^q}{sq}\left(\sum_{k=1}^{n} b_k^q\right)^{-1} \geq \frac{a_\nu^{1/s} b_\nu^{1/s}}{(\sum_{k=1}^{n} a_k^p)^{1/sp}(\sum_{k=1}^{n} b_k^q)^{1/sq}}$$

and

$$\frac{1}{sp}\frac{a_\nu^p}{\sum_{k=1}^{n} a_k^p} + \frac{1}{sq}\frac{b_\nu^q}{\sum_{k=1}^{n} b_k^q} \geq \frac{a_\nu^{1/s} b_\nu^{1/s}}{(\sum_{k=1}^{n} a_k^p)^{1/sp}(\sum_{k=1}^{n} b_k^q)^{1/sq}}.$$

Adding together these inequalities for $\nu = 1, 2, \ldots, n$, we have

$$\frac{1}{sp} + \frac{1}{sq} \geq \frac{\sum_{k=1}^{n} a_k^{1/s} b_k^{1/s}}{(\sum_{k=1}^{n} a_k^p)^{1/sp}(\sum_{k=1}^{n} b_k^q)^{1/sq}}.$$

For $\frac{1}{sp} + \frac{1}{sq} = 1$, we obtain the inequality (2.1).
Since equality holds in (2.3) if and only if $a^{sp} = b^{sq}$, we conclude, in virtue of (2.2), that there is equality in (2.1) if and only if $a_k^p(\sum_{k=1}^{n} a_k^p)^{-1} = b_k^q(\sum_{k=1}^{n} b_k^q)^{-1} \ for \ k = 1, 2, \ldots, n$, i.e., if and only if $\alpha a_k^p = \beta b_k^q \ for \ k = 1, 2, \ldots, n$. This completes the proof. □

Remark 2.1. a) If we put $s = 1$ in (2.1), we get Hölder's inequality.
b) If we put $s = 1$ and $p = q = 2$ in (2.1), we get Cauchy-Schwarz inequality.
c) From (1.1), the extension of (2.1) becomes, for $\sum_{j=1}^{m} \frac{1}{sp_j} = 1$

$$\sum_{i=1}^{n}\left(\prod_{j=1}^{m} x_{ij}\right)^{1/s} \leq \prod_{j=1}^{m}\left(\sum_{i=1}^{n} x_{ij}^{p_j}\right)^{1/sp_j}.$$

d) By (1.3), the integral form of inequality (2.1) becomes

$$\int_a^b \left(\prod_{j=1}^{m} f_j(x)\right)^{1/s} dx \leq \prod_{j=1}^{m}\left(\int_a^b f_j^{p_j}(x)\, dx\right)^{1/sp_j}. \tag{2.4}$$

e) By the inequality (2.3) in [28], we have for $s > 1 \ and \ x_i, y_i > 0$,

$$i = 1, 2, \ldots, n,$$

$$\left(\sum_{i=1}^{n} x_i^s\right)^{1/s} \leq \sum_{i=1}^{n} x_i.$$

Using the inequality above, from the Hölder's inequality and taking $x_i = a_k b_k$, we get, for $s > 1$

$$\left(\sum_{k=1}^{n} (a_k b_k)^s\right)^{1/s} \leq \sum_{k=1}^{n} a_k b_k \leq \left(\sum_{k=1}^{n} a_k^p\right)^{1/p}\left(\sum_{k=1}^{n} b_k^q\right)^{1/q},$$

and from this inequality we obtain

$$\sum_{k=1}^{n} (a_k b_k)^s \leq (\sum_{k=1}^{n} a_k b_k)^s \leq \left(\sum_{k=1}^{n} a_k^p\right)^{s/p}\left(\sum_{k=1}^{n} b_k^q\right)^{s/q},$$

which is another generalization of Hölder's inequality.
f) Let $\sum_{k=1}^{\infty} a_k^p$ and $\sum_{k=1}^{\infty} b_k^q$ be convergent series. Then, from the last inequalities in e), we have

$$\sum_{k=1}^{\infty}(a_k b_k)^s \leq \left(\sum_{k=1}^{\infty} a_k b_k\right)^s \leq \left(\sum_{k=1}^{\infty} a_k^p\right)^{\frac{s}{p}} \left(\sum_{k=1}^{\infty} b_k^q\right)^{\frac{s}{q}}.$$

Now, we give a generalization of Minkowski's inequality.

Theorem 2.2. *If* $a_k,\ b_k \geq 0,$ *for* $k = 1, 2, \ldots, n$ *and* $p > 1,$ *then*

$$\left(\sum_{k=1}^{n}(a_k + b_k)^p\right)^{1/mp} \leq \left(\sum_{k=1}^{n} a_k^p\right)^{1/mp} + \left(\sum_{k=1}^{n} b_k^p\right)^{1/mp}, \tag{2.5}$$

with equality holding if and only if the n-tuples $a = (a_1, a_2, \ldots, a_n)$ *and* $b = (b_1, b_2, \ldots, b_n)$ *are proportional, where* $m \in \mathbb{N}$.

Proof. We consider the identity

$$(a_k + b_k)^p = (a_k + b_k)^{1/m}(a_k + b_k)^{p-1/m}.$$

Using the inequality, for $a > 0,\ b > 0$ *and* $m \in \mathbb{N}$,

$$\sqrt[m]{a+b} \leq \sqrt[m]{a} + \sqrt[m]{b},$$

we obtain

$$(a_k + b_k)^p \leq (\sqrt[m]{a_k} + \sqrt[m]{b_k})(a_k + b_k)^{p-1/m}$$
$$\leq \sqrt[m]{a_k}(a_k + b_k)^{p-\frac{1}{m}} + \sqrt[m]{b_k}(a_k + b_k)^{p-1/m}.$$

Summing over $k = 1, 2, \ldots, n$, we get

$$\sum_{k=1}^{n}(a_k + b_k)^p \leq \sum_{k=1}^{n} \sqrt[m]{a_k}(a_k + b_k)^{p-\frac{1}{m}} + \sum_{k=1}^{n} \sqrt[m]{b_k}(a_k + b_k)^{p-1/m}.$$

By the inequality (2.1), for $\frac{1}{mp} + \frac{1}{mq} = 1$ *and* $p > 1$, we have

$$\sum_{k=1}^{n} \sqrt[m]{a_k}(a_k + b_k)^{p-\frac{1}{m}} \leq \left(\sum_{k=1}^{n} a_k^p\right)^{1/mp} \left(\sum_{k=1}^{n}(a_k + b_k)^{q(mp-1)}\right)^{1/mq}$$

and

$$\sum_{k=1}^{n} \sqrt[m]{b_k}(a_k + b_k)^{p-\frac{1}{m}} \leq \left(\sum_{k=1}^{n} b_k^p\right)^{1/mp} \left(\sum_{k=1}^{n}(a_k + b_k)^{q(mp-1)}\right)^{1/mq}.$$

Adding the last two relations, we obtain,

$$\sum_{k=1}^{n}(a_k + b_k)^p \leq \left[\left(\sum_{k=1}^{n} a_k^p\right)^{\frac{1}{mp}} + \left(\sum_{k=1}^{n} b_k^p\right)^{1/mp}\right] \left(\sum_{k=1}^{n}(a_k + b_k)^{q(mp-1)}\right)^{1/mq}.$$

Since $\frac{1}{mp} + \frac{1}{mq} = 1,$ *we get* $p = q(mp-1)$. Also, by dividing both sides of the inequality above by $\left(\sum_{k=1}^{n}(a_k + b_k)^p\right)^{1/mq}$, we obtain

$$\left(\sum_{k=1}^{n}(a_k + b_k)^p\right)^{1/mp} \leq \left(\sum_{k=1}^{n} a_k^p\right)^{1/mp} + \left(\sum_{k=1}^{n} b_k^p\right)^{1/mp},$$

which is required. □

Remark 2.2. a) If $m = 1$ is substituted into (2.5), we get Minkowski's inequality.
b) If $p = 2$ is substituted into (2.5), we get

$$\left(\sum_{k=1}^{n}(a_k+b_k)^2\right)^{1/2m} \leq \left(\sum_{k=1}^{n}a_k^2\right)^{1/2m} + \left(\sum_{k=1}^{n}b_k^2\right)^{1/2m}. \tag{2.6}$$

c) From (1.2), the extension of (2.5) becomes

$$\left(\sum_{i=1}^{n}\left(\sum_{j=1}^{k}x_{ij}\right)^p\right)^{1/mp} \leq \sum_{j=1}^{k}\left(\sum_{i=1}^{n}x_{ij}^p\right)^{1/mp}.$$

In the following theorems, we give the generalizations of the reverse Hölder inequality, Popoviciu's inequality, Lyapunov's inequality and Bellman's inequality respectively.

Theorem 2.3. *If $a_k,\ b_k > 0$ for $k = 1, 2, \ldots, n$ and $\frac{1}{sp} + \frac{1}{sq} = 1$ with $sp < 0$ or $sq < 0$, for $s \geq 1$, then*

$$\sum_{k=1}^{n}(a_kb_k)^{1/s} \geq \left(\sum_{k=1}^{n}a_k^p\right)^{1/sp}\left(\sum_{k=1}^{n}b_k^q\right)^{1/sq}, \tag{2.7}$$

with equality holding if and only if $\alpha a_k^p = \beta b_k^q$ for $k = 1, 2, \ldots n$, where α and β are real nonnegative constants such that $\alpha^2 + \beta^2 > 0$.

Proof. Let $sp < 0$ and put $P = -\frac{p}{sq},\ Q = \frac{1}{s^2q}. Then\ \frac{1}{sP} + \frac{1}{sQ} = 1$ with $sP > 0\ and\ sQ > 0$. Therefore, according to (2.1), we obtain

$$\left(\sum_{k=1}^{n}A_k^P\right)^{1/sP}\left(\sum_{k=1}^{n}B_k^Q\right)^{1/sQ} \geq \sum_{k=1}^{n}(A_kB_k)^{1/s},$$

where $A_k > 0\ and\ B_k > 0\ for\ k = 1, 2, \ldots, n$. The last inequality for $A_k = a_k^{-sq}$ and $B_k = a_k^{sq}b_k^{sq}$ becomes

$$\left(\sum_{k=1}^{n}a_k^p\right)^{-\frac{sq}{sp}}(\sum_{k=1}^{n}(a_kb_k)^{1/s})^{sq} \geq \sum_{k=1}^{n}(b_k^{sq})^{1/s}.$$

Hence, we have

$$\sum_{k=1}^{n}(a_kb_k)^{1/s} \geq \left(\sum_{k=1}^{n}a_k^p\right)^{1/sp}\left(\sum_{k=1}^{n}b_k^q\right)^{1/sq},$$

which is (2.7). □

Theorem 2.4. *Let a and b be two nonnegative n-tuples, p and q are real numbers such that $p, q \neq 0,\ s \geq 1$ and $\frac{1}{sp} + \frac{1}{sq} = 1$ and let $a_1^p - a_2^p - \cdots - a_n^p > 0$ and $b_1^q - b_2^q - \cdots - b_n^q > 0$. Then, we have for $p > 1$,*

$$(a_1^p - a_2^p - \cdots - a_n^p)^{1/s}(b_1^q - b_2^q - \cdots - b_n^q)^{1/s} \leq a_1^{\frac{1}{s}}b_1^{\frac{1}{s}} - a_2^{\frac{1}{s}}b_2^{\frac{1}{s}} - .. - a_n^{\frac{1}{s}}b_n^{\frac{1}{s}}. \tag{2.8}$$

If $p < 1$ $(p \neq 0)$, we have the reverse inequality.

Proof. Replacing a_1^p and b_1^q by $a_1^p - a_2^p - \cdots - a_n^p$ and $b_1^q - b_2^q - \cdots - b_n^q$, respectively, in (2.1), we have

$$(a_1b_1)^{1/s} \leq (a_1b_1)^{\frac{1}{s}} - (a_2b_2)^{\frac{1}{s}} - \cdots - (a_nb_n)^{\frac{1}{s}}.$$

Resubstituting, the last inequality becomes

$$(a_1^p - a_2^p - \cdots - a_n^p)^{1/s}(b_1^q - b_2^q - \cdots - b_n^q)^{1/s} \leq a_1^{\frac{1}{s}}b_1^{\frac{1}{s}} - a_2^{\frac{1}{s}}b_2^{\frac{1}{s}} - .. - a_n^{\frac{1}{s}}b_n^{\frac{1}{s}}.$$

which is (2.8). □

Remark 2.3. a) If we put $s=1$ in (2.8), we get Popoviciu's inequality.
b) If we put $s=1$ *and* $p=q=2$ in (2.8), we get Aczél's inequality.

Remark 2.4. From (2.1) for $u\geq 1$, we get

$$\sum_{k=1}^{n}(a_kb_k)^{1/u}\leq\left(\sum_{k=1}^{n}a_k^p\right)^{1/up}\left(\sum_{k=1}^{n}b_k^q\right)^{1/uq},$$

where $\frac{1}{up}+\frac{1}{uq}=1$. Substituting $p=\frac{r-t}{r-s}$, $q=\frac{r-t}{s-t}$ $(r>s>t>0)$, $a_k^p=p_kx_k^t$ *and* $b_k^q=p_kx_k^r$, $(p_k\geq 0,\ x_k\geq 0\ for\ k=1,2,\ldots,n\ )$ into the inequality above, we have

$$\sum_{k=1}^{n}(p_kx_k^t)^{\frac{r-s}{u(r-t)}}(p_kx_k^r)^{\frac{s-t}{u(r-t)}}\leq(\sum_{k=1}^{n}p_kx_k^t)^{\frac{r-s}{u(r-t)}}(\sum_{k=1}^{n}p_kx_k^r)^{\frac{s-t}{u(r-t)}}.$$

From the last inequality, we get

$$\sum_{k=1}^{n}\left(p_kx_k^{\frac{s}{u}}\right)^{u(r-t)}\leq(\sum_{k=1}^{n}p_kx_k^t)^{r-s}(\sum_{k=1}^{n}p_kx_k^r)^{s-t},\tag{2.9}$$

which is the generalization of Lyapunov's inequality. Letting $u=1$ in (2.9), we obtain Lyapunov's inequality.

Theorem 2.5. *Let a and b be n-tuples of nonnegative numbers such that* $a_1^{mp}-a_2^p-\cdots-a_n^p>0$ *and* $b_1^{mp}-b_2^p-\cdots-b_n^p>0$. *If* $p\geq 1$ $(or\ p<0)$, $then$

$$\left[(a_1^{mp}-a_2^p-\cdots-a_n^p)^{\frac{1}{p}}+(b_1^{mp}-b_2^p-\cdots-b_n^p)^{\frac{1}{p}}\right]^p$$
$$\leq(a_1+b_1)^{mp}-(a_2+b_2)^p-..-(a_n+b_n)^p.\tag{2.10}$$

If $0<\ p<1$, *then the reverse inequality in* (2.10) *holds, where* $m\in\mathbb{N}$.

Proof. Replacing a_1^p and b_1^p by $a_1^{mp}-a_2^p-\cdots-a_n^p$ and $b_1^{mp}-b_2^p-\cdots-b_n^p$, respectively, in (2.5), we have

$$(a_1+b_1)^p\leq(a_1+b_1)^{mp}-(a_2+b_2)^p-\cdots-(a_n+b_n)^p.$$

Resubstituting, the last inequality becomes

$$\left[(a_1^{mp}-a_2^p-\cdots-a_n^p)^{\frac{1}{p}}+(b_1^{mp}-b_2^p-\cdots-b_n^p)^{\frac{1}{p}}\right]^p$$
$$\leq(a_1+b_1)^{mp}-(a_2+b_2)^p-\cdots-(a_n+b_n)^p,$$

which is (2.10). □

Remark 2.5. If $m=1$ is substituted into (2.10), we get Bellman's inequality.

3. Applications

Now, using inequalities (2.5) and (2.6), we give some applications for the metric spaces, normed spaces, Banach spaces and sequence spaces. Furthermore, using inequality (2.4), we give an integral inequality.

Corollary 3.1. *Let* $d:\ \mathbb{R}^nx\mathbb{R}^n\rightarrow\mathbb{R}$ *be the function such that*

$$d(x,y)=\left[(x_1-y_1)^2+(x_2-y_2)^2+\cdots+(x_n-y_n)^2\right]^{1/2m}.\tag{3.1}$$

Then $(\mathbb{R}^n,d)$ *is a metric space for* $m\in\mathbb{N}$.

Proof. The properties (M1) and (M2) of the metric are obvious. Applying the inequality (2.6), we obtain for $x, y, z \in \mathbb{R}^n$

$$d(x,y) = (\sum_{k=1}^{n} |x_k - y_k|^2)^{1/2m} = (\sum_{k=1}^{n} |x_k - z_k + z_k - y_k|^2)^{1/2m}$$
$$\leq \left(\sum_{k=1}^{n} |x_k - z_k|^2\right)^{\frac{1}{2m}} + (\sum_{k=1}^{n} |z_k - y_k|^2)^{1/2m}$$
$$\leq d(x,z) + d(z,y),$$

which is (M3). □

Corollary 3.2. *The space* $\mathbb{R}^n$ *with the norm defined for* $x = (x_1, x_2, \ldots, x_n)$ *and* $m \in \mathbb{N}$ *by*

$$\|x\|_m = (\sum_{i=1}^{n} |x_i|^2)^{1/2m},$$

is a $\frac{1}{m}$*−normed vector space.*

Proof. The space $\mathbb{R}^n$ is an n-dimensional vector space, so we need to verify the properties of the norm. We have
(N1). $\|x\|_m = 0 \Leftrightarrow x = \theta$.
(N2). For $\alpha \epsilon \mathbb{R}$,

$$\|\alpha x\|_m = (\sum_{i=1}^{n} |\alpha x_i|^2)^{1/2m} = \alpha^{1/m} (\sum_{i=1}^{n} |x_i|^2)^{1/2m} = \alpha^{1/m} \|x\|_m.$$

(N3). Applying the inequality (2.6), we get

$$\|x+y\|_m = (\sum_{i=1}^{n} |x_i + y_i|^2)^{1/2m} \leq \left(\sum_{i=1}^{n} |x_i|^2\right)^{\frac{1}{2m}} + \left(\sum_{i=1}^{n} |y_i|^2\right)^{\frac{1}{2m}} = \|x\|_m + \|y\|_m.$$

Thus, $(\mathbb{R}^n, \|x\|_m)$ is a $\frac{1}{m}$−normed vector space. □

Corollary 3.3. *The metric space* $(\mathbb{R}^n, d)$ *is complete.*

Proof. Suppose that (x_m) is a Cauchy sequence in $\mathbb{R}^n$. Then, we have

$$d(x_m, x_k) \to 0 \qquad (m, k \to \infty).$$

Note that each member of the sequence $(x^{(m)})$ is itself a sequence
$x_m = \left(x_i^{(m)}\right) = \left(x_1^{(m)}, x_2^{(m)}, \ldots, x_n^{(m)}\right)$, for $m = 1, 2, 3 \ldots$
Now, for each $\varepsilon > 0$ there exists $n_o \in \mathbb{N}$ such that $d(x_m, x_k) < \varepsilon, \ \forall m, k \geq n_o$. By (3.1), we have

$$d(x_m, x_k) = \left(\sum_{i=1}^{n} \left(x_i^{(m)} - x_i^{(k)}\right)^2\right)^{1/2t} < \varepsilon, \quad for\ \forall m, k \geq n_o\ and\ t \in \mathbb{N}.$$

Since each term in the above inequality is positive,

$$\left|x_i^{(m)} - x_i^{(k)}\right| < \varepsilon^t \ \ for\ i = 1, 2, \ldots, n\ and\ \forall m, k \geq n_o.$$

Hence $\left(x_i^{(m)}\right) = \left(x_1^{(m)}, x_2^{(m)}, \ldots, x_n^{(m)}\right)$ is a Cauchy sequence in $\mathbb{R}$, $for\ i = 1, 2, \ldots, n$. Since $\mathbb{R}$ is complete, $\left(x_i^{(m)}\right)$ converges to x_i in $\mathbb{R}$ for $i = 1, 2, \ldots, n$. So,

$$\lim_{m \to \infty} x_i^{(m)} = x_i \quad for\ i = 1, 2, \ldots, n.$$

Let $x = (x_1, x_2, \ldots, x_n)\,,$ *then* $x \in \mathbb{R}^n$. We now prove that (x_m) converges to x.

$$d\left(x_m, x\right) = \left(\sum_{i=1}^{n}\left(x_i^{(m)} - x_i\right)^2\right)^{1/2t} = \lim_{k\to\infty}\left(\sum_{i=1}^{n}\left(x_i^{(m)} - x_i^{(k)}\right)^2\right)^{1/2t} < \varepsilon,\ \forall m \geq n_0\ .$$

Hence the Cauchy sequence (x_m) converges to $x \in \mathbb{R}^n$. Thus, $(\mathbb{R}^n, d)$ is a complete metric space. □

Corollary 3.4. *The vector space* $\mathbb{R}^n$ *with the norm defined for* $x = (x_1, x_2, \ldots, x_n)$ *and* $m \in \mathbb{N}$ *by*

$$\|x\|_m = (\sum_{i=1}^{n}|x_i|^2)^{1/2m},$$

is a $\frac{1}{m}-$*Banach space.*

Proof. Note that a Banach space is a normed linear space that is a complete metric space with respect to the metric derived from its norm. For this reason, the claim follows from Corollaries 3.2 and 3.3. □

Corollary 3.5. *Let* $d_{m,p}:\ l_p x l_p \to \mathbb{R}$ *be the function such that*

$$d_{m,p} = (\sum_{i=1}^{\infty}|x_i - y_i|^p)^{1/mp},$$

for $1 \leq p < \infty$, $m \in \mathbb{N}$ *and* $x = (x_1, x_2, \ldots)$. *Then* $(l_p, d_{m,p})$ *is a metric space.*

Proof. The properties (M1) and (M2) of the metric are obvious. The property (M3) follows from the inequality (2.5). □

Corollary 3.6. *The space* l_p *with the norm defined for* $1 \leq p < \infty$, $m \in \mathbb{N}$ *and* $x = (x_1, x_2, \ldots)$ *by*

$$\|x\|_{m,p} = (\sum_{i=1}^{\infty}|x_i|^p)^{1/mp},$$

is a $\frac{1}{m}-$*normed vector space.*

Proof. The properties (N1) and (N2) of the norm are obvious. The property (N3) follows from the inequality (2.5). □

Corollary 3.7. *The metric space* $(l_p, d_{m,p})$ *is complete.*

Proof. Let (x_n) be a Cauchy sequence in the space l_p, where $x_n = \left(x_i^{(n)}\right) = \left(x_1^{(n)}, x_2^{(n)}, \ldots\right)$. Let $\varepsilon > 0$ be a real number. Then, there exists a positive integer n_o such that

$$d_{m,p}\left(x_n, x_t\right) = \left(\sum_{i=1}^{n}\left(x_i^{(n)} - x_i^{(t)}\right)^p\right)^{1/mp} < \varepsilon\ \ , \tag{3.2}$$

$for\ all\ n, t \geq n_o\ and\ m \in \mathbb{N}$. This shows that $\left|x_i^{(n)} - x_i^{(t)}\right| < \varepsilon^m$, for all $n, t \geq n_o$ and consequently $\left(x_i^{(n)}\right) = \left(x_1^{(n)}, x_2^{(n)}, \ldots\right)$ is a Cauchy sequence in $\mathbb{K}$ ($\mathbb{R}$ *or* $\mathbb{C}$). Since these spaces are complete, $\left(x_i^{(n)}\right)$ converges to a point $x_i \in \mathbb{K}$. Also, for each $k \in \mathbb{N}$, the statement (3.2) gives

$$\sum_{i=1}^{k}\left|x_i^{(n)} - x_i^{(t)}\right|^p < \varepsilon^{mp}\ for\ all\ n, t \geq n_o. \tag{3.3}$$

From (3.3) with $t \to \infty$, we get

$$\sum_{i=1}^{k}\left|x_i^{(n)} - x_i\right|^p < \varepsilon^{mp}. \tag{3.4}$$

We need to prove that $x = (x_1, x_2, \dots)$ *is in* l_p. The inequalities (3.4) and (2.5) show that

$$\begin{aligned}(\sum_{i=1}^{k} |x_i|^p)^{1/mp} &= (\sum_{i=1}^{k} \left|x_i - x_i^{(n)} + x_i^{(n)}\right|^p)^{1/mp} \\ &\leq \left(\sum_{i=1}^{k} \left|x_i - x_i^{(n)}\right|^p\right)^{\frac{1}{mp}} + \left(\sum_{i=1}^{k} \left|x_i^{(n)}\right|^p\right)^{1/mp} \\ &< \varepsilon + \left(\sum_{i=1}^{k} \left|x_i^{(n)}\right|^p\right)^{1/mp}.\end{aligned}$$

Since $\left(x_i^{(n)}\right)$ is in l_p, the above inequality shows that $(\sum_{i=1}^{k} |x_i|^p)^{1/mp}$ is bounded and monotonically increasing, therefore the series $\sum_{i=1}^{k} |x_i|^p$ is convergent. Thus, x *is in* l_p. Also, it is obvious from (3.4) that (x_n) *converges to* x. Therefore, $(l_p, d_{m,p})$ is a complete metric space. □

Corollary 3.8. *The space* l_p *, with the norm defined for* $1 \leq p < \infty,\ m \in \mathbb{N}$ *and* $x = (x_1, x_2, \dots)$ *by*

$$\|x\|_{m,p} = (\sum_{i=1}^{\infty} |x_i|^p)^{1/mp},$$

is a $\frac{1}{m}$*−Banach space.*

Proof. Note that a Banach space is a normed linear space that is a complete metric space with respect to the metric derived from its norm. For this reason, the claim follows from Corollaries 3.6 and 3.7. □

Let $b_{p,1}^{r,s}$ be the binomial sequence space such that

$$b_{p,1}^{r,s} = \left\{ x = (x_k) \in w : \sum_{n} \left| \frac{1}{(s+r)^n} \sum_{i=0}^{n} \sum_{k=0}^{i} \binom{i}{k} s^{i-k} r^k x_k \right|^p < \infty \right\} \quad , 1 \leq p < \infty.$$

The space $b_{p,1}^{r,s}$ includes the spaces l_p and $b_p^{r,s}$. Hence we may give the following corollary.

Corollary 3.9. *The space* $b_{p,1}^{r,s}$ *with the norm defined for* $m \in \mathbb{N}$ *by*

$$\|x\|_{r,s,m} = \left(\sum_{n=0}^{\infty} \left| \frac{1}{(s+r)^n} \sum_{i=0}^{n} \sum_{k=0}^{i} \binom{i}{k} s^{i-k} r^k x_k \right|^p \right)^{1/mp},$$

is a $\frac{1}{m}$*−normed space.*

Proof. So, we need to verify the conditions (N1)-(N3) of the norm. We have
(N1). $\|x\|_{r,s,m} = 0 \Leftrightarrow x = \theta.$
(N2). For $\alpha \epsilon \mathbb{R}$,

$$\|\alpha x\|_{r,s,m} = \left(\sum_{n=0}^{\infty} \left| \frac{1}{(s+r)^n} \sum_{i=0}^{n} \sum_{k=0}^{i} \binom{i}{k} s^{i-k} r^k (\alpha x_k) \right|^p \right)^{1/mp} = \alpha^{1/m} \|x\|_{r,s,m}.$$

(N3). Applying the inequality (2.5), we get

$$\begin{aligned}\|x+y\|_{r,s,m} &= \left(\sum_{n=0}^{\infty} \left| \frac{1}{(s+r)^n} \sum_{i=0}^{n} \sum_{k=0}^{i} \binom{i}{k} s^{i-k} r^k (x_k + y_k) \right|^p \right)^{1/mp} \\ &\leq \left(\sum_{n=0}^{\infty} \left| \frac{1}{(s+r)^n} \sum_{i=0}^{n} \sum_{k=0}^{i} \binom{i}{k} s^{i-k} r^k x_k \right|^p \right)^{\frac{1}{mp}} +\end{aligned}$$

$$+\left(\sum_{n=0}^{\infty}\left|\frac{1}{(s+r)^{n}}\sum_{i=0}^{n}\sum_{k=0}^{i}\left(\begin{array}{c}i\\k\end{array}\right)s^{i-k}r^{k}y_{k}\right|^{p}\right)^{1/mp}$$

$$\leq \|x\|_{r,s,m}+\|y\|_{r,s,m}.$$

Thus, $\left(b_{p,1}^{r,s},\|x\|_{r,s,m}\right)$ is a $\frac{1}{m}-$normed space. □

Finally, we give an integral inequality:

Corollary 3.10. *Let f be a real valued function defined on $[a,b]\subset\mathbb{R}^{+}$ such that the functions $|f|^{p}$ and $|f|^{q}$ are integrable on $[a,b]$ and let*

$$I_{n/s}=\int_{a}^{b}f(x)^{n/s}dx,$$

then we have for $n>1$ and $s\in\mathbb{N}$

$$I_{2(n-1)/s}^{s}=\ \leq I_{np}^{\frac{1}{p}}I_{(n-2)q}^{\frac{1}{q}}.$$

Proof. Applying the inequality (2.4) for $j=1,2$, we obtain

$$I_{2(n-1)/s}^{s}=\left(\int_{a}^{b}f(x)^{2(n-1)/s}dx\right)^{s}=\left(\int_{a}^{b}f(x)^{n/s}f(x)^{(n-2)/s}dx\right)^{s}$$

$$\leq\left(\int_{a}^{b}f(x)^{np}dx\right)^{1/p}\left(\int_{a}^{b}f(x)^{(n-2)q}dx\right)^{1/q}\leq I_{np}^{\frac{1}{p}}I_{(n-2)q}^{\frac{1}{q}},$$

which is required. □

4. Conclusion

In this paper, the generalizations of Hölder's inequality and Minkowski's inequality have been presented. Furthermore, the generalizations of Aczél's, Popoviciu's, Lyapunov's and Bellman's inequalities have been given. Finally, some applications for the metric spaces, normed spaces, Banach spaces, sequence spaces and integral inequalities have been provided.

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