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THE STRUCTURE OF MATRIX POLYNOMIAL ALGEBRAS

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Dedicated to the memory of Professor Edmund R. Puczyłowski

Abstract. This work formally introduces and starts investigating the structure of matrix polynomial algebra extensions of a coefficient algebra by (elementary) matrix-variables over a ground polynomial ring in not necessary commuting variables. These matrix subalgebras of full matrix rings over polynomial rings show up in noncommutative algebraic geometry. We carefully study their (one-sided or bilateral) noetherianity, obtaining a precise lift of the Hilbert Basis Theorem when the ground ring is either a commutative polynomial ring, a free noncommutative polynomial ring or a skew polynomial ring extension by a free commutative term-ordered monoid. We equally address the natural but rather delicate question of recognising which matrix polynomial algebras are Cayley-Hamilton algebras, which are interesting noncommutative algebras arising from the study of Gl_n-varieties.

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1. Introduction

Of fundamental role in commutative algebra and commutative algebraic geometry are commutative polynomial rings over a base field [2,3,4]. This explains the rich investigation of their ideal theory and that of near skew polynomial rings. With the development of noncommutative algebraic geometry, various more or less specialized matrix subalgebras of the full matrix algebras over polynomial rings arise naturally as interesting classes of algebras. In the realm of noncommutative deformation theory as introduced by Laudal and Eriksen [5,12,13] and further refined by Siqveland [18, §5], free matrix polynomial algebras and their completions showed up as important tools in the computation of pro-representing hulls (formal moduli) [5, §5, p 105]. Motivated by this, an earlier rather informal attempt to matrix polynomial algebras appeared in [18, p 2] and was not further investigated. For a positive integer n and arising from Gl_n-equivariant algebraic geometry, an interesting class of noncommutative algebras investigated by [14] is formed by Cayley-Hamilton algebras of degree n over the field $\mathbb{C}$ of complex numbers. Here, free affine Cayley-Hamilton algebras are specialized matrix polynomial algebras in commuting variables, called the *trace algebra* [14, §1.4; §1.8, Theorem 1.16].

However on the algebra side of the story, it seems there had not been a systematic study of matrix polynomial algebras as compared to commutative polynomial rings. The need therefore arises to formalize and systematically investigate the structure theory and the geometry of matrix polynomial algebra extensions. In this work we focus on the structure of these algebras, their geometry being the object of a subsequent work. Briefly, letting $\Bbbk$ be a base commutative ring and given a non-zero $n \in \mathbb{N}$, a matrix polynomial algebra A is an extension of a coefficient $\Bbbk$-algebra R by set $\mathbb{X}$ of $n \times n$ matrix-variables $X = (x_{i,j})_{i,j}$ (together with some special constant matrices), with $x_{i,j} \in \boldsymbol{X} \cup \{0\}$ for all i, j, where $\boldsymbol{X}$ is a set of not necessarily commuting variables. Such an object is actually very general with a complex structure, and should model various polynomial-like algebras arising in noncommutative algebraic geometry. In order to access a relevant aspect of the structure and geometry of A, it is necessary to restrict attention to generic subclasses. As pointed out above, this was done for Cayley-Hamilton algebras by [14]. In this work we are concerned essentially with the case where A includes all the elementary idempotent matrices.

We shall now describe in more details the main contributions of this paper. In the first part (Sections 2, 3 and 4), we fix a ground polynomial ring extension of a coefficient $\Bbbk$-algebra by a set $\boldsymbol{X}$ of independent variables subject only to some commutativity relations between some variables; this includes the commutative and the free noncommutative polynomial rings. In this framework, Section 2 further unfolds the structure of matrix polynomial algebras, computes their centers and briefly illustrates their use in noncommutative deformation theory. Section 3 addresses the question of (one-sided or bilateral) noetherianity and the main contribution in this direction given by Theorem 3.12 is a precise analogue of the Hilbert Basis Theorem ([8, §7.10], [2, Thm 7.5]).

In Section 4 we study the intersection with Cayley-Hamilton algebras, obtaining in Theorem 4.4 that a matrix polynomial algebra A in commuting variables admits a trace map making it Cayley-Hamilton of degree $r \in [\![1, n]\!]$ precisely when the diagonal components of A coincide and the trace map is induced by the natural trace on matrices.

The final section 5 generalizes the framework by letting the ground ring be a skew polynomial ring extension of a coefficient algebra by a free commutative term-ordered monoid. This change yields a more complex structure for matrix skew polynomial extensions. We are able to achieve a precise generalisation of the Hilbert Basis Theorem to univariate matrix skew polynomial extensions (Theorem 5.4 and Corollary 5.5); the corresponding result in the multivariate case (Theorem 5.7) restricts the coefficient algebra to a division algebra.

2. Matrix polynomial algebras

As general settings, we fix a base commutative ring $\Bbbk$ and a coefficient $\Bbbk$-algebra R; every $\Bbbk$-algebra is assumed associative and unitary. We also fix a non-zero $n \in \mathbb{N}$ and suitably write $[\![p, q]\!] = \{i \in \mathbb{Z} \ : \ p \le i \le q\}$ for $p, q \in \mathbb{N}$. The canonical basis of the full matrix ring $\mathcal{M}_n(\Bbbk)$ and the subset of elementary idempotent matrices are given by:

$$\mathbb{E} = \{\boldsymbol{e}_{i,j} \ : \ i, j \in [\![1, n]\!]\} \text{ and } \boldsymbol{I} = \{\boldsymbol{e}_i = \boldsymbol{e}_{i,i} \ : \ 1 \le i \le n\}.$$

Each elementary matrix $\boldsymbol{e}_{i,j}$ has 1 as its (i, j)-entry, and 0's elsewhere.

From this section down to Section 4, we fix a ground polynomial ring $\boldsymbol{R} = R\langle \boldsymbol{X}; \boldsymbol{C}\rangle$ with $\boldsymbol{X}$ an algebraically R-independent set of not necessarily commuting variables, subject to some commutativity relations '$xy = yx$' for (x, y) running through a prescribed subset $\boldsymbol{C} \subset \boldsymbol{X} \times \boldsymbol{X}$. Thus the ambient monoid of terms

$$\langle \boldsymbol{X}; \boldsymbol{C}\rangle = \langle \boldsymbol{X} \ : \ xy = yx \text{ for all } (x, y) \in \boldsymbol{C}\rangle$$

is the quotient of the free monoid $\langle \boldsymbol{X}\rangle$ modulo some commutativity relations.

Definition 2.1. By a *matrix-variable* (over $\boldsymbol{X}$) is meant any non-zero $(n \times n)$-matrix X each of whose entries is either a variable from $\boldsymbol{X}$ or is equal to 0; it is called *elementary* if $X = \boldsymbol{e}_{i,j}x$ with $i, j \in [\![1, n]\!]$ and $x \in \boldsymbol{X}$.

Let $\mathbb{X}$ be a set of matrix-variables, to which is adjoined a multiplicatively closed subset $\boldsymbol{E} \subset \mathbb{E}$. The *matrix polynomial extension* of R by $(\boldsymbol{E}, \mathbb{X}, \boldsymbol{C})$ is the subalgebra

$$R\langle \boldsymbol{E}, \mathbb{X}; \boldsymbol{C}\rangle = R\langle \boldsymbol{e}, X \ : \ \boldsymbol{e} \in \boldsymbol{E}, X \in \mathbb{X}\,; \boldsymbol{C}\rangle$$

of the full matrix algebra $\mathcal{M}_n(\boldsymbol{R})$, generated over $R \cdot \boldsymbol{E} = \oplus_{\boldsymbol{e}\in\boldsymbol{E}} R\boldsymbol{e}$ by the matrix-variables in $\mathbb{X}$. We refer to $(\mathbb{X}, \boldsymbol{C})$ (or more precisely, $(\boldsymbol{E}, \mathbb{X}, \boldsymbol{C})$) as a *seed*.

From now and henceforth we fix a seed $(\boldsymbol{E}, \mathbb{X}, \boldsymbol{C})$ and write $\mathsf{A} = R\langle \boldsymbol{E}, \mathbb{X}; \boldsymbol{C}\rangle$ for our running matrix polynomial algebra. We also consider the generic case where A includes all the elementary idempotent matrices. Hence,

$$\boldsymbol{I} \subset \boldsymbol{E} \subset \mathbb{E} \text{ and } \mathbb{X} \text{ re-organizes as } \mathbb{X} = (\mathbb{X}_{i,j})_{i,j} = \bigcup_{1\le i,j\le n} \mathbb{X}_{i,j}\boldsymbol{e}_{i,j}.$$

And as matrix subalgebra of $\mathcal{M}_n(\boldsymbol{R})$ we have:

$$\mathsf{A} = \big(\mathsf{A}_{i,j}\big)_{i,j} = \bigoplus_{1\le i,j\le n} \mathsf{A}_{i,j}\boldsymbol{e}_{i,j} = \begin{pmatrix} \mathsf{A}_{1,1} & \cdots & \mathsf{A}_{1,n} \\ \vdots & & \vdots \\ \mathsf{A}_{n,1} & \cdots & \mathsf{A}_{n,n} \end{pmatrix}, \text{ and we set } \mathsf{A}_i = \mathsf{A}_{i,i},\ 1 \le i \le n.$$

Each diagonal component $\mathsf{A}_i \subset \boldsymbol{R}$ is a $\Bbbk$-algebra extension of R, each $\mathsf{A}_{i,j} \subset \boldsymbol{R}$ is an A_i-A_j-bimodule for $1 \le i, j \le n$.

Observe for all $i, j, k, l \in [\![1, n]\!]$ that the set $\boldsymbol{e}_{i,k} \cdot \mathbb{X}_{k,l}\boldsymbol{e}_{k,l} \cdot \boldsymbol{e}_{l,j} = \boldsymbol{e}_{i,j}\mathbb{X}_{k,l}$ already lies in A as soon as both $\boldsymbol{e}_{i,k}$ and $\boldsymbol{e}_{l,j}$ belong to $\boldsymbol{E}$. Thus we may henceforth assume from the start that $\mathbb{X}$ is $\boldsymbol{E}$-*saturated* in the sense that:

$$\text{for all } i, j, k, l \in [\![1, n]\!],\ \text{if } \boldsymbol{e}_{i,k}, \boldsymbol{e}_{l,j} \in \boldsymbol{E},\ \text{ then } \mathbb{X}_{k,l} \subset \mathbb{X}_{i,j}.$$

❍ When $\boldsymbol{E} = \boldsymbol{I}$, simplifying the notation one writes:

$$R\langle \mathbb{X}; \boldsymbol{C}\rangle = R\langle \boldsymbol{I}, \mathbb{X}; \boldsymbol{C}\rangle = R\langle \boldsymbol{e}_i,\ x\boldsymbol{e}_{i,j}\ :\ 1 \le i, j \le n,\ x \in \mathbb{X}_{i,j}\,; \boldsymbol{C}\rangle.$$

On the other hand, the full matrix algebra $\mathcal{M}_n(\boldsymbol{R})$ is recovered when $\boldsymbol{E} = \mathbb{E}$.

❍ For $\boldsymbol{C} = \varnothing$ and $\boldsymbol{E} = \boldsymbol{I}$, we get the *free matrix polynomial algebra extension*

$$R\langle \mathbb{X}\rangle = R\left\langle \begin{matrix} \mathbb{X}_{1,1} & \cdots & \mathbb{X}_{1,n} \\ \vdots & \ddots & \vdots \\ \mathbb{X}_{n,1} & \cdots & \mathbb{X}_{n,n} \end{matrix} \right\rangle = R\langle \boldsymbol{e}_i,\ x\boldsymbol{e}_{i,j}\ :\ 1 \le i, j \le n,\ x \in \mathbb{X}_{i,j}\rangle.$$

❍ When $\boldsymbol{C} = \boldsymbol{X} \times \boldsymbol{X}$ and $\boldsymbol{E} = \boldsymbol{I}$, we get the *matrix polynomial ring* (in commuting variables)

$$R[\mathbb{X}] = R \begin{bmatrix} \mathbb{X}_{1,1} & \cdots & \mathbb{X}_{1,n} \\ \vdots & \ddots & \vdots \\ \mathbb{X}_{n,1} & \cdots & \mathbb{X}_{n,n} \end{bmatrix} = R[\boldsymbol{e}_i,\ x\boldsymbol{e}_{i,j}\ :\ 1 \le i, j \le n,\ x \in \mathbb{X}_{i,j}].$$

With A is associated a set $\mathbb{T} = \mathbb{T}(\mathbb{X})$ of *elementary matrix-terms*, such that:

$$\begin{gathered} \mathbb{T} \cup \{0\} = \langle \boldsymbol{E}, \mathbb{X}; \boldsymbol{C}\rangle \text{ is the semigroup generated by } \boldsymbol{E} \text{ and the elementary matrix-} \\ \text{variables, the components } \mathbb{T}_{i,j} \text{ are such that } \boldsymbol{e}_i\mathbb{T}\boldsymbol{e}_j = \boldsymbol{e}_{i,j}\mathbb{T}_{i,j},\ i, j \in [\![1, n]\!]. \end{gathered} \tag{2.1}$$

For every $i, j \in [\![1, n]\!]$, the set $\mathbb{T}_i = \mathbb{T}_{i,i}$ is a submonoid of the (not necessarily commutative) monoid $\langle \boldsymbol{X}; \boldsymbol{C}\rangle = \langle \boldsymbol{X}\ :\ xy = yx \text{ for all } (x, y) \in \boldsymbol{C}\rangle$. For $i \ne j$, the set $\mathbb{T}_{i,j}$ includes the constant term 1 precisely when $\boldsymbol{e}_{i,j} \in \boldsymbol{E}$.

Both $\mathbb{X}$ and the structure of $\mathbb{T}$ and A are efficiently described by a labelled quiver:

$$\mathsf{Q}_{\mathbb{X}}:\quad \text{with set of points } [\![1, n]\!], \text{ and labelled arrows } i \xrightarrow{x} j \text{ for } x \in \mathbb{X}_{i,j},\ 1 \le i, j \le n.$$

Let us make precise some terminology on labelled quivers needed in the sequel.

i) For each pair $i, j \in [\![1, n]\!]$, the set $\mathsf{Q}_{\mathbb{X},1}(i, j)$ consists of all labelled arrows $i \xrightarrow{x} j$ with $x \in \mathbb{X}_{i,j}$. Alternatively (when $\mathbb{X}_{i,j} \neq \emptyset$), the set of all labelled arrows from i to j may be represented by a single labelled arrow $i \xrightarrow{\mathbb{X}_{i,j}} j$.

ii) For any non-zero $m \in \mathbb{N}$, an *m-length path* from i to j in $\mathsf{Q}_{\mathbb{X}}$ is any sequence of labelled arrows $\omega : i = i_1 \xrightarrow{x_1} i_2 \cdots i_m \xrightarrow{x_m} i_{m+1} = j$. The *empty path* (or *trivial path*) at i is identified with the idempotent $\boldsymbol{e}_i$. We denote by $\mathsf{Q}_{\mathbb{X}}(i, j)$ the set of all paths from i to j, and by $\mathsf{Q}_{\mathbb{X},m}(i, j)$ the subset of m-length paths from i to j.

For an m-length path $\omega \in \mathsf{Q}_{\mathbb{X}}(i, j)$ as before, the m-length *term of* ω and the *incidence set of* ω are defined by:

$$\underline{\omega} = x_1 \cdots x_m \in \mathbb{T}_{i,j} \quad \text{and} \quad \boldsymbol{i}_\omega = \{i_1, \ldots, i_m, i_{m+1}\}. \tag{2.2}$$

In particular, the incidence set of the empty path at each $i \in [\![1, n]\!]$ is $\boldsymbol{i}_{\boldsymbol{e}_i} = \{i\}$.

iii) A *cycle* is any *non-trivial path* ω whose source and target coincide. A *simple path* is any one that does not properly contain a cycle.

The next remark expands the description of the set $\mathbb{T}$ of elementary matrix-terms, highlighting the link between matrix polynomial algebras and usual path algebras.

Remark 2.2. (a) The ring $\mathsf{A} = R\mathbb{T} = R[\mathbb{T}]$ is a semigroup ring extension of R with multiplicative basis $\mathbb{T}$, that is, $\mathbb{T}$ is an R-central basis for A such that $\mathbb{T} \cup \{0\}$ is a *semigroup with zero.*

(b) For every $i, j \in [\![1, n]\!]$, $\mathbb{T}_{i,j}$ consists of terms $\underline{\omega}$ along all paths ω in $\mathsf{Q}_{\mathbb{X}}(i, j)$, together with the constant 1 if also $\boldsymbol{e}_{i,j} \in \boldsymbol{E}$.

(c) If the components of $\mathbb{X}$ are pairwise disjoint, then the free matrix polynomial algebra $R\langle\mathbb{X}\rangle$ coincides with the usual path algebra $R\mathsf{Q}_{\mathbb{X}}$ of an abstract quiver (as presented for instance in [1, ch II]).

Example 2.3. Let x, y, z be three independent variables over R, and take:
$\mathbb{X} = \left[\begin{smallmatrix} x & y,z \\ z & y \end{smallmatrix}\right]$, with associated labelled quiver $\mathsf{Q}_{\mathbb{X}}$:
$$\begin{tikzcd} 1 \arrow[loop left, "x"] \arrow[r, bend left, "{y,z}"] & 2 \arrow[l, bend left, "z"] \arrow[loop right, "y"] \end{tikzcd}$$

$$R[\mathbb{X}] = R\left[\begin{smallmatrix} x & y,z \\ z & y \end{smallmatrix}\right] = \begin{pmatrix} R[x] + \{yz, z^2\}S & \{y,z\}S \\ zS & R[y] + \{yz, z^2\}S \end{pmatrix} \text{ with } S = R[x, y, yz, z^2]$$

$$R\langle\mathbb{X}\rangle = R\left\langle\begin{smallmatrix} x & y,z \\ z & y \end{smallmatrix}\right\rangle = \begin{pmatrix} \mathsf{A}_1 & \mathsf{A}_{1,2} \\ \mathsf{A}_{2,1} & \mathsf{A}_2 \end{pmatrix} \text{ with } \begin{cases} \mathsf{A}_1 = R\langle x, y^{1+n}z, zy^n z \ :\ n \in \mathbb{N}\rangle, \\ \mathsf{A}_2 = R\langle y, zx^n y, zx^n z \ :\ n \in \mathbb{N}\rangle, \\ \mathsf{A}_{1,2} = \mathsf{A}_1 \cdot \{y,z\} \cdot \mathsf{A}_2, \ \ \mathsf{A}_{2,1} = \mathsf{A}_2 \cdot z \cdot \mathsf{A}_1. \end{cases}$$

While $R[\mathbb{X}]$ and $R\langle\mathbb{X}\rangle$ are finitely generated $\Bbbk$-algebras, their diagonal components are not.

The next proposition makes clear the fact that for matrix polynomial extensions with commutative diagonal components, one may assume the commuting variables context.

Proposition 2.4. *Every matrix polynomial algebra extension $R\langle \boldsymbol{E}, \mathbb{Y}; \boldsymbol{C}\rangle$ with commutative diagonal components is a natural quotient of a matrix polynomial algebra extension $R[\boldsymbol{E}, \mathbb{X}]$ with commuting variables where $\mathbb{X}$ and $\mathbb{Y}$ have equipotent components.*

Proof. We may consider a copy $\mathbb{X}$ of $\mathbb{Y}$ with pairwise disjoint components, together with bijections $\mathbb{X}_{i,j} \longrightarrow \mathbb{Y}_{i,j}$, $x \longmapsto \overline{x}$, for $i, j \in [\![1, n]\!]$. Thus the labelled quiver $\mathsf{Q}_{\mathbb{X}}$ is an ordinary abstract quiver since $\mathbb{X}$ has pairwise disjoint components, and elementary matrix-terms in the free matrix polynomial extension $R\langle \boldsymbol{E}, \mathbb{X}\rangle$ identify precisely with paths of the quiver $\mathsf{Q}_{\mathbb{X}}$. Obviously there is a natural surjective algebra morphism $\pi : R\langle \boldsymbol{E}, \mathbb{X}\rangle \longrightarrow R\langle \boldsymbol{E}, \mathbb{Y}; \boldsymbol{C}\rangle$ with

$$\pi(\boldsymbol{e}) = \boldsymbol{e} \text{ and } \pi(x) = \pi(x\boldsymbol{e}_{i,j}) = \overline{x}\boldsymbol{e}_{i,j} \text{ for all } \boldsymbol{e} \in \boldsymbol{E},\ i, j \in [\![1, n]\!] \text{ and } x \in \mathbb{X}_{i,j}.$$

Our aim is to show that the above map factors through $R[\boldsymbol{E}, \mathbb{X}]$. Thus arbitrarily given $i, j \in [\![1, n]\!]$ and paths $\omega, \omega' \in \mathsf{Q}_{\mathbb{X}}(i, j)$, and writing $\underline{\omega}$ for the term in commuting variables along ω, assuming that $\underline{\omega} = \underline{\omega'}$, we have to show that $\pi(\omega) = \pi(\omega')$. To this end we shall proceed by induction on the length $\ell(\omega)$ of ω. The base of the induction being trivial, suppose that $\ell(\omega)$ is positive and the result proved for all paths with length less than $\ell(\omega)$. One may write $\omega = xu$ and $\omega' = x'u'$ for some arrows $i \xrightarrow{x} k$ and $i \xrightarrow{x'} k'$ and for some subpaths $u \in \mathsf{Q}_{\mathbb{X}}(k, j)$ and $u' \in \mathsf{Q}_{\mathbb{X}}(k', j)$. There are the two following cases to consider.

a) *The case where $x' = x$.* Then, $k' = k$ and $\underline{u} = \underline{u'}$ so that by the induction hypothesis, $\pi(u) = \pi(u')$ and $\pi(\omega) = \overline{x}\pi(u) = \overline{x}\pi(u') = \pi(\omega')$.

b) *The case where $x' \neq x$.* Observe that the equality $\underline{\omega} = \underline{\omega'}$ (of terms in commuting variables) means that ω and ω' consist of exactly the same arrows in possibly different order in case these paths include a cyclic subpath. Since here the first arrow in ω' is distinct from the first arrow in ω, forcibly we have: $\omega' = (x'v')(xv)v''$ for some cyclic subpaths $x'v', xv \in \mathsf{Q}_{\mathbb{X}}(i, i)$ and some path $v'' \in \mathsf{Q}_{\mathbb{X}}(i, j)$. Now the path $w'' = (xv)(x'v')v'' \in \mathsf{Q}_{\mathbb{X}}(i, j)$ shares the same first arrow with ω, and the words $\underline{\omega}$ and $\underline{\omega''}$ in commuting variables are still equal. Hence by point (a) above and by the fact that the diagonal components of $R\langle \boldsymbol{E}, \mathbb{X}; \boldsymbol{C}\rangle$ are commutative, we compute:

$$\pi(\omega) = \pi(\omega'') = \pi(xv)\pi(x'v')\pi(v'') = \pi(x'v')\pi(xv)\pi(v'') = \pi(\omega').$$

Thus π factors through $R[\boldsymbol{E}, \mathbb{X}]$, completing the proof of the proposition. □

In order to describe the center $\mathsf{Z}(\mathsf{A})$ of A, for every $\Lambda = \{i_1, \ldots, i_p\} \subset [\![1, n]\!]$ we let:

$$\mathbb{T}_\Lambda = \mathbb{T}_{i_1} \cdots \mathbb{T}_{i_p} \text{ and } \mathbb{T}'_\Lambda = \langle \underline{\omega} \ :\ \omega \text{ is a cycle in } \mathsf{Q}_{\mathbb{X}} \text{ with } \boldsymbol{i}_\omega \subset \Lambda \rangle \subset \langle \boldsymbol{X}; \boldsymbol{C} \rangle. \qquad (2.3)$$

Proposition 2.5. *Let* $\mathsf{A} = \Bbbk[\boldsymbol{E}, \mathbb{X}]$ *be a matrix polynomial ring, and set* $\boldsymbol{S} = \Bbbk[\mathbb{T}_{[\![1,n]\!]}]$.

(a) *Then the monoid* $\mathbb{T}'_\Lambda$ *for each* $\Lambda \subset [\![1, n]\!]$ *is generated by terms along simple cycles* σ *in* $\mathsf{Q}_{\mathbb{X}}$ *with* $\boldsymbol{i}_\sigma \subset \Lambda$.

(b) *Suppose that* $\mathsf{Q}_{\mathbb{X}}$ *is connected. Then* $\mathsf{Z}(\mathsf{A})$ *identifies with* $\mathsf{Z} = \Bbbk[\cap_{i=1}^n \mathbb{T}_i]$. *Moreover when* $\mathsf{Q}_{\mathbb{X}}$ *has only finitely many simple cycles, there are finite subsets* $\boldsymbol{\tau}_\Lambda$ *for* Λ *running through any family* C *of subsets in* $[\![1, n]\!]$ *with* $\mathbb{T}_1 = \{\underline{\omega} \ :\ \omega \in \mathsf{Q}_{\mathbb{X}}(1,1)$ *and* $\boldsymbol{i}_\omega \in C\}$, *such that:*

$$\boldsymbol{\tau}_{[\![1,n]\!]} \cdot \mathbb{T}_{[\![1,n]\!]} \subset \bigcap_{i=1}^n \mathbb{T}_i,\ \ \boldsymbol{\tau}_\Lambda \subset \bigcap_{i=1}^n \mathbb{T}_i \cap \{\underline{\omega} \ :\ \omega \in \mathsf{Q}_{\mathbb{X}}(1,1) \text{ and } \boldsymbol{i}_\omega = \Lambda\} \subset \boldsymbol{\tau}_\Lambda \cdot \mathbb{T}'_\Lambda.$$

In particular, $\Bbbk[\cup_{\Lambda\in C} \boldsymbol{\tau}_\Lambda] + \boldsymbol{\tau}_{[\![1,n]\!]} \cdot \boldsymbol{S} \subset \mathsf{Z} \subset \Bbbk + \sum_{\Lambda \in C} \boldsymbol{\tau}_\Lambda \cdot \Bbbk[\mathbb{T}'_\Lambda]$.

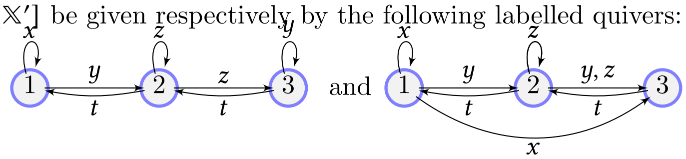
Proof. Notice that the monoid $\mathbb{T}_{[\![1,n]\!]} = \mathbb{T}_1 \cdots \mathbb{T}_n$ obviously coincides with the monoid $\mathbb{T}'_{[\![1,n]\!]}$ generated by the terms $\underline{\omega}$ for all $\omega \in \mathsf{Q}_{\mathbb{X}}(i,i)$ with $i \in [\![1, n]\!]$. For each given non-empty subset $\Lambda \subset [\![1, n]\!]$ and any non-simple cycle ω in $\mathsf{Q}_{\mathbb{X}}$ with $\boldsymbol{i}_\omega \subset \Lambda$, it is clear that $\omega = \omega'\sigma\omega''$ for some simple cycle σ and some cycle $\omega'\omega''$ in $\mathsf{Q}_{\mathbb{X}}$ with $\boldsymbol{i}_\sigma, \boldsymbol{i}_{\omega'\omega''} \subset \Lambda$. Thus $\underline{\omega} = \underline{\sigma} \cdot \underline{\omega'\omega''}$ with $\ell(\omega'\omega'') < \ell(\omega)$. Hence by the way of induction on the length of paths, it follows that $\underline{\omega}$ is a product of terms along simple cycles whose incidence sets are contained in Λ. Thus, $\mathbb{T}'_\Lambda$ is generated by terms along simple cycles σ in $\mathsf{Q}_{\mathbb{X}}$ with $\boldsymbol{i}_\sigma \subset \Lambda$.

Turning to part (b), suppose that $\mathsf{Q}_{\mathbb{X}}$ is connected. For the sake of briefness, set $\mathbb{T}_{\mathsf{c}} = \cap_{i=1}^n \mathbb{T}_i$. On one hand, every $u \in \mathsf{Z}$ identifies with $u \cdot \mathbb{1}_n = u \cdot (\boldsymbol{e}_1 + \cdots + \boldsymbol{e}_n)$ lying in $\mathsf{Z}(\mathsf{A})$. On the other hand, since the matrix algebra A includes all the elementary idempotent matrices, every $u \in \mathsf{Z}(\mathsf{A})$ has the form: $u = \sum_{i=1}^n u_i \boldsymbol{e}_i$ with $u_i \in \mathsf{A}_i = R[\mathbb{T}_i]$ for each $i \in [\![1, n]\!]$. For any $i, j \in [\![1, n]\!]$ with $\mathbb{X}_{i,j} \neq \varnothing$, picking any $x \in \mathbb{X}_{i,j}$ one must have: $x u_i \boldsymbol{e}_{i,j} = u \cdot x\boldsymbol{e}_{i,j} = x\boldsymbol{e}_{i,j} \cdot u = x u_j \boldsymbol{e}_{i,j}$, showing that $u_i = u_j$ lies in $\mathsf{A}_i \cap \mathsf{A}_j$. Hence, the assumption that $\mathsf{Q}_{\mathbb{X}}$ is connected yields that the u_i's are all equal to a common $v \in \cap_{i=1}^n \mathsf{A}_i$. This proves that $\mathsf{Z}(\mathsf{A})$ identifies with $\mathsf{Z} = \Bbbk[\mathbb{T}_{\mathsf{c}}]$. Next, additionally assume that $\mathsf{Q}_{\mathbb{X}}$ has only finitely many simple cycles and let C be any family of subsets in $[\![1,n]\!]$ with $\mathbb{T}_1 = \{\underline{\omega} \ : \ \omega \in \mathsf{Q}_{\mathbb{X}}(1,1) \text{ and } \boldsymbol{i}_\omega \in C\}$. Thus for every term $\tau \in \mathbb{T}_{\mathsf{c}}$, there is some $\Lambda \in C$ and $\omega \in \mathsf{Q}_{\mathbb{X}}(1,1)$ with $\tau = \underline{\omega}$ and $\boldsymbol{i}_\omega = \Lambda$. Part (a) shows that the commutative monoid $\mathbb{T}'_\Lambda$ is finitely generated (by terms along simple cycles σ with $\boldsymbol{i}_\sigma \subset \Lambda$). So, Dickson's Lemma [3, §4, Theorem 5, Exercises 7] applies yielding a finite subset $\boldsymbol{\tau}_\Lambda$ with

$$\boldsymbol{\tau}_\Lambda \subset \mathbb{T}_{\mathsf{c}} \cap \{\underline{\omega} \ : \ \omega \in \mathsf{Q}_{\mathbb{X}}(1,1) \text{ and } \boldsymbol{i}_\omega = \Lambda\} \subset \boldsymbol{\tau}_\Lambda \cdot \mathbb{T}'_\Lambda.$$

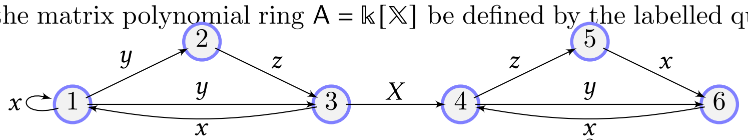
To complete the proof of (b), it only remains to observe that $\boldsymbol{\tau}_{[\![1,n]\!]} \cdot \mathbb{T}_{[\![1,n]\!]} \subset \mathbb{T}_{\mathsf{c}}$. But then the latter is clear because variables mutually commute. Thus every element in $\boldsymbol{\tau}_{[\![1,n]\!]} \cdot \mathbb{T}_{[\![1,n]\!]}$ is still a term along some cycle ω with $\boldsymbol{i}_\omega = [\![1,n]\!]$, while for any such cycle ω the term $\underline{\omega}$ lives in $\mathbb{T}_{\mathsf{c}} = \cap_{i=1}^{n} \mathbb{T}_i$. ◻

Example 2.6. (a) Let the matrix polynomial ring extensions $\mathsf{A} = \Bbbk[\mathbb{X}]$ and $\mathsf{A}' = \Bbbk[\mathbb{X}']$ be given respectively by the following labelled quivers:



- ❍ We have $\boldsymbol{S} = \Bbbk[x,y,z,yt,zt]$ and computing, we get $\mathsf{Z} = \Bbbk \oplus yzt \cdot \Bbbk[z] \oplus yzt^2 \cdot \boldsymbol{S}$.
- ❍ Likewise for A', we have $\boldsymbol{S}' = \Bbbk[x,z,yt,zt,xt^2]$, while a more involved computation yields that $\mathsf{Z}' = \Bbbk \oplus yt \cdot \Bbbk[z] \oplus \boldsymbol{\tau}_{[\![1,2,3]\!]} \cdot \boldsymbol{S}'$ with $\boldsymbol{\tau}_{[\![1,2,3]\!]} = \{yzt^2, y^2t^2, xt^2\}$.

(b) Let the matrix polynomial ring $\mathsf{A} = \Bbbk[\mathbb{X}]$ be defined by the labelled quiver:



where X is any set of variables. Thus $\boldsymbol{S} = \Bbbk[x, xy, xyz, x^2z]$, and computing we obtain that $\mathsf{Z} = \Bbbk \oplus x^3yz \cdot \Bbbk[x^3yz, xy]$.

Usage of free matrix polynomial algebras in deformation theory of modules. Assume here a base field $\Bbbk$. For every family $(V_{i,j})_{1 \le i,j \le n}$ of finite-dimensional $\Bbbk$-vector spaces, the associated 'free matrix ring' of [5, §5, p. 105] is the free matrix polynomial algebra $\Bbbk((V_{i,j})_{i,j}) = \Bbbk\langle\mathbb{X}\rangle$ where each $\mathbb{X}_{i,j} = \{x^{(s)}_{i,j} \ : \ 1 \le s \le d_{i,j}\}$ is a basis of $V_{i,j}$, $i,j \in [\![1,n]\!]$. The corresponding *formal matrix ring* $\Bbbk\langle\langle\mathbb{X}\rangle\rangle$ consists of infinite linear combinations of elementary matrix-terms from $\mathbb{T}$. Given a family $\boldsymbol{V} = (V_1, \dots, V_n)$ of left modules over an algebra A, [11] introduced a noncommutative deformation functor $\mathsf{Def}_{\boldsymbol{V}} : \mathrm{alg}_n \longrightarrow \mathsf{Set}$ from the category of finite-dimensional n-pointed algebras to the category of sets. Under the condition that the cohomological $\Bbbk$-vector spaces $\mathsf{Ext}^1_A(V_j, V_i)$ and $\mathsf{Ext}^2_A(V_j, V_i)$ are finite-dimensional, [5, Theorem 5.2] computes a *pro-representing hull* $\widehat{H}(\boldsymbol{V})$ for $\mathsf{Def}_{\boldsymbol{V}}$ as a quotient of a formal matrix ring $\Bbbk\langle\langle\mathbb{X}\rangle\rangle$, where each $\mathbb{X}_{i,j}$ is a $\Bbbk$-basis for the dual vector space $\left(\mathsf{Ext}^1_A(V_j, V_i)\right)^*$ for all $i, j \in [\![1, n]\!]$. In some favourable cases, $\widehat{H}(\boldsymbol{V})$ is a quotient of a matrix polynomial algebra as illustrated by the next borrowed example.

Example 2.7 ([17, Theorem 19])**.** For a base field $\Bbbk$, the $\Bbbk$-algebra of the 'affine moduli' (pro-representing hull) for the $\mathrm{GL}_3(\Bbbk)$-orbits of $\mathcal{M}_3(\Bbbk)$ was computed as a quotient of the noetherian matrix polynomial algebra

$$\begin{pmatrix} \Bbbk[x_1, x_2, x_3] & \Bbbk\langle y^{(1)}_{1,2}, y^{(2)}_{1,2}, y^{(3)}_{1,2}\rangle & \Bbbk\langle z^{(1)}_{1,3}, z^{(2)}_{1,3}, z^{(3)}_{1,3}\rangle \\ 0 & \Bbbk[y_1, y_2] & \Bbbk\langle z^{(1)}_{2,3}, z^{(2)}_{2,3}\rangle \\ 0 & 0 & \Bbbk[z] \end{pmatrix}.$$

3. Noetherian matrix polynomial extensions

The main objective here is to investigate conditions under which the ring $\mathsf{A} = R\langle \boldsymbol{E}, \mathbb{X}; \boldsymbol{C}\rangle$ is left (resp., right or bilateral) noetherian.

3.1. Noetherianity for semigroup modules. Notice that the elementary idempotent matrices $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ form a complete system of orthogonal idempotents for A: $1_{\mathsf{A}} = \boldsymbol{e}_1 + \cdots + \boldsymbol{e}_n$ and $\boldsymbol{e}_i\boldsymbol{e}_j = 0$ for all $i \neq j$ in $[\![1, n]\!]$. We start with the following lemma, whose proof we shall drop relies on basic well-know facts from abstract algebra.

Lemma 3.1. *Let A be any $\Bbbk$-algebra together with a complete system of orthogonal idempotents $\mathsf{e}_1, \ldots, \mathsf{e}_n$, set $A_i = \mathsf{e}_i A \mathsf{e}_i$ for every $i \in [\![1, m]\!]$. Then a given left A-module M is noetherian precisely when each left A_i-module $\mathsf{e}_i M$ is, for $i \in [\![1, n]\!]$. If moreover each ring A_i for $i \in [\![1, n]\!]$ is left noetherian, then M is noetherian precisely when it is finitely generated.*

Of course for any $\Bbbk$-algebras A and B, the above lemma also applies to right A-modules (since these are just left modules over the opposite algebra A° of A), as well as to bilateral A-B-modules (since the latter are just left modules over the algebra $A \otimes_\Bbbk B^\circ$).

We shall need to apply this lemma to the set $\mathbb{T}$ of elementary matrix-terms, so we make precise the notion of noetherianity for (subsets in) semigroups.

Definition 3.2. Let S, S' be two (multiplicative) semigroups.

(a) A *bilateral* S-S'*-module* (or briefly, a *bilateral semigroup module*) is any non-empty set M together with *multiplications* $\mathsf{S} \times M \longrightarrow M$, $(\lambda, u) \longmapsto \lambda \cdot u$ and $M \times \mathsf{S} \longrightarrow M$, $(u, \lambda') \longmapsto u \cdot \lambda'$ such that for all $u \in M$, $\lambda, \mu \in \mathsf{S}$ and

$\lambda', \mu' \in \mathsf{S}'$,

$$(\lambda\mu) \cdot u = \lambda \cdot (\mu \cdot u), \ \ u \cdot (\lambda'\mu') = (u \cdot \lambda') \cdot \mu', \ \ (\lambda \cdot u) \cdot \lambda' = \lambda \cdot (u \cdot \lambda').$$

We require the *identity axiom* whenever the given semigroups are monoids. This definition specializes to one-sided modules.

(b) Let M be a bilateral S-S'-module, and $T \subset M$. A *left submodule* of T is any non-empty subset $U \subset T$ with $(\mathsf{S} \cdot U) \cap T \subset U$, where $\mathsf{S} \cdot U = \{\lambda \cdot u : \lambda \in \mathsf{S}, u \in U\}$. And we say that U is generated by some $X \subset U$ provided $U = X \cup ((\mathsf{S} \cdot X) \cap T)$; where one should notice that when S is a monoid it is already granted that $X \subset \mathsf{S} \cdot X$ since we require the identity axiom in this case.

Likewise, one gets right submodules and bilateral submodules in T.

(c) A subset T of a bilateral S-S'-module M is called *left* (or *right, bilateral*) *noetherian* provided every left (or resp., right, bilateral) submodule in T is finitely generated.

The analogue of the next proposition in abstract algebra, stated here for semigroup modules, is an easy and standard fact.

Proposition 3.3. *Let* S *be a semigroup, and* $M_1, \dots, M_r$ *be left (or right, bilateral) semigroup* S*-modules for some non-zero* $r \in \mathbb{N}$*. Then, the disjoint union* $\biguplus_{i=1}^r M_i$ *is noetherian precisely when each of the* M_i*'s is. In particular if the semigroup* S *is left (or right, bilateral) noetherian, then every left (or resp., right, bilateral) semigroup* S*-module is noetherian precisely when it is finitely generated.*

Proof. It is enough to consider only the case of left semigroup modules; the reader would also want to notice that coproducts in the category of S-modules are simply given by disjoint unions. For the first claim of the proposition, it readily follows by Definition 3.2(c) that every submodule of a noetherian left S-module is noetherian; so if $\biguplus_{i=1}^r M_i$ is noetherian, then the same holds for each of the M_i's. Conversely, if the semigroup S-modules $M_1, \dots, M_r$ are noetherian, then the disjoint union $\biguplus_{i=1}^r M_i$ is noetherian as well because every submodule in $\biguplus_{i=1}^r M_i$ arises as a disjoint union $T = \biguplus_{i \in \Lambda} T_i$ for some $\Lambda \subset [\![1, r]\!]$ and submodules $T_i \subset M_i$, $i \in \Lambda$.

We then continue with the proof of the second claim of the proposition. By definition, every noetherian S-module is already finitely generated. Conversely, suppose that the semigroup S is left noetherian and let a left S-module M be generated by a finite subset $X = \{x_1, \dots, x_r\}$. Thus $M = X \cup \mathsf{S}X$, and we have to prove that M is noetherian. By the above paragraph, the coproduct of r copies of S

(as semigroup left module over itself) is the noetherian left S-module given by the disjoint union $\cup_{i=1}^{r} \mathsf{S} \times \{i\}$. We get a surjective morphism $f : \cup_{i=1}^{r} \mathsf{S} \times \{i\} \longrightarrow \mathsf{S}X$ with $f(\lambda, i) = \lambda x_i$ for all $\lambda \in \mathsf{S}$ and $i \in [\![1, r]\!]$, showing that the S-module $\mathsf{S}X$ is noetherian as an homomorphic image of a noetherian module, whence the module $M = X \cup \mathsf{S}X$ is noetherian as well since also X is finite. $\square$

In view of Proposition 3.3, Lemma 3.1 applies to semigroups, yielding the next remark.

Remark 3.4. In the semigroup $\mathbb{T} \cup \{0\}$ of elementary matrix-terms, $\mathbb{T}$ is left (right or bilateral) noetherian precisely when each monoid $\mathbb{T}_i$ is left (right or bilateral) noetherian while each bilateral $\mathbb{T}_i$-$\mathbb{T}_j$-module $\mathbb{T}_{i,j}$ is finitely generated as left (or resp., right, bilateral) module, $1 \leq i \neq j \leq n$.

A (left, right or bilateral) noetherian semigroup needs not be finitely generated as semigroup; the following crucial lemma sheds more light.

Lemma 3.5. *Let S be a semigroup with a length function $\ell : \mathsf{S} \longrightarrow \mathbb{N}$ such that the set $\mathsf{S}_0 = \{\varepsilon \in \mathsf{S} : \ell(\varepsilon) = 0\}$ is finite and for all $\alpha \in \mathsf{S} \backslash \mathsf{S}_0$ and $\beta \in \mathsf{S}$, $\ell(\beta) < \min(\ell(\beta\alpha), \ell(\alpha\beta))$. If S is bilateral noetherian, then S is finitely generated as semigroup. Consequently, every bilateral noetherian submonoid in the monoid $\langle \boldsymbol{X}; \boldsymbol{C} \rangle$ is a finitely generated monoid.*

Proof. The last statement in the lemma is an immediate application of the main part by considering the usual additive length function on words. Turning to the main part, since by assumption the subset S_0 of length-0 elements is finite, we may assume that S_0 does not coincide with S. The assumption on $\ell : \mathsf{S} \longrightarrow \mathbb{N}$ ensures that $\mathsf{S} \backslash \mathsf{S}_0$ is a bilateral ideal of S. Thus when S is bilateral noetherian, it comes that the ideal $\mathsf{S} \backslash \mathsf{S}_0$ is generated by finitely many elements $\alpha_1, \ldots, \alpha_n \in \mathsf{S} \backslash \mathsf{S}_0$. We will now show that S coincides with $\mathsf{S}_0[\alpha_1, \ldots, \alpha_n]$, the sub-semigroup generated over S_0 by $\alpha_1, \ldots, \alpha_n$. Given $\alpha \in \mathsf{S}$, we will proceed by induction on $\ell(\alpha)$. The base of induction being obviously given, suppose that $\ell(\alpha) \geq 1$ and the result proved for all $\beta \in \mathsf{S}$ with $\ell(\beta) < \ell(\alpha)$. We may further assume that α is not already one of the elements $\alpha_1, \ldots, \alpha_n$; then $\alpha \in \mathsf{S} \backslash (\mathsf{S}_0 \cup \{\alpha_1, \ldots, \alpha_n\})$ and for some $\beta, \beta' \in \mathsf{S}$ and $s \in [\![1, n]\!]$, we have $\alpha \in \{\beta\alpha_s,\ \alpha_s\beta,\ \beta\alpha_s\beta'\}$. But the hypotheses of the lemma ensures that $\ell(\beta) < \ell(\alpha)$ (in case $\alpha \in \{\beta\alpha_s,\ \alpha_s\beta\}$) or $\max(\ell(\beta), \ell(\beta')) < \ell(\alpha)$ (in case $\alpha = \beta\alpha_s\beta'$). Whence, the induction hypothesis applies yielding that β or both β and β' belong to $\mathsf{S}_0[\alpha_1, \ldots, \alpha_n]$, so that, α belongs to $\mathsf{S}_0[\alpha_1, \ldots, \alpha_n]$ as well. Hence $\mathsf{S} = \mathsf{S}_0[\alpha_1, \ldots, \alpha_n]$, completing the proof of the lemma. $\square$

We include the following lemma which specializes to one-sided modules, whose proof readily follows from the definition of semigroup ring extensions and is omitted.

Lemma 3.6. *Let $A = R[\mathcal{B}]$ be any semigroup ring extension of R, together with sub-extensions $R'[\mathcal{B}']$ and $R''[\mathcal{B}'']$ for some $\mathcal{B}', \mathcal{B}'' \subset \mathcal{B}$ and some subalgebras $R', R'' \subset R$. If A is a bilateral noetherian $R'[\mathcal{B}']$-$R''[\mathcal{B}'']$-module, then $\mathcal{B}$ is bilateral noetherian (for the bilateral $\mathcal{B}'$-$\mathcal{B}''$-module structure on $\mathcal{B} \cup \{0\}$).*

We must now draw attention to the point that non-familiarity with pitfalls in attempts to generalize the Hilbert Basis Theorem ([8, §7.10], [2, Theorem 7.5]) to noncommutative monoid ring extensions may lead to a wrong impression that the converse of Lemma 3.6 above would be true and easily provable. The rest of this section investigates the converse of Lemma 3.6 for matrix polynomial algebra extensions.

3.2. Lifting the Hilbert Basis Theorem to matrix polynomial algebras. For the rest of this section, we shall restrict the shape of commutativity relations imposed on some variables to central relations:

$$\left| \begin{array}{l} \boldsymbol{C} = \boldsymbol{X}_c \times \boldsymbol{X} \text{ for some } \boldsymbol{X}_c \subset \boldsymbol{X}, \text{ meaning that } \boldsymbol{X} \text{ splits as } \boldsymbol{X} = \boldsymbol{X}_c \cup \boldsymbol{X}_{nc} \text{ where} \\ \boldsymbol{X}_c \text{ consists of central variables and } \boldsymbol{X}_{nc} \text{ of free non-commuting variables.} \end{array} \right| \tag{3.1}$$

A deep analysis of the structure of the semigroup $\mathbb{T} \cup \{0\}$ of elementary matrix-terms is necessary to reach a precise statement about the noetherianity of $\mathbb{T}$ and of $\mathsf{A} = R\langle \boldsymbol{E}, \mathbb{X}; \boldsymbol{C}\rangle$. We start by characterizing one-sided noetherianity for submonoids in the ambient monoid $\langle \boldsymbol{X}; \boldsymbol{C}\rangle$.

Lemma 3.7. *Let T be a submonoid in $\langle \boldsymbol{X}; \boldsymbol{C}\rangle$. Then T is left (or, right) noetherian if and only if T is commutative and finitely generated.*

Proof. Every finitely generated commutative monoid is clearly noetherian, as a quotient of a free commutative monoid in finitely many variables. Moreover, every noetherian submonoid in $\langle \boldsymbol{X}; \boldsymbol{C}\rangle$ must be finitely generated by virtue of Lemma 3.5. Assuming that the submonoid $T \subset \langle \boldsymbol{X}; \boldsymbol{C}\rangle$ is left or right noetherian, it remains to prove that T is also commutative. We consider the case that T is left noetherian; the case that T is right noetherian shall be seen to be similar. Recall by our assumption (3.1) that the center of $\langle \boldsymbol{X}; \boldsymbol{C}\rangle$ is the free commutative submonoid over the subset $\boldsymbol{X}_c \subset \boldsymbol{X}$, while $\boldsymbol{X}_{nc} = \boldsymbol{X} \backslash \boldsymbol{X}_c$ consists of free non-commuting variables. So $\langle \boldsymbol{X}_{nc}\rangle$ stands for the free monoid in non-commuting variables from $\boldsymbol{X}_{nc}$, and it holds that:

every $t \in \langle \boldsymbol{X}; \boldsymbol{C} \rangle$ can be expressed in a unique way as $t = \delta\tau$ with $\mathsf{var}(\delta) \subset \boldsymbol{X}_c$ and $\tau \in \langle \boldsymbol{X}_{nc} \rangle$.

Now let $\omega = \delta\tau$ and $\omega' = \delta'\tau'$ in T, with $\mathsf{var}(\delta), \mathsf{var}(\delta') \subset \boldsymbol{X}_c$ and $\tau, \tau' \in \langle \boldsymbol{X}_{nc} \rangle$, and with $\ell(\tau) \leq \ell(\tau')$. If $\ell(\tau) = 0$ (so that, $\tau = 1$), then there is nothing to show. So we may assume that $\ell(\tau') \geq \ell(\tau) \geq 1$. Then we will show that the non-trivial words τ and τ' (in non-commuting variables) do commute. By assumption, the left ideal $T\{\omega'\omega^k : 1 \leq k \in \mathbb{N}\} = \{t\omega'\omega^k : t \in T, 1 \leq k \in \mathbb{N}\}$ must be generated by finitely many elements $\omega'\omega^1, \ldots, \omega'\omega^m$ for some natural number m. In particular for $k \geq \ell(\tau') + m$, there exist some $s \in [\![1, m]\!]$ and a term $t = u\gamma \in T$ with $\mathsf{var}(u) \subset \boldsymbol{X}_c$ and $\gamma \in \langle \boldsymbol{X}_{nc} \rangle$ such that

$$\omega'\omega^k = t\omega'\omega^s, \text{ that is, } \delta'\delta^k\tau'\tau^k = u\delta'\delta^s\gamma\tau'\tau^s.$$

Since $k - s \geq \ell(\tau') + m - s \geq \ell(\tau') \geq 1$, it holds that: $u = \delta^{k-s}$ and $\tau'\tau^{k-s} = \gamma\tau'$. By Euclidean division we may write $\ell(\tau') = q\ell(\tau) + r$ for some $q, r \in \mathbb{N}$ with $0 \leq r < \ell(\tau)$. But, $\ell(\tau) \leq \ell(\tau') \leq k - s \leq \ell(\tau^{k-s})$, so that, $1 \leq q \leq k - s$ and the relation $\tau'\tau^{k-s} = \gamma\tau'$ shows that τ' is a suffix of τ^{k-s} and:

$$\tau' = \lambda'\tau^q \text{ and } \tau = \lambda\lambda' \text{ for some } \lambda, \lambda' \in \langle \boldsymbol{X}_{nc} \rangle \text{ with } \lambda' \neq \tau.$$

Next, applying once more the left noetherianity of T to the left ideal generated by the set $\{\omega'^2\omega^k : 1 \leq k \in \mathbb{N}\}$, as before and for a sufficiently large $l \in \mathbb{N}$, we must have:

$$\tau'^2\tau^l\tau^{q+2} = \gamma'\tau'^2 \text{ for some } \gamma' \in \langle \boldsymbol{X}_{nc} \rangle, \text{ that is,}$$
$$\tau'^2\tau^l \cdot \lambda \cdot (\lambda'\lambda\lambda') \cdot \tau^q = \gamma' \cdot (\lambda'\tau^q)\lambda'\tau^q = \gamma' \cdot (\lambda'\tau^{q-1}) \cdot (\lambda\lambda'\lambda')\tau^q.$$

Given that $(\lambda'\lambda\lambda') \cdot \tau^q$ and $(\lambda\lambda'\lambda')\tau^q$ are both suffixes with common length for the word $\tau'^2\tau^l\tau^{q+2} = \gamma'\tau'^2$, they must be equal, forcing that $\lambda'\lambda = \lambda\lambda'$. Therefore, the terms $\tau = \lambda\lambda'$ and $\tau' = \lambda'\tau^q$ commute. This completes the proof of the lemma. □

We then continue with a complete description of commutative submonoids in $\langle \boldsymbol{X}; \boldsymbol{C} \rangle$.

Proposition 3.8. *Let $\tau, \tau' \in \langle \boldsymbol{X}_{nc} \rangle$.*

(a) *If τ and τ' commute in $\langle \boldsymbol{X}_{nc} \rangle$ then they arise as powers of a common subterm.*

(b) *If $\tau^k = \tau'^{k'}$ for some non-zero $k, k' \in \mathbb{N}$, then both τ and τ' are powers of a common subterm.*

(c) *Every commutative submonoid T in $\langle \boldsymbol{X}; \boldsymbol{C} \rangle$ is contained in the submonoid generated by $\boldsymbol{X}_c \cup \{\lambda\}$ for some $\lambda \in \langle \boldsymbol{X}_{nc} \rangle$.*

Proof. Giving commuting terms $\tau, \tau' \in \langle \boldsymbol{X}_{nc} \rangle$, with $\ell(\tau) \leq \ell(\tau')$, we will proceed by induction on $\ell(\tau\tau')$ to show that τ and τ' are powers of a common subword. If $\ell(\tau) = 0$, then obviously $\tau = 1 = \tau'^0$. So, suppose that $1 \leq \ell(\tau) \leq \ell(\tau')$. Then the relation $\tau\tau' = \tau'\tau$ shows clearly that τ is a prefix of τ' and we may write $\tau' = \tau\tau''$ for some $\tau'' \in \langle \boldsymbol{X}_{nc} \rangle$. If $\tau'' = 1$, then τ and τ' coincide and the result is obvious. Next, supposing that $\tau'' \neq 1$, the equation $\tau\tau\tau'' = \tau\tau''\tau$ yields that $\tau\tau'' = \tau''\tau$ with $\ell(\tau\tau'') < \ell(\tau\tau')$. Hence the induction hypothesis yields some $\lambda \in \langle \boldsymbol{X}_{nc} \rangle$ with $\tau = \lambda^k$ and $\tau'' = \lambda^{k'}$, so that $\tau' = \lambda^{k+k'}$ as well. This proves part (a) of the proposition.

Accepting that part (b) holds, we will prove (c). Thus let T be a commutative submonoid in $\langle \boldsymbol{X}; \boldsymbol{C} \rangle$. If T is contained in the center of $\langle \boldsymbol{X}; \boldsymbol{C} \rangle$, then we are done. Otherwise, one can find a term $\lambda \in \langle \boldsymbol{X}_{nc} \rangle$ with the following properties:

❍ λ is non-trivial and is not a proper power of another word,
❍ it holds for some term δ with $\mathsf{var}(\delta) \subset \boldsymbol{X}_c$ and non-zero $l \in N$ that $\delta\lambda^l \in T$.

Now, every $\omega \in T$ can be expressed as: $\omega = u\tau$ with $\mathsf{var}(u) \subset \boldsymbol{X}_c$ and $\tau \in \langle \boldsymbol{X}_{nc} \rangle$. Since $\omega \cdot \delta\lambda^l = \delta\lambda^l \cdot \omega$, it follows that λ^l commutes with τ, so that by part (a) of the proposition, $\lambda^l = \gamma^k$ and $\tau = \gamma^r$ for some $\gamma \in \langle \boldsymbol{X}_{nc} \rangle$ and some non-zero $k, r \in \mathbb{N}$. But then, since λ is non-trivial and is not a proper power of another word, part (b) applied to the equality $\lambda^l = \gamma^k$ (with $l \geq 1$) forces γ to be a power of λ, so that τ is a power of λ as well, proving that T is, as desired, contained in the submonoid of $\langle \boldsymbol{X}; \boldsymbol{C} \rangle$ generated by $\boldsymbol{X}_c \cup \{\lambda\}$.

We shall now prove part (b). Assume that $\tau, \tau' \in \langle \boldsymbol{X}_{nc} \rangle$ are terms with $\ell(\tau) \leq \ell(\tau')$, such that $\tau^k = \tau'^{k'}$ for some non-zero $k, k' \in \mathbb{N}$. Then $k\ell(\tau) = k'\ell(\tau')$ and $k \geq k'$. We may assume that k and k' are co-prime. Indeed, if d is the greatest common divisor of k and k' and we set $l = k/d$ and $l' = k'/d$, then $l\ell(\tau) = l'\ell(\tau')$ and the equation

$$\tau^l \cdot (\tau^l)^{d-1} = \tau'^{l'} \cdot (\tau'^{l'})^{d-1}$$

shows that $\tau^l = \tau'^{l'}$. So, we suppose that k and k' are co-prime. The relation $k\ell(\tau) = k'\ell(\tau')$ shows that k' divides $\ell(\tau)$, so letting $p = \ell(\tau)/k'$, we may write $\tau = \tau_1 \cdots \tau_{k'}$ as product of k' subterms with common length p. Next, writing the Euclidean division of k by k' as $k = qk' + r$, we have that $q \geq 1$ (since $k \geq k'$), $1 \leq r < k'$ (since k and k' are co-prime), $\ell(\tau') = k\ell(\tau)/k' = q\ell(\tau) + rp$, and the relation $\tau^k = \tau'^{k'}$ rewrites as:

$$\tau^q \cdot (\tau_1 \cdots \tau_r) \cdot (\tau_{r+1} \cdots \tau_{k'}) \cdot \tau^{k-q-1} = \tau'^{k'} = \tau^{k-q-1} \cdot (\tau_1 \cdots \tau_{k'-r}) \cdot (\tau_{k'-r+1} \cdots \tau_{k'}) \cdot \tau^q.$$

The latter implies that $\tau^q \cdot (\tau_1 \cdots \tau_r) = \tau' = (\tau_{k'-r+1} \cdots \tau_{k'})\tau^q$, and using the fact that $\tau = \tau_1 \cdots \tau_{k'}$ is a product of k' subterms with common length, it follows that

$$(\tau_1, \ldots\ldots, \tau_{k'-r}, \tau_{k'-r+1}, \ldots, \tau_{k'}) = (\tau_{k'-r+1}, \ldots, \tau_{k'}, \tau_1, \ldots, \tau_{k'-r}),$$

showing that the k'-length list $(\tau_{k'-r+1}, \ldots, \tau_{k'}, \tau_1, \ldots, \tau_{k'-r})$ is r-permutable. Here,

a list $\boldsymbol{\lambda} = (\lambda_0, \ldots, \lambda_{m-1})$ (of objects) is called *r-permutable* when

$$\boldsymbol{\lambda} = (\lambda_0, \ldots, \lambda_{r-1}, \ldots, \lambda_{m-1}) = (\lambda_r, \ldots\ldots, \lambda_{m-1}, \lambda_0, \ldots, \lambda_{r-1}).$$

But since $\mathsf{gcd}(k', r) = 1$, we claim that $\tau_1 = \tau_2 = \cdots = \tau_{k'}$, showing that $\tau = \tau_1^{k'}$ and $\tau' = \tau_1^k$, as desired. For our last claim, let $1 \le r < m$ be two integers, and $\boldsymbol{\lambda} = (\lambda_0, \ldots, \lambda_{m-1})$ an r-permutable list of objects with $\mathsf{gcd}(m, r) = 1$. The proof that $\lambda_0 = \cdots = \lambda_{m-1}$ proceeds by induction on m where for the induction step (with $m \ge 2$ and $r \ge 2$), one uses the Euclidean division of m by r. □

We shall need a few additional notations for the sequel. For each $i \in [\![1, n]\!]$,

$\mathbb{T}_i^{\mathrm{s}}$ stands for the submonoid of $\mathbb{T}_i$ generated by all $\underline{\sigma}$ for simple cycles $\sigma \in \mathsf{Q}_{\mathbb{X}}(i, i)$.

We recall that a monoid S is *left cancellative* (or, *right cancellative*) provided for all $u, v \in T$, $\lambda \in \mathsf{S}$ and $\lambda' \in \mathsf{S}'$,

if $\lambda \cdot u = \lambda \cdot v \in T$ then $u = v$, or respectively, if $u \cdot \lambda' = v \cdot \lambda' \in T$ then $u = v$.

T is *cancellative* when it is left cancellative and right cancellative.

Given a finite subset $\boldsymbol{\tau} = \{\tau_1, \ldots, \tau_m\}$ in some (multiplicative) monoid, for each $\Lambda \subset \mathbb{N}^m$, we let $\boldsymbol{\tau}^\Lambda = \{\boldsymbol{\tau}^\alpha = \tau_1^{\alpha_1} \cdots \tau_m^{\alpha_m} \ : \ \alpha = (\alpha_1, \ldots, \alpha_m) \in \Lambda\}$. We then consider the following condition for pairs $\mathsf{T}' \subset \mathsf{T}$ of submonoids of a commutative cancellative monoid S, together with a subset $\boldsymbol{\tau} = \{\tau_1, \ldots, \tau_m\} \subset \mathsf{S} \backslash \mathsf{T}'$.

$$1 \notin (\mathsf{S}\backslash\{1\})\mathsf{S},\ \mathsf{T}\backslash\mathsf{T}' \subset (\mathsf{T}'\backslash\{1\})[\boldsymbol{\tau}] \text{ and } \lambda_k \tau_k^{\mathbb{N}} \cap \mathsf{T} \text{ is infinite for some } \lambda_k \in \mathsf{T}, 1 \le k \le m. \tag{3.2}$$

In the commuting variables context, recall from (2.3) the notation $\mathbb{T}_\Lambda = \mathbb{T}_{i_1} \cdots \mathbb{T}_{i_p}$ for each $\Lambda = \{i_1 \cdots i_p\} \subset [\![1, n]\!]$; we shall also let $\mathbb{T}_\Lambda^{\mathrm{s}} = \mathbb{T}_{i_1}^{\mathrm{s}} \cdots \mathbb{T}_{i_p}^{\mathrm{s}}$.

Lemma 3.9. *Assume the commuting variables context, and that $\mathsf{Q}_{\mathbb{X}}$ contains only finitely many simple cycles. Then for the evidently cancellative monoid $\mathsf{S} = \mathbb{T}_{[\![1,n]\!]}$ and for every $\Lambda \subset [\![1, n]\!]$, the triple $(\mathbb{T}_\Lambda^{\mathrm{s}}, \mathbb{T}_\Lambda, \mathsf{S})$ satisfies condition (3.2) for the finite subset $\boldsymbol{\tau}$ of all terms $\underline{\sigma}$ such that σ is a simple cyclic subpath of some cycle in $\mathsf{Q}_{\mathbb{X}}(i, i)$ with $i \in \Lambda$ and $\underline{\sigma} \notin \mathbb{T}_\Lambda^{\mathrm{s}}$.*

Proof. Write $\mathcal{T}$ for the free commutative monoid over $\boldsymbol{X}$. It is evident that $1 \notin (\mathsf{S}\backslash\{1\}) \cdot \mathsf{S}$ since $\mathsf{S} \subset \mathcal{T}$ and this holds for $\mathcal{T}$. Write $\Lambda = \{i_1 \cdots i_p\}$ for $p \in [\![1, n]\!]$. By definition, $\boldsymbol{\tau}$ consists of finitely many terms $\tau_k = \underline{\sigma_k}$ with $1 \le k \le m$ for some $m \in \mathbb{N}$, where each σ_k is a simple cyclic subpath of some cycle $v_k \in \mathsf{Q}_{\mathbb{X}}(j_k, j_k)$ with $j_k \in \{i_1, \dots, i_p\}$. Thus for $\lambda_k = \underline{v_k}$ for each $k \in [\![1, m]\!]$, we see that $\lambda_k \tau_k^{\mathbb{N}} \subset \mathbb{T}_\Lambda$. Every element $u \in \mathbb{T}_\Lambda$ can be expressed as $u = u_1 \cdots u_p$ with $u_k = \underline{w_k}$ for some cycle or empty path $\omega_k \in \mathsf{Q}_{\mathbb{X}}(i_k, i_k)$, $1 \le k \le p$. Assuming that $u \notin \mathbb{T}^{\mathrm{s}}_\Lambda$, there must exist some $k \in [\![1, p]\!]$ such that w_k properly contains a cyclic subpath σ with $\underline{\sigma} \notin \mathbb{T}^{\mathrm{s}}_\Lambda$. So $\omega_k = v\sigma v'$ for some cycle $\omega' = vv' \in \mathsf{Q}_{\mathbb{X}}(i_k, i_k)$, and setting $u' = \prod_{s \neq k} u_s$, it holds that:

$$\underline{\sigma} \in \boldsymbol{\tau},\ u = u' \cdot \underline{\omega'}\underline{\sigma} \text{ with } \omega' \in \mathsf{Q}_{\mathbb{X}}(i_k, i_k),\ \underline{\omega'} \neq 1 \text{ and } u'\underline{\omega'}\underline{\sigma}^{\mathbb{N}} = \{u'\underline{v\sigma^r v'} \ :\ r \in \mathbb{N}\} \subset \mathbb{T}_\Lambda.$$

It follows by the way of induction on the length of paths that for every $u \in \mathbb{T}_\Lambda \backslash \mathbb{T}^{\mathrm{s}}_\Lambda$, there is a non-zero multi-index $\delta \in \{0,1\}^m$ together with a term $\lambda \in \mathbb{T}^{\mathrm{s}}_\Lambda$ such that

$$\lambda \neq 1 \text{ and } u \in \lambda \boldsymbol{\tau}^{\mathbb{N}^m_\delta} \subset \mathsf{T} \text{ with } \mathbb{N}^m_\delta = \{\alpha \in \mathbb{N}^m \ :\ \alpha_k \neq 0 \text{ if and only if } \delta_k \neq 0,\ 1 \le k \le m\}.$$

This proves that $(\mathbb{T}^{\mathrm{s}}_\Lambda, \mathbb{T}_\Lambda, \mathsf{S})$ satisfies a more stronger version of condition (3.2). □

Beside the main objective of this section, we obtain a precise characterisation of commutative noetherian monoids arising as products of diagonal components of $\mathbb{T}$.

Lemma 3.10. *Let $\mathsf{T}' \subset \mathsf{T}$ be submonoids of a commutative cancellative monoid S, and $\boldsymbol{\tau} = \{\tau_1, \dots, \tau_m\} \subset \mathsf{S}\backslash\mathsf{T}'$. Assume that the monoid T is noetherian and let $\lambda, \tau \in \mathsf{S}$ such that $\lambda\tau^{\mathbb{N}} \cap \mathsf{T}$ is infinite. Then a positive power of τ lies in T. If additionally $(\mathsf{T}', \mathsf{T}, \mathsf{S})$ satisfies* (3.2)*, then a positive power of each of the terms $\tau, \tau_1, \dots, \tau_m$ lies in T'.*

Proof. We assume that T is noetherian and let $\lambda, \tau \in \mathsf{S}$ with $\lambda\tau^{\mathbb{N}} \cap \mathsf{T}$ infinite. Then the T-submodule in T generated by the set $\lambda\tau^{\mathbb{N}} \cap \mathsf{T}$ must be generated by finitely many terms $\lambda\tau^{k_1}, \dots, \lambda\tau^{k_p}$ for some $p \in \mathbb{N}$ and integers $k_1 < k_2 < \cdots < k_p$. By assumption there is some integer $k > k_p$ with $\lambda\tau^k \in \mathsf{T}$, and it holds that $\lambda\tau^k = t \cdot \lambda\tau^{k_s}$ for some $t \in \mathsf{T}$ and $s \in [\![1, p]\!]$. Since by hypothesis the monoid S is cancellative, we get that $\tau^{k-k_s} = t$ is a positive power of τ lying in T.

Next, suppose in addition that $(\mathsf{T}', \mathsf{T}, \mathsf{S})$ satisfies (3.2). The latter shows that $\mathsf{T} \subset \mathsf{T}'[\tau_1, \dots, \tau_m]$ while $\lambda_k \tau_k^{\mathbb{N}} \cap \mathsf{T}$ is infinite for some $1 \neq \lambda_k \in \mathsf{T}$, $1 \le k \le m$, and the previous paragraph already yields a positive integer $a_k \in \mathbb{N}$ such that $\tau_k^{a_k} \in \mathsf{T}$. Let us prove that a positive power of each of the terms $\tau_1, \dots, \tau_m$ already lies in T'. But then, assuming for some $m' \in [\![1, m]\!]$ that for every $k \in [\![1, m]\!]$ a positive power $\tau_k^{p_k}$ lives in $\mathsf{T}'[\tau_1, \dots, \tau_{m'}]$, it is enough to show that a positive power $\tau_k^{p'_k}$ of each τ_k lies in

$\mathsf{T}'[\tau_1,\dots,\tau_{m'-1}]$ as well. We already have that $\tau_{m'}^a \in \mathsf{T}$ for some non-zero $a = a_{m'} \in \mathbb{N}$. If $\tau_{m'}^a \in \mathsf{T}'$, then take $p'_{m'} = a$. Otherwise, (3.2) yields that $\tau_{m'}^a = t\tau_{k_1}^{\alpha_1}\cdots\tau_{k_r}^{\alpha_r}$ for some term $1 \neq t \in \mathsf{T}'$, some indices $1 \le k_1 < k_2 < \ldots < k_r \le m$ and positive integers $\alpha_1,\dots,\alpha_r$. Then if $k_r \le m'-1$, then $\tau_{m'}^a$ is already in $\mathsf{T}'[\tau_1,\dots,\tau_{m'-1}]$ and once again one lets $p'_{m'} = a$. If not, let $s \in [\![1,r]\!]$ be the least integer with $k_s \ge m'$. Since by assumption $\tau_k^{p_k}$ belongs to $\mathsf{T}'[\tau_1,\dots,\tau_{m'}]$ for all $k \in [\![1,m]\!]$, letting $a' = p_{k_s}\cdots p_{k_r}$ shows that $\tau_{m'}^{aa'} = \lambda t^{a'}\tau_1^{n_1}\cdots\tau_{m'}^{n_{m'}}$ for some $\lambda \in \mathsf{T}'$ and some $n_1,\dots,n_{m'} \in \mathbb{N}$. But then, $\lambda t^{a'} \neq 1$ because $t \neq 1$ and $1 \notin \mathsf{S}\cdot(\mathsf{S}\backslash\{1\})$ according to (3.2). Since moreover the monoid S is cancellative, one must have $aa' > n_{m'}$, so that putting $p'_{m'} = aa' - n_{m'}$ yields a positive power $\tau_{m'}^{p'_{m'}}$ of $\tau_{m'}$ belonging to $\mathsf{T}'[\tau_1,\dots,\tau_{m'-1}]$. Thus in all cases, we get a positive power $\tau_{m'}^{p'_{m'}}$ lying in $\mathsf{T}'[\tau_1,\dots,\tau_{m'-1}]$. Next for any other $k \in [\![1,m]\!]$ with $k \neq m'$, by assumption $\tau_k^{p_k} \in \mathsf{T}'[\tau_1,\dots,\tau_{m'}]$, so that letting $p'_k = p_k p'_{m'}$ yields that $\tau_k^{p'_k} = (\tau_k^{p_k})^{p'_{m'}}$ is a positive power of τ_k that belongs to $\mathsf{T}'[\tau_1,\dots,\tau_{m'-1}]$. This completes the proof that a positive power of each of the terms $\tau_1,\dots,\tau_m$ lies in T'.

In particular now letting $0 \neq p_k \in \mathbb{N}$ with $\tau_k^{p_k} \in \mathsf{T}'$, and (in view of the first paragraph) letting $0 \neq p \in \mathbb{N}$ with $\tau^p \in \mathsf{T} \subset \mathsf{T}'[\tau_1,\dots,\tau_m]$, it holds that $\tau^{pp_1\cdots p_m} \in \mathsf{T}'$ as well. This completes the proof the lemma. □

We include a converse of Lemma 3.10 in the next proposition of a more general and independent interest where, when specialized to monoid ring extensions $\boldsymbol{R} = \mathbb{k}[\mathsf{T}]$ for a submonoid T of a commutative monoid S, one should observe that a term $\tau \in \mathsf{S}$ is integral over $\boldsymbol{R}$ if and only if a positive power of τ lives in T.

Proposition 3.11. *For a commutative $\mathbb{k}$-algebra $\boldsymbol{S}$ and a finite subset $\boldsymbol{v} \subset \boldsymbol{S}$, let $\boldsymbol{R}' \subset \boldsymbol{R} \subset \boldsymbol{R}'[\boldsymbol{v}]$ be $\mathbb{k}$-subalgebras in $\boldsymbol{S}$ such that $\boldsymbol{v}$ is integral over $\boldsymbol{R}$. Then the algebra $\boldsymbol{R}$ is noetherian or respectively finitely generated if $\boldsymbol{R}'$ is.*

Proof. Let $\boldsymbol{v} = \{v_1,\dots,v_m\} \subset \boldsymbol{S}$; by hypotheses there are monic polynomials $f_i = x^{n_i} + \sum_{j=0}^{n_i-1} f_{i,j}x^j \in \boldsymbol{R}[x]$ with $f_i(v_i) = 0$. Put $\boldsymbol{F} = \{f_{i,j} \ : \ 1 \le i \le m \text{ and } 1 \le j \le n_i - 1\}$ and consider the $\mathbb{k}$-subalgebra $\boldsymbol{S}' = \boldsymbol{R}'[\boldsymbol{F}] \subset \boldsymbol{R}$. Since $\boldsymbol{v}$ is integral over $\boldsymbol{S}'$, it follows (by [7, Theorem VIII.5.2]) that the ring $\boldsymbol{S}'[\boldsymbol{v}]$ is a finitely generated $\boldsymbol{S}'$-module.

Now assuming that the ring $\boldsymbol{R}'$ is noetherian, it follows (by the Hilbert Basis Theorem) that the commutative ring extension $\boldsymbol{S}' = \boldsymbol{R}'[\boldsymbol{F}]$ is noetherian as well (because $\boldsymbol{F}$ is finite), and the finitely generated $\boldsymbol{S}'$-module $\boldsymbol{S}'[\boldsymbol{v}]$ is noetherian. But then, $\boldsymbol{R}$ is a $\boldsymbol{S}'$-submodule of the noetherian $\boldsymbol{S}'$-module $\boldsymbol{S}'[\boldsymbol{v}]$, so $\boldsymbol{R}$ is a noetherian (and hence, a finitely generated) $\boldsymbol{S}'$-submodule; in particular the ring $\boldsymbol{R}$ is noetherian. Likewise if $\boldsymbol{R}'$ is finitely generated over $\mathbb{k}$, then it is already noetherian and the previous discussion shows that $\boldsymbol{R}$ is generated as an $\boldsymbol{S}'$-module by some finite set U, so that, $\boldsymbol{R} = \boldsymbol{S}'[U] = \boldsymbol{R}'[\boldsymbol{F} \cup U]$ is a finitely generated $\Bbbk$-algebra as a finite type extension of a finitely generated $\Bbbk$-algebra. □

We can now state the main theorem for this section, which precisely lifts the Hilbert Basis Theorem to matrix polynomial algebras under the assumption (3.1) that the set of variables splits between central variables and free non-commuting variables. When a given path ω in $\mathsf{Q}_{\mathbb{X}}(i,j)$ contains a simple cycle σ, one may define the multiplicity $\mathsf{m}_\sigma(\omega)$ of σ in ω as the largest $m \in \mathbb{N}$ such that

$$\omega = \omega_1\sigma\omega_2\sigma\cdots\omega_m\sigma\omega_{m+1} \text{ for some paths } \omega_1,\dots,\omega_{m+1}.$$

Below, for prescribed integers $0 \neq \mathsf{m}_\sigma \in \mathbb{N}$ for all simple cycles σ in $\mathsf{Q}_{\mathbb{X}}$, we set:

$$\mathrm{Q}_{i,j} = \{\omega \in \mathsf{Q}_{\mathbb{X}}(i,j) \ : \ \mathsf{m}_\sigma(\omega) < \mathsf{m}_\sigma \text{ for any simple cyclic subpath } \sigma \text{ of } \omega\},\ 1 \le i,j \le n.$$

Theorem 3.12. *The ring $R\langle \boldsymbol{E}, \mathbb{X}; \boldsymbol{C}\rangle$ is left noetherian precisely when R and $\mathbb{T}$ are; and $\mathbb{T}$ is left noetherian if and only if $\mathbb{X}$ is finite and for all $i,j \in [\![1,n]\!]$, each $\mathbb{T}_i$ is commutative while for any simple cycle $\sigma \in \mathsf{Q}_{\mathbb{X}}(j,j)$ with $\mathsf{Q}_{\mathbb{X}}(i,j) \neq \varnothing$, there is a prescribed $0 \neq \mathsf{m}_\sigma \in \mathbb{N}$ such that a permutation of $\underline{\sigma}^{\mathsf{m}_\sigma}$ lies in $\mathbb{T}_i^{\mathrm{s}}$ while $\underline{\omega}\underline{\sigma}^{\mathsf{m}_\sigma} \in \mathbb{T}_i^{\mathrm{s}}\underline{\omega}$ for every $\omega \in \mathrm{Q}_{i,j}$.*

While the corresponding statement for right noetherianity is immediate, it's worth observing that the theorem refines in the commuting variables context as it follows.

Remark 3.13. The ring $R[\boldsymbol{E}, \mathbb{X}]$ is left (right, or bilateral) noetherian precisely when R is while $\mathbb{X}$ is finite and for all $i,j \in [\![1,n]\!]$ and any simple cycle $\sigma \in \mathsf{Q}_{\mathbb{X}}(k,k)$ with $\mathsf{Q}_{\mathbb{X}}(i,k), \mathsf{Q}_{\mathbb{X}}(k,j) \neq \varnothing$, a positive power of $\underline{\sigma}$ lies in $\mathbb{T}_i^{\mathrm{s}}$ (or resp., in $\mathbb{T}_j^{\mathrm{s}}$ or $\mathbb{T}_i^{\mathrm{s}} \cdot \mathbb{T}_j^{\mathrm{s}}$).

So the only fragment still left open is the following question.

Problem 3.14. Suppose that $\mathbb{T}$ is bilateral noetherian while the subset $\boldsymbol{X}_{nc} \subset \boldsymbol{X}$ of free non-commuting variables is non-empty. Is it true that $R\langle \boldsymbol{E}, \mathbb{X}; \boldsymbol{C}\rangle$ is bilateral noetherian, or can one prove that each diagonal monoid $\mathbb{T}_i$ is commutative?

The proof of Theorem 3.12 and Remark 3.13. One first observes that the left (resp, right, bilateral) noetherianity of the matrix ring extension $R \cdot \boldsymbol{E}$, (with finite R-basis given by the multiplicatively closed subset $\boldsymbol{E} \subset \mathbb{E}$ of basic elementary matrices), is equivalent to that of the coefficient algebra R. One recalls in view of Remark 2.2 (2.2) that $\mathsf{A} = R \cdot \mathbb{T}$ is the semigroup ring with multiplicative R-basis $\mathbb{T}$. We shall now continue by splitting the proof of Theorem 3.12 and Remark 3.13 in several statements.

Statement 3.1. *The ring* $\mathsf{A} = R\langle \boldsymbol{E}, \mathbb{X}; \boldsymbol{C}\rangle$ *is left noetherian precisely when* R *and* $\mathbb{T}$ *are. The same holds for bilateral noetherianity in the commuting variables context.*

Proof. If A is left (or resp., right, bilateral) noetherian, then so is $\mathbb{T}$ according to Lemma 3.6; the same holds for the algebra $R{\cdot}\boldsymbol{E}$ (and hence for R) since $\mathsf{A} = R{\cdot}\boldsymbol{E} \oplus_{\Bbbk} \mathsf{A}\mathbb{X}$ where $\mathsf{A}\mathbb{X} = \mathbb{X}\mathsf{A}$ is the ideal generated by elementary matrix-variables.

Conversely, assuming that both R and $\mathbb{T}$ are left noetherian, we want to show that A is left noetherian. By virtue of Remark 3.4, each submonoid $\mathbb{T}_i \subset \langle \boldsymbol{X}; \boldsymbol{C}\rangle$ is left noetherian and $\mathbb{T}_{i,j}$ is a finitely generated left $\mathbb{T}_i$-module. But then, thanks to Lemma 3.7, each monoid $\mathbb{T}_i$ is commutative and finitely generated and, since also each $\mathbb{T}_{i,j}$ is already a finitely generated left $\mathbb{T}_i$-module for all i, j, it follows that $\mathbb{X}$ is finite. Letting $i, j \in [\![1, n]\!]$, it also follows from the previous discussion that the diagonal component $\mathsf{A}_i = R\mathbb{T}_i$ of A is left noetherian as a quotient of an ordinary polynomial ring extension of the left noetherian ring R by finitely many commuting independent variables. For $i \neq j$, since $\mathsf{A}_{i,j} = R \cdot \mathbb{T}_{i,j}$ while the left $\mathbb{T}_i$-module $\mathbb{T}_{i,j}$ is finitely generated, it readily follows that the left A_i-module $\mathsf{A}_{i,j}$ is finitely generated, and hence, left noetherian because A_i is already a noetherian ring. We conclude by virtue of Lemma 3.1 that A is left noetherian.

Likewise in the commuting variables context, if R and $\mathbb{T}$ are bilateral noetherian, then so is the ring $R[\boldsymbol{E}, \mathbb{X}]$ by repeating the same arguments as before with the expression 'left noetherian' replaced by 'bilateral noetherian', while for all $i, j \in [\![1, n]\!]$, the expression 'left $\mathbb{T}_i$-module' is replaced by 'bilateral $\mathbb{T}_i$-$\mathbb{T}_j$-module'. □

For the proof of the second part of Theorem 3.12 (and of Remark 3.13), beside the ring $\mathsf{A} = R\langle \boldsymbol{E}, \mathbb{X}; \boldsymbol{C}\rangle$ it shall be necessary to consider at the same time the matrix polynomial ring $R[\boldsymbol{E}, \mathbb{X}]$ in commuting variables. So for the sake of clarity we need (only in this proof) alternative notations for terms in commutating variables: let $i, j \in [\![1, n]\!]$; for all $\omega \in \mathsf{Q}_{\mathbb{X}}(i, j)$ and every term $\lambda = \underline{\omega} \in \mathbb{T}_{i,j}$ (in not necessarily commuting variables),

- we write $\underset{\sim}{\lambda} = \underset{\sim}{\omega}$ for the corresponding term in commuting variables,
- we also write T for the set of elementary matrix-terms in commuting variables; $\mathsf{T}_i^{\mathrm{s}} \subset \mathsf{T}_i$ stands for the submonoid generated by terms along simple cycles at point i.

Thus T is naturally a quotient of $\mathbb{T}$ with canonical projection $\mathbb{T} \longrightarrow \mathsf{T}$, $\lambda \boldsymbol{e}_{i,j} \longmapsto \underset{\sim}{\lambda} \boldsymbol{e}_{i,j}$ for all $i, j \in [\![1, n]\!]$ and $\lambda \in \mathbb{T}_{i,j}$.

Statement 3.2. *Assuming that $\mathbb{T}$ is left noetherian, its shape is as described by Theorem 3.12. Likewise, if $\mathbb{T}$ is bilateral noetherian while we are in the commuting variables context, then its shape is as described by Remark 3.13.*

Proof. With the assumption that $\mathbb{T}$ is left noetherian, the same holds for T as well. From the previous paragraph, it is already proved that $\mathbb{X}$ is finite and for every $i, j \in [\![1, n]\!]$, each $\mathbb{T}_i$ is a finitely generated commutative monoid while each bilateral $\mathbb{T}_i$-$\mathbb{T}_j$-module $\mathbb{T}_{i,j}$ is left noetherian. Now fixing i, Proposition 3.8(c) also grants that there is some term $v_i \in \langle \boldsymbol{X}_{nc} \rangle$ such that $\mathbb{T}_i$ is contained in the submonoid $\langle \boldsymbol{X}_c \cup \{v_i\} \rangle$ of $\langle \boldsymbol{X}; \boldsymbol{C} \rangle$ generated by $\boldsymbol{X}_c \cup \{v_i\}$. Here one recalls that $\boldsymbol{X}_c$ consists of central variables while $\boldsymbol{X}_{nc} = \boldsymbol{X} \backslash \boldsymbol{X}_c$ consists of free non-commuting variables; in particular the submonoid $\langle \boldsymbol{X}_{nc} \rangle$ is also the free non-commutative monoid generated over $\boldsymbol{X}_{nc}$. It follows for all $\omega, \omega' \in \mathsf{Q}_{\mathbb{X}}(i, i)$ that if $\underset{\sim}{\omega} = \underset{\sim}{\omega}'$, then it already holds in $\mathbb{T}_i$ that $\underline{\omega} = \underline{\omega'}$. Hence, the canonical projection $\mathbb{T}_i \longrightarrow \mathsf{T}_i$, $\lambda \longmapsto \underset{\sim}{\lambda}$ is an isomorphism of monoids.

Let $\sigma \in \mathsf{Q}_{\mathbb{X}}(j, j)$ be any simple cycle with $\mathsf{Q}_{\mathbb{X}}(i, j) \neq \emptyset$ and pick any path $\omega \in \mathsf{Q}_{\mathbb{X}}(i, j)$. We want to show that there is a positive integer m (depending on both σ and ω) such that $\underset{\sim}{\sigma}^m$ lies in $\mathbb{T}_i^{\mathrm{s}}$ up to a permutation of variables while $\underline{\omega \sigma^m} \in \mathbb{T}_i^{\mathrm{s}} \underline{\omega}$. But having in view the canonical isomorphism $\mathbb{T}_i \xrightarrow{\sim} \mathsf{T}_i$, $\lambda \longmapsto \underset{\sim}{\lambda}$, we have to prove that there is a positive integer m together with a term $v \in \mathbb{T}_i^{\mathrm{s}}$ such that $\underset{\sim}{\sigma}^m = \underset{\sim}{v} \in \mathsf{T}_i^{\mathrm{s}}$ and $\underline{\omega \sigma^m} = v \underline{\omega}$. The term $\underline{\sigma}$ satisfies $\underline{\omega} \underline{\sigma}^r = \underline{\omega \sigma^r} \in \mathbb{T}_{i,j}$ for all $r \in \mathbb{N}$. Since by assumption $\mathbb{T}_{i,j}$ is a noetherian $\mathbb{T}_i$-module, the $\mathbb{T}_i$-submodule generated by the set $\underline{\omega} \underline{\sigma}^{\mathbb{N}} = \{\underline{\omega} \underline{\sigma}^l : l \in \mathbb{N}\}$ must be generated by finitely many elements from $\underline{\omega} \underline{\sigma}^{\mathbb{N}}$, implying that there exists some $\lambda \in \mathbb{T}_i$ and some positive $r \in \mathbb{N}$ with $\underline{\omega} \underline{\sigma}^r = \lambda \underline{\omega}$. In particular passing to T_i, it holds that $\underset{\sim}{\sigma}^r = \underset{\sim}{\lambda}$ and the term λ is, up to a permutation of variables, a positive power of the term $\underline{\sigma}$ lying in $\mathbb{T}_i$. But then, for the commutative monoid S generated by terms along all simple cycles in $\mathsf{Q}_{\mathbb{X}}$, Lemma 3.9 yields that $(\mathsf{T}_i^{\mathrm{s}}, \mathsf{T}_i, \mathsf{S})$ satisfies condition (3.2) and relying on Lemma 3.10, we deduce that a positive power $\underset{\sim}{\sigma}^m$ belongs to $\mathsf{T}_i^{\mathrm{s}}$; one may take $m = pr$ to be a multiple of r for some $p \geq 1$. It follows that

$$\underline{\omega \sigma^m} = \underline{\omega} \underline{\sigma}^{pr} = \underline{\omega} (\underline{\sigma}^r)^p = \lambda^p \underline{\omega} \text{ with } \underset{\sim}{\lambda}^p = \underset{\sim}{\sigma}^m \in \mathsf{T}_i^{\mathrm{s}}.$$

Thus as desired, m and the term $v = \lambda^p$ are such that $\underset{\sim}{\sigma}^m = \underset{\sim}{v} \in \mathsf{T}_i^{\mathrm{s}}$ while $\underline{\omega \sigma^m} = v \underline{\omega}$.

Since once again the left $\mathbb{T}_i$-module $\mathbb{T}_{i,j}$ must be generated by a finite subset $\Sigma \subset \mathbb{T}_{i,j}$ (while $\mathbb{T}_i$ is also commutative), applying the previous paragraph for ω running over the finite set Σ shows that there exists a positive integer m_σ (now depending only on σ) such that $\underline{\sigma}^{\mathsf{m}_\sigma}$ lies in $\mathbb{T}_i^{\mathrm{s}}$ up to a permutation of variables while $\underline{\omega}\underline{\sigma}^{\mathsf{m}_\sigma} \in \mathbb{T}_i^{\mathrm{s}}\underline{\omega}$ for all $\omega \in \mathsf{Q}_{\mathbb{X}}(i,j)$. This completes the proof that every left noetherian $\mathbb{T}$ has its shape described as in the second part of Theorem 3.12.

Likewise, specializing to the commuting variables context (so that, $\mathsf{A} = R[\boldsymbol{E}, \mathbb{X}]$ and in this case the alternative notation T coincides with $\mathbb{T}$), suppose that $\mathbb{T}$ is bilateral noetherian. Then the technical issue due in the general case to the presence of non-necessarily commuting variables disappears. For all $i, j \in [\![1, n]\!]$ and any cycle $\sigma \in \mathsf{Q}_{\mathbb{X}}(k,k)$ with $\mathsf{Q}_{\mathbb{X}}(i,k), \mathsf{Q}_{\mathbb{X}}(k,j) \neq \varnothing$, as before (in the second paragraph of this proof), the bilateral noetherianity of $\mathbb{T}_{i,j}$ yields more easily that a positive power of $\underline{\sigma}$ lies in $\mathbb{T}_i\mathbb{T}_j$. But again, the triple $(\mathsf{T}_i^{\mathrm{s}}\mathsf{T}_j^{\mathrm{s}}, \mathbb{T}_i\mathbb{T}_j, \mathsf{S})$ satisfies condition (3.2) and relying on Lemma 3.10, one deduces that a positive power of $\underline{\sigma}$ lies in $\mathsf{T}_i^{\mathrm{s}}\mathsf{T}_j^{\mathrm{s}}$, showing that the shape of $\mathbb{T}$ is, as desired, described by Remark 3.13. □

We now turn to the sufficiency in the second part of Theorem 3.12 and Remark 3.13.

Statement 3.3. *Assume that $\mathbb{X}$ is finite and each $\mathbb{T}_i$ is commutative for all $i \in [\![1, n]\!]$.*

(a) *If moreover for all $i, j \in [\![1, n]\!]$ with $\mathsf{Q}_{\mathbb{X}}(i,j) \neq \varnothing$ and for any simple cycle $\sigma \in \mathsf{Q}_{\mathbb{X}}(j,j)$, there is a positive $\mathsf{m}_\sigma \in \mathbb{N}$ such that $\underset{\sim}{\sigma}^{\mathsf{m}_\sigma} \in \mathsf{T}_i^{\mathrm{s}}$ while $\underline{\omega}\underline{\sigma}^{\mathsf{m}_\sigma} \in \mathbb{T}_i^{\mathrm{s}}\underline{\omega}$ for every $\omega \in \mathsf{Q}_{i,j}$, then $\mathbb{T}$ is left noetherian.*

(b) *Likewise in the commuting variables context, if for all $i, j, k \in [\![1, n]\!]$ with $\mathsf{Q}_{\mathbb{X}}(i,k), \mathsf{Q}_{\mathbb{X}}(k,j) \neq \varnothing$ and for any simple cycle $\sigma \in \mathsf{Q}_{\mathbb{X}}(k,k)$, there is a positive $\mathsf{m}_\sigma \in \mathbb{N}$ such that $\underline{\sigma}^{\mathsf{m}_\sigma} \in \mathbb{T}_i^{\mathrm{s}} \cdot \mathbb{T}_j^{\mathrm{s}}$, then $\mathbb{T}$ is bilateral noetherian.*

Proof. Since for the bilateral noetherianity claimed by part (b) the additional assumption is weaker than the one in part (a), we start with the proof of (b) and we shall see how this proof specializes and extends to a proof of the one-sided noetherianity claimed by part (a). Thus we are in the commuting variables context, and having in view Lemma 3.7 and Remark 3.4, arbitrarily fixing $i, j \in [\![1, n]\!]$ we have to show that:

the commutative monoid $\mathbb{T}_i$ and the bilateral $\mathbb{T}_i$-$\mathbb{T}_j$-module $\mathbb{T}_{i,j}$ are finitely generated.

The finiteness of $\mathbb{X}$ implies that the number of simple paths in $\mathsf{Q}_{\mathbb{X}}$ is finite. In particular for all $k, k' \in [\![1, n]\!]$, the monoids $\mathbb{T}_k^{\mathrm{s}} \subset \mathbb{T}_k$ and $\mathbb{T}_k^{\mathrm{s}}\mathbb{T}_{k'}^{\mathrm{s}}$ are finitely generated. And letting $\boldsymbol{\tau}$ consist of all terms $\underline{\sigma}$ with σ a simple cyclic subpath of some cycle in $\mathsf{Q}_{\mathbb{X}}(k,k)$, and $\boldsymbol{\upsilon}$ consist of all terms $\underline{\sigma}$ with σ a simple cyclic subpath of some cycle in $\mathsf{Q}_{\mathbb{X}}(r,r)$ for $r \in \{k,k'\}$, it is already granted by Lemma 3.9 that

$$\mathbb{T}_k^{\mathrm{s}} \subset \mathbb{T}_k \subset \mathbb{T}_k^{\mathrm{s}}[\boldsymbol{\tau}] \text{ and } \mathbb{T}_k^{\mathrm{s}}\mathbb{T}_{k'}^{\mathrm{s}} \subset \mathbb{T}_k\mathbb{T}_{k'} \subset (\mathbb{T}_k^{\mathrm{s}}\mathbb{T}_{k'}^{\mathrm{s}})[\boldsymbol{v}].$$

It follows that the monoids $\mathbb{T}_i$ and $\mathbb{T}_i\mathbb{T}_j$ are already finitely generated by relying on Proposition 3.11 (applied to the monoid ring extensions $\Bbbk[\mathbb{T}_i]$ and $\Bbbk[\mathbb{T}_i\mathbb{T}_j]$). Thus the only case not (concretely) dealt with by Proposition 3.11 concerns the semigroup module $\mathbb{T}_{i,j}$ when $i \neq j$. So continuing with the case that $i \neq j$, it suffices for our purpose to show that each $\mathbb{T}_{i,j}$ is finitely generated as $\mathbb{T}_i^{\mathrm{s}}\mathbb{T}_j^{\mathrm{s}}$-module (or, just as $\mathbb{T}_i\mathbb{T}_j$-module). For every simple cycle σ appearing as subpath of a path from i to j, we let m_σ be the smallest non-zero natural number such that $\underline{\sigma}^{\mathsf{m}_\sigma}$ lies in $\mathbb{T}_i^{\mathrm{s}} \cdot \mathbb{T}_j^{\mathrm{s}}$. Recall that the multiplicity of a simple cycle σ in a given path $\omega \in \mathsf{Q}_{\mathbb{X}}(i,j)$ has been defined as the largest $m \in \mathbb{N}$ such that

$$\omega = \omega_1\sigma\omega_2\sigma\cdots\omega_m\sigma\omega_{m+1} \text{ for some paths } \omega_1,\ldots,\omega_{m+1}.$$

Since there are only finitely many simple paths, we get a finite subset $\mathrm{Q}_{i,j} \subset \mathsf{Q}_{\mathbb{X}}(i,j)$ consisting of all paths ω from i to j such that the multiplicity of any simple cyclic subpath σ in ω is less than m_σ. Write $\mathrm{T}_{i,j} \subset \mathbb{T}_{i,j}$ for the finite subset of all terms $\underline{\omega}$ with $\omega \in \mathrm{Q}_{i,j}$. Let $\omega \in \mathsf{Q}_{\mathbb{X}}(i,j)$ be arbitrarily given. If ω lies in $\mathrm{Q}_{i,j}$, then it is already granted that $\underline{\omega} \in \mathrm{T}_{i,j}$. Assuming that ω does not lie in $\mathrm{Q}_{i,j}$, it holds that ω contains a simple cycle σ whose multiplicity in ω is greater or equal to $m = \mathsf{m}_\sigma$. Thus we have:

$$\omega = \omega_1\sigma\omega_2\sigma\cdots\omega_m\sigma\omega_{m+1} \text{ for some paths } \omega_1,\ldots,\omega_{m+1}.$$

Clearly, $\omega' = \omega_1\cdots\omega_{m+1}$ is still a path from i to j with $\ell(\omega') < \ell(\omega)$, and it holds that $\underline{\omega} = \underline{\sigma}^m \cdot \underline{\omega'}$ with $\underline{\sigma}^m \in \mathbb{T}_i^{\mathrm{s}} \cdot \mathbb{T}_j^{\mathrm{s}}$ (because $m = \mathsf{m}_\sigma$). Whence, the proof that the term $\underline{\omega}$ for every $\omega \in \mathsf{Q}_{\mathbb{X}}(i,j)$ lies in $\mathbb{T}_i^{\mathrm{s}}\mathbb{T}_j^{\mathrm{s}} \cdot \mathrm{T}_{i,j}$ is completed by the way of induction on the length of paths. Thus, we have proved that each commutative monoid $\mathbb{T}_i$ is finitely generated and each $\mathbb{T}_{i,j}$ is a finitely generated $\mathbb{T}_i\mathbb{T}_j$-module, so that $\mathbb{T}$ is bilateral noetherian.

We now turn to the proof of part (a). Here $\mathsf{A} = R\langle \boldsymbol{E}, \mathbb{X}; \boldsymbol{C}\rangle$ and the variables do not necessarily commute, thus we shall also make use of the alternative notation T for the 'semigroup' of elementary matrix-terms in commuting variables. The specific assumption is that: for all $i,j \in [\![1,n]\!]$ with $\mathsf{Q}_{\mathbb{X}}(i,j) \neq \varnothing$ and for any simple cycle $\sigma \in \mathsf{Q}_{\mathbb{X}}(j,j)$, there is a positive $\mathsf{m}_\sigma \in \mathbb{N}$ with $\underset{\sim}{\sigma}^{\mathsf{m}_\sigma} \in \mathsf{T}_i^{\mathrm{s}}$ and $\underline{\omega}\underline{\sigma}^{\mathsf{m}_\sigma} \in \mathbb{T}_i^{\mathrm{s}}\underline{\omega}$ for every $\omega \in \mathrm{Q}_{i,j}$. But then, it is already granted by the above paragraph that each T_i is finitely generated, so that the same is true for $\mathbb{T}_i$ because of the canonical isomorphism $\mathbb{T}_i \xrightarrow{\sim} \mathsf{T}_i$, $\lambda \longmapsto \underset{\sim}{\lambda}$. We now proceed as above to show that each bilateral module $\mathbb{T}_{i,j}$ is generated as left module by the finite subset

$$\mathrm{T}_{i,j} = \{\underline{\omega} \;:\; \omega \in \mathrm{Q}_{i,j}\}.$$

We will proceed by induction on the length of paths. Let $\omega \in \mathsf{Q}_{\mathbb{X}}(i,j)$ be arbitrarily given. If ω lies in $\mathrm{Q}_{i,j}$, then it is already granted that $\underline{\omega} \in \mathrm{T}_{i,j}$. Assume that ω does not lie in $\mathrm{Q}_{i,j}$ and the desired result holds for all paths $\omega' \in \mathsf{Q}_{\mathbb{X}}(k,k')$ with $\ell(\omega') < \ell(\omega)$. It follows that ω contains a simple cycle $\sigma \in \mathsf{Q}_{\mathbb{X}}(k,k)$ whose multiplicity in ω is greater or equal to $m = \mathsf{m}_\sigma$. Thus we have: $\omega = \omega_1\sigma\omega_2\sigma\cdots\omega_m\sigma\omega_{m+1}$ for some paths $\omega_1 \in \mathsf{Q}_{\mathbb{X}}(i,k)$, $\omega_2\ldots,\omega_m \in \mathsf{Q}_{\mathbb{X}}(k,k)$ and $\omega_{m+1} \in \mathsf{Q}_{\mathbb{X}}(k,j)$. But since each monoid $\mathbb{T}_k$ is commutative, it holds that

$$\underline{\omega} = \underline{\omega_1}\,\underline{\sigma}^m\underline{\omega'} \text{ with } \omega' = \omega_2\cdots\omega_m\omega_{m+1}.$$

Now because $\ell(\omega_1) < \ell(\omega)$, the induction hypothesis applies, showing that $\underline{\omega_1} = \lambda\underline{\omega''}$ for some $\lambda \in \mathbb{T}_i$ and some $\omega'' \in \mathrm{Q}_{i,k}$. And by assumption, there exists some $\upsilon \in \mathbb{T}_i$ such that $\underline{\omega''}\,\underline{\sigma}^m = \upsilon\underline{\omega''}$ (because $m = \mathsf{m}_\sigma$ while $\omega'' \in \mathrm{Q}_{i,k}$). So we get:

$$\underline{\omega} = \underline{\omega_1}\,\underline{\sigma}^m\underline{\omega'} = \lambda\upsilon\underline{\omega''\omega'}, \text{ with } \ell(\omega''\omega') < \ell(\omega).$$

Thus another application of the induction hypothesis yields that $\underline{\omega''\omega'}$ lies in $\mathbb{T}_i\cdot\mathrm{T}_{i,j}$, implying that $\underline{\omega}$ lies in $\mathbb{T}_i\cdot\mathrm{T}_{i,j}$ as well. Hence each bilateral module $\mathbb{T}_{i,j}$ is as claimed a finitely generated $\mathbb{T}_i$-module. This completes the proof that $\mathbb{T}$ is left noetherian. The proof of Theorem 3.12 and Remark 3.13 is now finished. □

4. Matrix (polynomial) algebras with Cayley-Hamilton structures

For this section, the base ring $\Bbbk$ is a field of characteristic zero. The noncommutative geometry of Cayley-Hamilton algebras studied by [14] is well known: every affine Cayley-Hamilton algebra A of degree n over the field of complex numbers may be reconstructed as witness algebra of the commutative geometry space of trace-preserving n-dimensional representations of A. We address the obvious but rather delicate question of recognizing which matrix(-polynomial) algebras admit any Cayley-Hamilton structure.

It shall be necessary for our purpose to fully recall the setup for Cayley-Hamilton structures. With $0 \neq n \in \mathbb{N}$ fixed as usual, for countably many variables x_k, $1 \leq k \in \mathbb{N}$ we recall the *elementary symmetric functions* $\upsigma_k$ and the *power sum functions* s_k:

$$\upsigma_k = \upsigma_k(x_1,\ldots,x_n) = \sum_{1\leq i_1<\cdots<i_k\leq n} x_{i_1}\cdots x_{i_k},\ \mathsf{s}_k = \mathsf{s}_k(x_1,\ldots,x_n) = \sum_{i=1}^n x_i^k, 1 \leq k \leq n.$$

Let S_n be the symmetric group on n symbols; it is classical that both sets $\{\sigma_1, \ldots, \sigma_n\}$ and $\{\mathsf{s}_1, \ldots, \mathsf{s}_n\}$ are two algebra bases of the $\Bbbk$-algebra of symmetric polynomials

$$\Bbbk[x_1, \ldots, x_n]^{\mathsf{S}_n} = \{f \in \Bbbk[x_1, \ldots, x_n] \ : \ \forall \tau \in \mathsf{S}_n, \, f(x_{\tau(1)}, \ldots, x_{\tau(n)}) = f(x_1, \ldots, x_n)\}.$$

The well-known Newton identities [14, Eq. (1.2)] relating the σ_k's and the s_k's are:

$$(-1)^k k\sigma_k + \sum_{i=1}^{k-1} (-1)^i \sigma_i \mathsf{s}_{k-i} = 0, \text{ with } \sigma_0 = 1.$$

Now, recursively expressing the elementary symmetric polynomials in terms of the Newton power sum polynomials, one deduces the first item of the next crucial remark.

Remark 4.1. (a) There are universal polynomials $\hbar_k \in \mathbb{Q}[x_1, \ldots, x_k]$ with

$$\hbar_0 = 1, \ (-1)^k k \hbar_k = \sum_{i=0}^{k-1} (-1)^{i+1} \hbar_i x_{k-i} \text{ and } \sigma_k = \hbar_k(\mathsf{s}_1, \ldots, \mathsf{s}_k) \text{ for } 1 \leq k \leq n, \tag{4.1}$$

where with each fixed n and for all $k \in [\![1, n]\!]$, it is understood that $\sigma_k = \sigma_k(x_1, \ldots, x_n)$ and $\mathsf{s}_k = \mathsf{s}_k(x_1, \ldots, x_n)$.

(b) If $M \in \mathcal{M}_n(\Bbbk)$ is any matrix with n eigenvalues $\lambda_1, \ldots, \lambda_n$ (counted with multiplicities), then the characteristic polynomial of M expands as

$$\begin{aligned} \chi_M(t) = \det(t\mathit{1}_n - M) &= t^n + \sum_{k=1}^{n} (-1)^k \sigma_k(\lambda_1, \ldots, \lambda_n) t^{n-k} \\ &= t^n + \sum_{k=1}^{n} (-1)^k \hbar_k(\mathsf{tr}(M), \ldots, \mathsf{tr}(M^k)) t^{n-k}, \end{aligned}$$

where $\mathsf{tr} : \mathcal{M}_n(\Bbbk) \longrightarrow \Bbbk$ is the natural trace map on matrices.

Definition 4.2. (a) A *trace map* on an algebra A is a $\Bbbk$-linear map $tr : A \longrightarrow \mathsf{Z}(A)$ satisfying the following additional properties for all $a, b \in A$:

$$tr(ab) = tr(ba), \ tr(\mathsf{tr}(a)b) = tr(a)\mathsf{tr}(b).$$

In this case, there is for every $a \in A$ a *formal Cayley-Hamilton polynomial of degree n*:

$$\chi_a(t) = \chi_{a,tr}(t) = t^n + \sum_{k=1}^{n} (-1)^k \hbar_k(tr(a), \ldots, tr(a^k)) t^{n-k}.$$

(b) An algebra A with trace map tr is called *Cayley-Hamilton of degree n* if:

$$tr(\mathit{1}_A) = n\mathit{1}_A \text{ and } \chi_a(a) = 0 \text{ for all } a \in A.$$

Let (A, tr) be any Cayley-Hamilton algebra of degree n, generated over $tr(A)$ by at most m elements. View $\mathcal{M}_n^m = \mathcal{M}_n(\Bbbk) \oplus \cdots \oplus \mathcal{M}_n(\Bbbk)$, the direct sum of m-copies of the full matrix ring $\mathcal{M}_n = \mathcal{M}_n(\Bbbk)$, as a commutative geometric space with coordinate ring given by the polynomial ring $\Bbbk[\mathcal{M}_n^m] = \Bbbk[x_{i,j}^{(k)} : 1 \le i, j \le n, 1 \le k \le m]$ in $m \cdot n^2$ commuting variables. Then both $\Bbbk[\mathcal{M}_n^m]$ and $\mathcal{M}_n(\Bbbk[\mathcal{M}_n^m])$ carry a natural Gl_n-action. The invariant theoretical presentation theorem [14, Theorem 1.17] states that there is a canonical ideal $N_A \subset \Bbbk[\mathcal{M}_n^m]$ such that A and $tr(A)$ arise as Gl_n-invariant algebras

$$A = \mathcal{M}_n(\Bbbk[\mathcal{M}_n^m]/N_A)^{\mathrm{Gl}_n} \text{ and } tr(A) = \big(\Bbbk[\mathcal{M}_n^m]/N_A\big)^{\mathrm{Gl}_n}.$$

Thus the degree n free Cayley-Hamilton algebra, trace-generated by at most m elements, is just the matrix polynomial extension of $\Bbbk$ by m generic matrix-variables $X_k = (x_{i,j}^{(k)})_{i,j}$, $k = 1, \ldots, m$, in the ordinary commuting variables $x_{i,j}^{(k)}$ for $1 \le i, j \le n$, $1 \le k \le m$.

For finite-dimensional $\mathbb{C}$-algebras A, the algebraic reconstruction theorem can be used to describe all Cayley-Hamilton structures on A while restricting the range of the trace map to the base field, [14, Proposition 2.13]. However the arguments from [14, Proposition 2.13] do not lift to the present context, as stressed by the following remark.

Remark 4.3. The invariant theoretical and the geometric reconstruction theorems of Cayley-Hamilton algebras do not provide a mean to recognize whether a given matrix polynomial algebra admits a trace map making it Cayley-Hamilton.

For our purpose, it shall be necessary to resort to the abstract definition of Cayley-Hamilton algebras and delicately work out the recurrence relation (4.1) defining the universal polynomials involved in the Cayley-Hamilton identities. Our main contribution in this section is the following characterisation result which parallels [14, Proposition 2.13].

Theorem 4.4. *Let the coefficient algebra R be a domain and $\mathsf{A} = R[\boldsymbol{E}, \mathbb{X}]$ a connected matrix polynomial ring extension. Then A is Cayley-Hamilton of degree $r \in [\![1, n]\!]$ for some trace map $tr : \mathsf{A} \longrightarrow \mathsf{A}$ precisely when $r = n$, the diagonal components of A coincides and the trace map tr is induced as the restriction of the natural trace map on matrices.*

The rest of this section is devoted to essentially proving the above theorem. The sufficiency in the theorem is easy and follows by the fact that for any subalgebra A of a full matrix algebra $\mathcal{M}_n(R)$ over a commutative ring R such that A is stable under the natural trace map $\mathsf{tr} : \mathcal{M}_n(R) \longrightarrow R$, the algebra A endowed with the restriction $A \longrightarrow A, a \longmapsto \mathsf{tr}(a) \cdot \mathbb{1}_n$ is Cayley-Hamilton of degree n.

For the necessity part in Theorem 4.4, we need some preparation before we can uncover the specific effect of an eligible trace map providing A with a Cayley-Hamilton structure. Returning to Cayley-Hamilton identities, we define for every variable z the polynomial:

$$\pi_n(z) = \prod_{k=0}^{n-1}(z-k) = z\cdot(z-1)\cdots(z-(n-1)).$$

Lemma 4.5. *For each $k \in \mathbb{N}$, set $\hbar_0(z) = \hslash_0(z) = 1$ and $\hbar_k(z) = \hslash_k(z,\dots,z)$ (with k components all equal to z). Then the $\hbar_k$'s satisfy the following relations for all $n \in \mathbb{N}$:*

$$n!\hbar_n(z) = \pi_n(z) = \prod_{k=0}^{n-1}(z-k), \quad (-1)^n n! \sum_{k=0}^{n}(-1)^k \hbar_k(z) = \frac{1}{z}\pi_{n+1}(z) = \prod_{k=1}^{n}(z-k). \tag{4.2}$$

Proof. The polynomials $\hslash_k = \hslash_k(x_1,\dots,x_k) \in \mathbb{Q}[x_1,\dots,x_k]$, for $k \in \mathbb{N}$, satisfy by virtue of Remark 4.1 the recurrence relation:

$$\hslash_0 = 1,\ \ (-1)^k k\hslash_k = \sum_{i=0}^{k-1}(-1)^{i+1}\hslash_i x_{k-i} \text{ for } k \geq 1.$$

So the $\hbar_k$'s satisfy the recurrence relation:

$$(-1)^k k\hbar_k(z) = z\sum_{i=0}^{k-1}(-1)^{i+1}\hbar_i(z) \text{ for } k \geq 1.$$

We will prove the validity of both identities given by (4.2) using induction on n. We have $\hbar_0(z) = \hslash_0 = 1$, and $\hbar_1(z) = \hslash_1(z) = z = \sum_{k=0}^{1-1}(z-k)$, while $(-1)^1\cdot 1\cdot \sum_{k=0}^{1}(-1)^k\hbar_k(z) = -(1+(-z)) = (z-1)$, showing that (4.2) holds for $n = 1$. Turning to the induction step, assuming that (4.2) has been proved for a given $n \geq 1$, let us prove that (4.2) holds for $n+1$. Since $(-1)^{n+1}(n+1)\hbar_{n+1}(z) = -z\sum_{k=0}^{n}(-1)^k\hbar_k(z)$, on one hand we have:

$$\begin{aligned}(-1)^{n+1}(n+1)!\hbar_{n+1}(z) &= -zn!\sum_{k=0}^{n}(-1)^k\hbar_k(z)\\ &= -z(-1)^n\prod_{k=1}^{n}(z-k) = (-1)^{n+1}\prod_{k=0}^{n}(z-k),\end{aligned}$$

so that, $(n+1)!\hbar_{n+1}(z) = \prod_{k=0}^{n}(z-k)$. On the other hand, we get the following computations (where in the second row, one invokes the induction hypothesis and the already computed value of $(n+1)!\hbar_{n+1}(z)$):

$$\begin{aligned}(-1)^{n+1}(n+1)!\sum_{k=0}^{n+1}(-1)^k\hbar_k(z) &= -(n+1)(-1)^n n!\sum_{k=0}^{n}(-1)^k\hbar_k(z)\\ &\qquad + (-1)^{n+1}(n+1)!\cdot(-1)^{n+1}\hbar_{n+1}(z)\\ &= -(n+1)\prod_{k=1}^{n}(z-k)+\prod_{k=0}^{n}(z-k)=\prod_{k=1}^{n}(z-k)\cdot\big(-(n+1)+z\big)\\ &= \prod_{k=1}^{n+1}(z-k).\end{aligned}$$

Hence, (4.2) holds for $n+1$ and this completes the proof of the lemma. □

The next key lemma computes for an eligible trace tr on an arbitrary algebra A the possible values of $tr(e)$ for every idempotent element $e \in A$.

Lemma 4.6. *Let A be any $\Bbbk$-algebra endowed with a trace map tr. Let $1 \leq n \in \mathbb{N}$ and write χ_a for the associated degree n formal Cayley-Hamilton polynomial of $a \in A$.*

(a) *For every idempotent $e \in A$, it holds that*

$$(-1)^n n!\chi_e(e) = (tr(e) - ne)\cdot\prod_{k=1}^{n-1}(tr(e)-k).$$

(b) *Suppose that (A, tr) is Cayley-Hamilton of degree n, torsion-free as a module over the $\Bbbk$-subalgebra $tr(A)$, while $1 = e_1 + \cdots + e_p$ is a sum of non-zero orthogonal idempotents for some $p \in \mathbb{N}$. Then, either $tr(e_1), \ldots, tr(e_n) \in [\![1, n-1]\!]$ and $p \leq \sum_{i=1}^{p} tr(e_i) = n$, or $tr(e_i) = ne_i$ for all $i \in [\![1, p]\!]$ and the idempotents e_i's are necessarily central.*

Proof. Remember that since $\Bbbk$ is a field with characteristic zero, the field $\mathbb{Q}$ of rational numbers identifies with a subring of the center of the $\Bbbk$-algebra A. The degree n formal Cayley-Hamilton polynomial associated with tr is given for every $a \in A$ by:

$$\chi_a(x) = x^n + \sum_{k=1}^{n}(-1)^k \hbar_k(tr(a), \ldots, tr(a^k))x^{n-k}.$$

So for an idempotent $e \in A$, since $e^k = e$ for all $k \geq 1$ and recalling by definition that $\hbar_k(z) = \hbar_k(z, \ldots, z)$ for all k, with $\hbar_0(z) = \hbar_0 = 1$, we get that:

$$\chi_e(e) = e\cdot\sum_{k=0}^{n-1}(-1)^k\hbar_k(tr(e)) + (-1)^n\hbar_n(tr(e)).$$

Now multiplying the above by $(-1)^n n!$ and invoking Lemma 4.5, we get:

$$\begin{aligned}(-1)^n n!\chi_e(e) &= -ne\cdot(-1)^{n-1}(n-1)!\sum_{k=0}^{n-1}(-1)^k\hbar_k(tr(e)) + n!\hbar_n(tr(e))\\ &= -ne\prod_{k=1}^{n-1}(tr(e)-k)+\prod_{k=0}^{n-1}(tr(e)-k).\end{aligned}$$

Thus,

$$(-1)^n n!\chi_e(e) = (tr(e)-ne)\cdot\prod_{k=1}^{n-1}(tr(e)-k). \tag{$\dagger$}$$

Let us now assume that (A, tr) is Cayley-Hamilton of degree n, torsion-free as a module over the $\Bbbk$-subalgebra $tr(A)\subset \mathsf{Z}(A)$. Also let $1 = e_1+\cdots+e_p$ be a decomposition as a sum of non-zero orthogonal idempotents $e_1,\dots,e_p$ for some $p\in\mathbb{N}$. For every idempotent $e\in A$, since by assumption A is a torsion-free module over $tr(A)$, the Cayley-Hamilton identity $\chi_e(e)=0$, together with the relation ($\dagger$) above, show that

$$tr(e)\in\{1,2\dots,(n-1),ne\}.$$

Still because (A, tr) is Cayley-Hamilton of degree n, we know that $tr(1)=n$. Putting $\lambda_i = tr(e_i)$ for $1\le i\le p$, we have to solve the problem given by the following system:

$$\begin{cases}\lambda_1+\lambda_2+\cdots+\lambda_p = n\\ \lambda_i\in\{1,2\dots,(n-1),ne_i\} \text{ for } 1\le i\le p.\end{cases}\tag{$\ddagger$}$$

We claim that: either $\lambda_i = ne_i$ for all $i\in[\![1,p]\!]$, or the λ_i's are all positive integers in $[\![1,n-1]\!]$. Indeed, assuming the contrary, we have that both subsets $\Lambda=\{i\in[\![1,p]\!] \,:\, 1\le\lambda_i\le n-1\}$ and $\Lambda' = [\![1,p]\!]\backslash\Lambda$ are non-empty. Then ($\ddagger$) yields that:

$$\sum_{i\in\Lambda}\lambda_i + n\sum_{j\in\Lambda'}e_j = n = n\cdot 1 = n\sum_{k=1}^{p}e_k,$$

and it follows that $\sum_{i\in\Lambda}\lambda_i = n\sum_{k\in\Lambda}e_k$. Thus piking any $j\in\Lambda'$ and multiplying the last equation by e_j yields that $(\sum_{i\in\Lambda}\lambda_i)e_j = ne_j\sum_{k\in\Lambda}e_k = 0$ while $(\sum_{i\in\Lambda}\lambda_i)$ is a non-zero positive integer, which implies that $e_j=0$ and this last relation is a contradiction. Hence our claim holds: either $(\lambda_1,\dots,\lambda_p) = (ne_1,\dots,ne_p)$ and in this case the e_i's are necessarily central elements, or $\lambda_1,\dots,\lambda_p\in[\![1,n-1]\!]$ and $p\le\sum_{i=1}^p\lambda_i = n$. □

Returning to the statement of Theorem 4.4, let $tr:\mathsf{A}\longrightarrow\mathsf{A}$ be any trace map. By definition, $tr(\mathsf{A})$ is contained in the center $\mathsf{Z}(\mathsf{A})$ of A. But A is by assumption connected, that is, the associated labelled quiver $\mathsf{Q}_{\mathbb{X}}$ is connected; Proposition 2.5(b) shows that $\mathsf{Z}(\mathsf{A}) = \mathsf{Z}\cdot\mathbb{1}_n$ with $\mathsf{Z} = \cap_{i=1}^n\mathsf{A}_i$. Hence any given trace map on A arises as a trace map $tr:\mathsf{A}\longrightarrow\mathsf{Z}\cong\mathsf{Z}\cdot\mathbb{1}_n\hookrightarrow\mathsf{A}$ while A does not reduces to its diagonal algebra in case $n \geq 2$. With this observation, the necessity part of Theorem 4.4 becomes the object of the next slightly more general proposition, where a matrix subalgebra in a full matrix ring is always assumed to include all the basic elementary idempotent matrices.

Proposition 4.7. *Let R be any $\Bbbk$-algebra without zero-divisors, and $A = (A_{i,j})_{i,j} \subset \mathcal{M}_n(R)$ any matrix subalgebra together with a trace map $\mathit{tr} : A \longrightarrow R$ such that A is Cayley-Hamilton of some degree $r \in [\![1, n]\!]$, with $A \neq \operatorname{diag}(A)$ in case $n \geq 2$. Then necessarily, $r = n$, the trace map tr is induced as the restriction of the natural trace map on matrices while the diagonal components of A coincide with a subalgebra $Z \subset R$ acting centrally on A.*

Proof. Since R does not contain zero-divisors, the matrix algebra $\mathcal{M}_n(R)$, as well the matrix subalgebra A, are torsion-free as left R-modules. Recall that we write the basic elementary matrices as $\boldsymbol{e}_{i,j}$, with $\boldsymbol{e}_i = \boldsymbol{e}_{i,i}$, and the diagonal components of A as $A_i = A_{i,i}$, $1 \leq i, j \leq n$. Write the identity matrix as $\mathbb{1}_n = \boldsymbol{e}_1 + \cdots + \boldsymbol{e}_n$. It holds for all $i \neq j$ in $[\![1, n]\!]$ and any $\alpha \in A_{i,j}$ that $\mathit{tr}(\alpha \boldsymbol{e}_{i,j}) = \mathit{tr}(\boldsymbol{e}_i \cdot \alpha \boldsymbol{e}_{i,j}) = \mathit{tr}(\alpha \boldsymbol{e}_{i,j} \cdot \boldsymbol{e}_i) = \mathit{tr}(0) = 0$. Hence the formal trace of every matrix $\boldsymbol{a} = \sum_{1 \leq i,j \leq n} a_{i,j} \boldsymbol{e}_{i,j} \in A$ depends only on its diagonal components:

$$\mathit{tr}(\boldsymbol{a}) = \sum_{1 \leq i \leq n} \mathit{tr}(a_{i,i} \boldsymbol{e}_i).$$

Next, for countably many variables $x_1, x_2, x_3, x_4, \ldots$, and having in view Remark 4.1, recall that there are universal polynomials $\hbar_k = \hbar_k(x_1, \ldots, x_k) \in \mathbb{Q}[x_1, \ldots, x_k]$, $k \in \mathbb{N}$, defined by the recurrence relations:

$$\hbar_0 = 1, \ (-1)^m m \hbar_m = \sum_{k=0}^{m-1} (-1)^{k+1} \hbar_k x_{m-k}, \ m \geq 1. \tag{I}$$

If $n = 1$, then $r = 1$ and the degree-one Cayley-Hamilton identity for every $a \in A$ is given by: $0 = \chi_a(a) = a - \mathit{tr}(a)$, which amounts to the equality $\mathit{tr}(a) = a$, so that the desired result is trivial. We continue the proof with $n \geq 2$. Thus $\mathbb{1}_n = \boldsymbol{e}_1 + \cdots + \boldsymbol{e}_n$ is a sum of orthogonal idempotent matrices $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$, at least one of which is non-central because in this case, by assumption A does not reduce to its diagonal $\operatorname{diag}(A)$. Since moreover A is torsion-free as a left R-module while by assumption tr takes its values in R (which identifies to the subring $R \cdot \mathbb{1}_n \subset A$), part (b) of Lemma 4.6 shows that $\mathit{tr}(e_1), \ldots, \mathit{tr}(e_n)$ are n positive integers whose sum equals r. But then since $r \leq n$, the only possible solution for r and for the $\mathit{tr}(e_i)$'s is given by: $r = n$ and $\mathit{tr}(\boldsymbol{e}_i) = 1$ for all $i \in [\![1, n]\!]$.

Fixing any $i \in [\![1, n]\!]$ and any non-zero element $\alpha \in A_i$, our purpose is to prove that $\mathit{tr}(\alpha \boldsymbol{e}_i)$ is necessarily given by α. In view of Remark 4.1, the degree n formal

Cayley-Hamilton polynomial of $\alpha\boldsymbol{e}_i$ is given by

$$\chi_{\alpha\boldsymbol{e}_i}(x) = x^n + \sum_{k=1}^{n} \mu_k x^{n-k} \text{ with } \mu_k = (-1)^k \hbar_k(\mathit{tr}(\alpha\boldsymbol{e}_i), \ldots, \mathit{tr}(\alpha^k\boldsymbol{e}_i)),\ 1 \le k \le n.$$

Using a backward induction we will show that $\mu_m = 0$ for $m = n, n-1, \ldots, 2$. The degree n Cayley-Hamilton identity $\chi_{\alpha\boldsymbol{e}_i}(\alpha\boldsymbol{e}_i) = 0$ expands as:

$$\alpha^n\boldsymbol{e}_i + \sum_{k=1}^{n-1} \mu_k \alpha^{n-k}\boldsymbol{e}_i + \mu_n 1_A = \alpha^n\boldsymbol{e}_i + \sum_{k=1}^{n-1} \mu_k \alpha^{n-k}\boldsymbol{e}_i + \mu_n\boldsymbol{e}_i + \mu_n \cdot \sum_{j\neq i} \boldsymbol{e}_j = 0.$$

Since the elementary matrices form an R-basis of $\mathcal{M}_n(R)$, it follows that $\mu_n = 0$. Next, letting $m \in [\![2, n-1]\!]$ and assuming that $\mu_k = 0$ for $m+1 \le k \le n$, let us show that $\mu_m = 0$ as well. Since moreover A is torsion-free as a left R-module while α is a non-zero element of R, the degree n Cayley-Hamilton equation for $\alpha\boldsymbol{e}_i$ simplifies to the equation:

$$\alpha^m\boldsymbol{e}_i + \sum_{k=1}^{m-1} \mu_k \alpha^{m-k}\boldsymbol{e}_i + \mu_m\boldsymbol{e}_i = 0.$$

Applying the map tr to the last equation and using the already granted fact that $\mathit{tr}(\boldsymbol{e}_i) = 1$ together with the property that tr is $\mathit{tr}(A)$-linear, we get:

$$\mathit{tr}(\alpha^m\boldsymbol{e}_i) + \sum_{k=1}^{m-1} \mu_k\, \mathit{tr}(\alpha^{m-k}\boldsymbol{e}_i) + \mu_m = 0. \tag{II}$$

Substituting the μ_k's by their values in terms of the $\hbar_k$'s and using the recurrence relation given by (I), we compute:

$$\begin{aligned}
\mathit{tr}(\alpha^m\boldsymbol{e}_i) + \sum_{k=1}^{m-1} \mu_k\, \mathit{tr}(\alpha^{m-k}\boldsymbol{e}_i) &= \sum_{k=0}^{m-1} (-1)^k \hbar_k(\mathit{tr}(\alpha\boldsymbol{e}_i), \ldots, \mathit{tr}(\alpha^k\boldsymbol{e}_i))\, \mathit{tr}(\alpha^{m-k}\boldsymbol{e}_i) \\
&= -(-1)^m m \hbar_m(\mathit{tr}(\alpha\boldsymbol{e}_i), \ldots, \mathit{tr}(\alpha^m\boldsymbol{e}_i)) \\
&= -m\mu_m.
\end{aligned}$$

So equation (II) becomes $(1-m)\mu_m = 0$, showing that $\mu_m = 0$ because $m \ge 2$. We conclude that $\mu_m = 0$ for all $m \in [\![2, n]\!]$, and the degree n Cayley-Hamilton equation for $\alpha\boldsymbol{e}_i$ simplifies to the equation $0 = \alpha\boldsymbol{e}_i - \mathit{tr}(\alpha\boldsymbol{e}_i)\boldsymbol{e}_i = \big(\alpha - \mathit{tr}(\alpha\boldsymbol{e}_i)\big)\boldsymbol{e}_i$, forcing that $\mathit{tr}(\alpha\boldsymbol{e}_i) = \alpha$ since $\mathcal{M}_n(R)$ is torsion-free as a left R-module. Recalling that by the definition of a trace map we must have $\mathit{tr}(A) \cdot \mathbb{1}_n \subset \mathsf{Z}(A)$, it follows that each element $\alpha\mathbb{1}_n = \mathit{tr}(\alpha\boldsymbol{e}_i)\mathbb{1}_n$ must live in $\mathsf{Z}(A)$ and α necessarily belongs to each diagonal component A_j for all $j \in [\![1, n]\!]$. Since $i \in [\![1, n]\!]$ and $\alpha \in A_i$ were arbitrary, we have therefore proved that the diagonal components of A necessarily coincide with a commutative subalgebra $Z \subset R$ acting centrally on A, while for every matrix $a = \sum_{1\le i,j\le n} a_{i,j}\boldsymbol{e}_{i,j} \in A$ it holds that $\mathit{tr}(a) = \sum_{i=1}^n \mathit{tr}(a_{i,i}\boldsymbol{e}_i) = \sum_{i=1}^n a_{i,i}$. This completes the proof of the proposition, as well as the proof of Theorem 4.4. □

5. Matrix skew polynomial extensions

Generalizing the framework of the three preceding sections, our purpose is to examine the structure of matrix polynomial extensions over a skew polynomial ring. By an R-ring or a ring extension of R is meant any $\Bbbk$-algebra containing R as a $\Bbbk$-subalgebra. Given a finite set $\boldsymbol{x} = \{x_1, \ldots, x_m\}$ of m variables, we denote the set of all terms in $\boldsymbol{x}$ by:

$$\boldsymbol{T}(\boldsymbol{x}) = \boldsymbol{T}(x_1, \ldots, x_m) = \{\boldsymbol{x}^{\boldsymbol{r}} = x_1^{r_1} \cdots x_m^{r_m} \ : \ \boldsymbol{r} = (r_1, \ldots, r_m) \in \mathbb{N}^m\}.$$

5.1. A more complex structure of matrix skew polynomial extensions. It helps recalling from the general framework of [16, §3] the following concept.

Definition 5.1. An R-ring $R[x; \alpha, \delta]$, where x is a variable, $\alpha : R \longrightarrow R$ a *conjugation map* and $\delta : R \longrightarrow R$ an *α-derivation*, is called a (univariate) *skew polynomial extension* or an *Ore extension* of R provided,

❍ as left R-module, $R[x; \alpha, \delta]$ is (not necessarily freely) generated by the set $\boldsymbol{T}(x)$,

❍ the associative multiplication of $R[x; \alpha, \delta]$, explicitly written using a symbol, say, $\star$, is defined such that for every $k, l \in \mathbb{N}$ and $a \in R$ we have:

$$x^k x^l = x^{k+l}, \ a \star x^k = ax^k \text{ and the Ore-rule: } \ x \star a = \alpha(a)x + \delta(a).$$

A skew polynomial ring $R[x; \alpha]$ with a zero derivation is called a *graded skew polynomial ring*. When the variable x is algebraically independent over R, one recovers ordinary Ore extension [6]. One can iterate the above process to form *multivariate iterated skew polynomial rings*.

We start by illustrating on a simple example how the structure of a matrix skew polynomial algebra may be very hard to predict.

Example 5.2. Consider three independent variables x, y, t over $\Bbbk$, and let the set of elementary matrix-variables $\mathbb{X}$ be given by the labelled quiver:

$$\mathsf{Q}_{\mathbb{X}} : \begin{tikzcd} 1 \arrow[loop left, "x"] \arrow[r, bend left, "y"] & 2 \arrow[l, bend left, "x"] \arrow[loop right, "y"] \end{tikzcd}$$

(a) Over $R = \Bbbk[t]$, the matrix polynomial ring $R[\mathbb{X}]$ in commuting variables is neither left nor right noetherian; one easily computes:

$$R[\mathbb{X}] = R\left[\begin{smallmatrix} x & y \\ x & y \end{smallmatrix}\right] = \begin{pmatrix} R[x] + xyS & yS \\ xS & R[y] + xyS \end{pmatrix} \text{ with } \boldsymbol{S} = R[x, y] = \Bbbk[t, x, y].$$

a) Let [(b)] $\boldsymbol{R} = \Bbbk[t][x; \mathsf{id}, \frac{\mathrm{d}}{\mathrm{d}t}][y; \mathsf{id}, \frac{\mathrm{d}}{\mathrm{d}t}]$ be the iterated skew polynomial ring with trivial conjugation maps and where the derivations are given by the derivative with respect to the variable t. Then the matrix skew polynomial extension A of $\Bbbk[t]$, generated as subalgebra of $\mathcal{M}_2(\boldsymbol{R})$ by $\{\boldsymbol{e}_1, \boldsymbol{e}_2\} \cup \mathbb{X}$, is quickly checked to coincide with all of $\mathcal{M}_2(\boldsymbol{R})$; but this is something not directly captured by $\mathsf{Q}_{\mathbb{X}}$.

(c) Now choosing some $0 \neq e \in \mathbb{N}$ and $0 \neq c \in \Bbbk$, let the ground polynomial ring be a graded skew polynomial ring $\boldsymbol{R} = \Bbbk[x][y; \alpha]$, where α is the $\Bbbk$-algebra map with $\alpha(x) = cx^e + 1$. If $c = 1$ then $\boldsymbol{R}$ is simply the monoid ring extension of $\Bbbk$ by the non-commutative monoid $(\Gamma, \circ) = \langle x, y \,:\, yx = x^e y\rangle$; in general $\boldsymbol{R}$ is a *skew monoid ring extension* of $\Bbbk$ by Γ, see Definition 5.6 below. Let $\mathsf{A} = \Bbbk[\mathbb{X}; \alpha]$ be the matrix skew polynomial extension of $\Bbbk$ by $\mathbb{X}$. For instance, the cycle $1 \xrightarrow{y} 2 \xrightarrow{x} 1$ yields in A_1 the element: $y \star x = \alpha(x)y = (cx^e + 1)y$. One would naturally expect that A_1 be described as a skew monoid ring extension of $\Bbbk$ by the submonoid $\mathbb{T}_1 \subset \Gamma$ given by

$$\mathbb{T}_1 = \{1 = \underline{\boldsymbol{e}_1},\ \underline{\omega} = x_1 \circ \cdots \circ x_m \text{ with } \omega : 1 \xrightarrow{x_1} i_2 \cdots i_m \xrightarrow{x_m} 1 \text{ a cycle in } \mathsf{Q}_{\mathbb{X}}\}.$$

However when $c \in \Bbbk$ is not a zero-divisor, it appears that for every $0 \neq a \in \Bbbk$ and $1 \neq \tau \in \mathbb{T}_1$, the monomial $a\tau$ does not even live in A_1! For instance, by computing in A the products of elementary matrix-variables along all cycles $\omega \in \mathsf{Q}_{\mathbb{X}}(1,1)$ containing at most two occurrences of the variable y while not containing the only loop at 1, we see that:

$$y \star x = \alpha(x)y, y^2 \star x = \alpha^2(x)y^2 = (c\alpha(x)^e + 1)y^2, (y \star x)^2 = \alpha(x)\alpha^2(x)y^2,$$

indicating that for all $0 \neq a \in \Bbbk$ and $m \in \mathbb{N}$, none of the monomials $ax^m y, ax^m y^2$ lies in A_1. Actually it will require developing sophisticated tools for a rigorous proof that $a\tau \notin \mathsf{A}_1$ for every $0 \neq a \in \Bbbk$ and $1 \neq \tau \in \mathbb{T}_1$. This reveals how complex the structure of a diagonal component of A is!

5.2. Noetherianity of matrix skew polynomial extensions in one variable. With the above illustration, it's worth starting with a careful examination of the structure of matrix skew polynomial extensions A over a univariate skew polynomial ring $\boldsymbol{R} = R[x; \alpha, \delta]$. The diagonal components of A are special R-subrings in $\boldsymbol{R}$, and we definitely need to investigate some aspect of their arithmetic. As in Section 3, this subsection and the following one aim at investigating the noetherianity.

Since we allow $\boldsymbol{T}(x)$ to be only a (not necessarily free) left R-generating set for $\boldsymbol{R}$, we do not immediately have a well defined degree function over $\boldsymbol{R}$. However, we may still consider the following filtration: $\boldsymbol{R}_m = R_m[x; \alpha, \delta] = R\{x^s \,:\, 0 \leq s \leq m\}$ for $m \in \mathbb{N}$.

Definition 5.3. Let $m \in \mathbb{N}$. For each algebraic expression $\xi = \sum_{s=0}^{m} a_s x^s$ of an element $f \in \boldsymbol{R}$ with $a_0, \ldots, a_m \in R$ and $a_m \neq 0$, we loosely write:

$$\mathrm{lc}(f) = \mathrm{lc}(\xi) = a_m \text{ and } \mathrm{M}(f) = \mathrm{M}(\xi) = a_m x^m$$

for the *leading coefficient* and the *maximal* (or *leading*) *monomial* of the expression ξ.

Thus rigorously, the leading coefficient function and the maximal monomial function are defined not directly as maps over $\boldsymbol{R}$, but they are understood as well defined maps on algebraic '*expressions*'. We obtain the following theorem, extending the subbilateral noetherianity result of [16, Theorem 3.6] to R-subrings of $\boldsymbol{R}$.

Theorem 5.4. *Let $\boldsymbol{S} \subset R[x;\alpha,\delta]$ be any R-subring of a skew polynomial ring, such that x^p lies in $\boldsymbol{R}_{p-1} + \boldsymbol{S}$ for some positive $p \in \mathbb{N}$. Then, if R is right noetherian while α is surjective then every right $\boldsymbol{S}$-submodule of $\boldsymbol{R}$ is finitely generated. Likewise, if R is left noetherian while α is bijective then every left $\boldsymbol{S}$-submodule of $\boldsymbol{R}$ is finitely generated.*

Of course, this theorem already includes the statement that the given R-subring $\boldsymbol{S}$ is right noetherian whenever R is while α is surjective; and $\boldsymbol{S}$ is left noetherian if the same holds for R while α is bijective.

Proof. Let us denote by '$\circledast$' the multiplication of the graded skew polynomial ring $R[x;\alpha]$ associated with $\boldsymbol{R} = R[x;\alpha,\delta]$. Recall that we set $\boldsymbol{R}_m = \sum_{s=0}^{m} R x^s$. For all $a, b \in R$ and $m, m' \in \mathbb{N}$, the Ore rule of Definition 5.1 shows that

$$ax^m \star bx^{m'} = ax^m \circledast bx^{m'} + v = a\alpha^m(b)x^{m+m'} + v \text{ for some } v \in \boldsymbol{R}_{m+m'-1},$$

where one also sets $\boldsymbol{R}_{-1} = 0$; in particular, $\boldsymbol{R}_m \star \boldsymbol{R}_{m'} \subset \boldsymbol{R}_{m+m'}$. As an R-subring of $\boldsymbol{R}$, the ring $\boldsymbol{S}$ contains the coefficient algebra R. By assumption,

$$\text{we may let } 0 \neq p \in \mathbb{N} \text{ and } \xi = x^p + \zeta \in \boldsymbol{S} \text{ with } \zeta \in \boldsymbol{R}_{p-1}. \qquad (\star)$$

One could choose p as the smallest positive integer with the desired property, but this does not matter for the rest of the proof. The commutative monoid $\boldsymbol{T}(x^p)$ is obviously noetherian; by Euclidean division by p, the $\boldsymbol{T}(x^p)$-module $\boldsymbol{T}(x)$ is generated by the subset $\{1, x, x^2, \ldots, x^{p-1}\}$, hence $\boldsymbol{T}(x)$ is a noetherian $\boldsymbol{T}(x^p)$-module.

Let $0 \neq M \subset R[x;\alpha,\delta]$ be any left or right $\boldsymbol{S}$-submodule; assume that α is surjective for the right module context, or bijective for the left module context. In view of Definition 5.3 recall that: for every element $f \in \boldsymbol{R}$ given by an algebraic expression $f = ax^m + v$ with $0 \neq a \in R$ and $v \in \boldsymbol{R}_{m-1}$, we write:

$$\mathrm{lc}(f) = a \text{ and } \mathrm{M}(f) = ax^m.$$

We define the following filtration of M:

$$M_0 = M \cap R \text{ and } M_m = \{ax^m + v \in M \ :\ 0 \neq a \in R \text{ and } v \in \boldsymbol{R}_{m-1}\} \text{ for } 1 \leq m \in \mathbb{N}.$$

With M is associated a set $\mathrm{M}(M)$ of maximal monomials and a subset of terms $\mathsf{T} \subset \boldsymbol{T}(x)$:

$$\mathrm{M}(M) = \{\mathrm{M}(f) = ax^m \ :\ m \in \mathbb{N}, f = ax^m + v \in M \text{ for some } 0 \neq a \in R \text{ and } v \in \boldsymbol{R}_{m-1}\},$$

$$\mathsf{T} = \{x^m \ :\ m \in \mathbb{N} \text{ and } ax^m \in \mathrm{M}(M) \text{ for some } 0 \neq a \in R\}.$$

Then, T is a submodule of the noetherian $\boldsymbol{T}(x^p)$–module $\boldsymbol{T}(x)$. Indeed let $m \in \mathbb{N}$ with $x^m \in \mathsf{T}$; so there is some $f = ax^m + v \in M$ with $0 \neq a \in R$ and $v \in \boldsymbol{R}_{m-1}$. We get that:

- ○ when M is a right $\boldsymbol{S}$-module, then $f \star \xi = (ax^m + v) \star (x^p + \zeta) = ax^{m+p} + v' \in M$ with $v' = ax^m \star \zeta + v \star \xi \in \boldsymbol{R}_{m+p-1}$, showing that $x^{m+p} \in \mathsf{T}$;
- ○ when M is a left $\boldsymbol{S}$-module (while α is injective), then $\xi \star f = \alpha^p(a)x^{m+p} + v' \in M$ with $\alpha^p(a) \neq 0$ while $v' = (x^p \star ax^m - x^p \circledast ax^m) + x^p \star v + \zeta \star f$ rewrites as a polynomial in $\boldsymbol{R}_{m+p-1}$, hence $x^{m+p} \in \mathsf{T}$.

Next, since the $\boldsymbol{T}(x^p)$–module $\boldsymbol{T}(x)$ is noetherian, its $\boldsymbol{T}(x^p)$-submodule T is generated by a finite subset $\Sigma = \{x^{p_1}, \ldots, x^{p_r}\} \subset \mathsf{T}$ for some positive $r \in \mathbb{N}$ and some integers $0 \leq p_1 < \cdots < p_r$, pairwise non congruent modulo p. We have now gathered enough facts to adapt the generalized proof about the (subbilateral) noetherianity of the skew polynomial ring $\boldsymbol{R} = R[x;\alpha,\delta]$ ([16, Theorem 3.6]) to the case of its R-subring $\boldsymbol{S}$.

Starting with the right module context, let $M \subset \boldsymbol{R}$ be any right $\boldsymbol{S}$-submodule. Assuming that the map $\alpha : R \longrightarrow R$ is surjective and choosing a right inverse map (not necessarily a ring morphism) $\alpha' : R \longrightarrow R$ with $\alpha\alpha' = \mathsf{id}$, we have to prove that M is finitely generated. For each $k \in [\![1, r]\!]$ and every $l \in \mathbb{N}$, we define:

$$\mathfrak{C}_{k,l} = \{\mathrm{lc}(f) = a \ :\ a \neq 0 \text{ and } f \in (ax^{p_k+lp} + \boldsymbol{R}_{p_k+lp-1}) \cap M\}R \subset R. \tag{†}$$

For a fixed k, since T is a $\boldsymbol{T}(x^p)$-submodule of $\boldsymbol{T}(x)$ we get an increasing sequence of right ideals: $\mathfrak{C}_{k,0} \subset \mathfrak{C}_{k,1} \subset \cdots \subset \mathfrak{C}_{k,l} \subset \mathfrak{C}_{k,l+1} \cdots$, which by the right noetherianity of the ring R must stabilize. So it holds for some $m_k \in \mathbb{N}$ that $\mathfrak{C}_{k,l} = \mathfrak{C}_{k,m_k}$ for every integer $l \geq m_k$. Still by the right noetherianity of R, for each $l \in [\![0, m_k]\!]$ there is a positive $n_{k,l} \in \mathbb{N}$ together with a finite subset $F_{k,l} = \{f_{k,l,s} \ :\ 1 \leq s \leq n_{k,l}\} \subset M_{p_k+lp}$ such that $\mathfrak{C}_{k,l} = \{\mathsf{c}_{k,l,s} = \mathrm{lc}(f_{k,l,s}) \ :\ 1 \leq s \leq n_{k,l}\}R$. By construction, $M = \cup_{m\in\mathbb{N}} M_m$; we will show that the finite set $F = \cup_{k=1}^{r} \cup_{s=0}^{m_k} F_{k,s}$ is a generating set for the right $\boldsymbol{S}$-module M. To this end, given $m \in \mathbb{N}$ and any $0 \neq f \in M_m$, we shall proceed by induction on m to prove that f lies in $F \star \boldsymbol{S}$.

i) The base case '$m = 0$'. Then $f \in M_0 = M \cap R$ is a non-zero constant polynomial; so $1 = x^0 \in T$ and forcibly $p_1 = 0$ and since $\mathfrak{C}_{0,0} = F_{0,0}R \subset R$, it follows that f already lies in $F_{0,0}R \subset F \star \boldsymbol{S}$.

Moving to the induction step, suppose that $m \geq 1$ and that $M_{m'} \subset F \star \boldsymbol{S}$ for all $0 \leq m' \leq m-1$. There are $k \in [\![1, r]\!]$ and $l \in \mathbb{N}$ with:

$$m = p_k + lp \text{ and } f = ax^m + v \text{ for some } 0 \neq a = \mathrm{lc}(f) \in R \text{ and } v \in \boldsymbol{R}_{m-1}.$$

We distinguish the two following cases and recall that we write '$\circledast$' for the multiplication of the graded skew polynomial extension $R[x;\alpha]$.

ii) The case '$0 \leq l \leq m_k$'. Here, $F_{k,l} \subset F$, $a \in \mathfrak{C}_{k,l}$ and for some $a_s \in R$, $1 \leq s \leq n_{k,l}$, we get:

$$a = \sum_{s=1}^{n_{k,l}} \mathsf{c}_{k,l,s} a_s \text{ and } \mathrm{M}(f) = \sum_{s=1}^{n_{k,l}} \mathsf{c}_{k,l,s} a_s x^m = \sum_{s=1}^{n_{k,l}} \mathrm{M}(f_{k,l,s}) \circledast \alpha'^m(a_s).$$

Thus the polynomial $h = f - \sum_{s=1}^{n_{k,l}} f_{k,l,s} \star \alpha'^m(a_s)$ lies in M and rewrites as an element in $M_{m'}$ for some $m' \in [\![0, m-1]\!]$. So the induction hypothesis shows that $h \in F \star \boldsymbol{S}$, so that, $f = h + \sum_{s=1}^{n_{k,l}} f_{k,l,s} \star \alpha'^m(a_g)$ lies in $F \star \boldsymbol{S}$ as well.

iii) The case '$l > m_k$'. Here, $a \in \mathfrak{C}_{k,l} \subset \mathfrak{C}_{k,m_k}$ and for some $a_s \in R$, $1 \leq s \leq m_k$, it holds that $a = \sum_{s=1}^{n_{k,m_k}} \mathsf{c}_{k,m_k,s} a_s$, (and computing in $R[x;\alpha]$) we get:

$$\begin{aligned}\mathrm{M}(f) &= \sum_{s=1}^{n_{k,m_k}} \mathsf{c}_{k,m_k,s} a_s x^{p_k + m_k p} x^{(l-m_k)p} \\ &= \sum_{s=1}^{n_{k,m_k}} \mathsf{c}_{k,m_k,s} x^{p_k+m_k p} \circledast \alpha'^{p_k+m_k p}(a_s) x^{(l-m_k)p} \\ &= \sum_{s=1}^{n_{k,m_k}} \mathrm{M}(f_{k,m_k,s}) \circledast \alpha'^{p_k+m_k p}(a_s) x^{(l-m_k)p}.\end{aligned}$$

In view of ($\star$), recall that $\xi = x^p + \zeta \in \boldsymbol{S}$ with $\zeta \in \boldsymbol{R}_{p-1}$; so $\xi^{l-m_k} \in \boldsymbol{S} \cap \boldsymbol{R}_{(l-m_k)p}$ and $\mathrm{M}(\xi^{l-m_k}) = x^{(l-m_k)p}$. Hence letting $h = f - \sum_{s=1}^{n_{k,m_k}} f_{k,m_k,s} \star \alpha'^{p_k+m_k p}(a_s)\xi^{l-m_k}$ gives a rightful element of M which rewrites as a polynomial in $M_{m'}$ for some $m' \in [\![0, m-1]\!]$. The induction hypothesis shows that h lies in $F \star \boldsymbol{S}$, thus the polynomial $f = h + \sum_{s=1}^{m_k} f_{k,m_k,s} \star \alpha'^{p_k+m_k p}(a_s)\xi^{l-m_k}$ lives in $F \star \boldsymbol{S}$ as well.

This finishes the proof that every right $\boldsymbol{S}$-submodule of $\boldsymbol{R}$ is finitely generated.

Turning to the statement of the theorem on left modules, we use the following strategy to convert it to a statement about right modules. Here the specific assumption is that the conjugation map $\alpha : R \longrightarrow R$ is an automorphism, so we let $\alpha^{-1} : R \longrightarrow R$ for its inverse. For every $\Bbbk$-algebra B with ring multiplication written as '$\star$'; the opposite algebra B° is still the $\Bbbk$-module B whose ring multiplication written as '$\star^\circ$' is given by: $a \star^\circ b = b \star a$ for all $a, b \in B$. Now the opposite ring of $\boldsymbol{R} = R[x; \alpha, \delta]$ is checked to coincide with the univariate skew polynomial extension $\boldsymbol{R}^\circ = R^\circ[x; \alpha^{-1}, -\delta\alpha^{-1}]$. Hence for every left $\boldsymbol{S}$-submodule $M \subset \boldsymbol{R}$, that is, M is a right $\boldsymbol{S}^\circ$-submodule of $\boldsymbol{R}^\circ$, the first part of the theorem yields that M is a finitely generated right $\boldsymbol{S}^\circ$-module, which means that M is a finitely generated left $\boldsymbol{S}$-module. □

Corollary 5.5. *Let $\mathsf{A} = R\langle \boldsymbol{E}, \mathbb{X}; \alpha, \delta\rangle$ be any matrix skew polynomial extension over a univariate skew polynomial ring $\boldsymbol{R} = R[x; \alpha, \delta]$. Then A is left noetherian if R is left noetherian while α is bijective and for all $i, j \in [\![1, n]\!]$ with $\mathsf{Q}_{\mathbb{X}}(i, j), \mathsf{Q}_{\mathbb{X}}(j, j) \neq \varnothing$ it also holds that $\mathsf{Q}_{\mathbb{X}}(i, i) \neq \varnothing$. A is right noetherian if R is right noetherian while α is surjective and for all $i, j \in [\![1, n]\!]$ with $\mathsf{Q}_{\mathbb{X}}(i, i), \mathsf{Q}_{\mathbb{X}}(i, j) \neq \varnothing$ it also holds that $\mathsf{Q}_{\mathbb{X}}(j, j) \neq \varnothing$.*

Proof. Arbitrarily let $i, j \in [\![1, n]\!]$ with $j \neq i$. Given the corresponding assumption for the statement about the one-sided (left or right) noetherianity, by virtue of Lemma 3.1 we have to prove that each diagonal component A_i is one-sided noetherian while each bilateral A_i-A_j-module $\mathsf{A}_{i,j}$ is finitely generated as a one-sided module. Each A_i is an R-subring of $\boldsymbol{R} = R[x; \alpha, \delta]$. If $\mathsf{Q}_{\mathbb{X}}$ contains no cycle at i, then A_i coincides with the coefficient algebra R, otherwise, A_i contains a positive power of the variable x. Thus, Theorem 5.4 applies showing that A_i is left noetherian (or right noetherian) if the same holds for R while the conjugation map is bijective (or resp., surjective).

Next, each $\mathsf{A}_{i,j}$ is a bilateral A_i-A_j-submodule of $\boldsymbol{R}$, to which is attached (just as in the proof of Theorem 5.4) the following subset of terms in $\boldsymbol{T}(x)$:

$$\begin{aligned} \mathsf{T}_{i,j} = \mathsf{T}(\mathsf{A}_{i,j}) &= \{x^m \ : \ \exists a \in R\backslash\{0\} \text{ with } ax^m \in \mathsf{A}_{i,j} + \boldsymbol{R}_{m-1}\} \\ &\supseteq \{\underline{\omega} \ : \ \omega \in \mathsf{Q}_{\mathbb{X}}(i, j)\}, \end{aligned}$$

where $\boldsymbol{R}_m = \sum_{s=0}^m Rx^s$ for every $m \in \mathbb{N}$. We have the two following exclusive cases to consider.

i) *The case that $\mathsf{Q}_{\mathbb{X}}(i, j)$ is finite.* Let m be the maximal length of paths in $\mathsf{Q}_{\mathbb{X}}(i, j)$. Then $\mathsf{T}_{i,j} \subset \{1, x, x^2, \ldots, x^m\}$, and $\mathsf{A}_{i,j}$ is a submodule of the bilateral R-submodule $\boldsymbol{R}_m$ generated as a one-sided R-module by the finite set $\{1, x, x^2, \ldots, x^m\}$. Thus when R is left or right noetherian, then so is $\boldsymbol{R}_m$ and consequently $\mathsf{A}_{i,j}$ is already finitely generated as a left (or resp., as a right) R-module.

ii) *The case that* $\mathsf{Q}_{\mathbb{X}}(i,j)$ *is infinite.* Since $i \neq j$, the set $\mathsf{Q}_{\mathbb{X}}(i,j)$ contains a non-simple path. For the statement of the corollary about left noetherianity, the hypothesis on $\mathbb{X}$ yields that A_i contains a positive power of the variable x; hence Theorem 5.4 applies showing that $\mathsf{A}_{i,j}$ is a finitely generated left A_i-module. And for the statement of the corollary about right noetherianity, the hypothesis on $\mathbb{X}$ yields that A_j contains a positive power of the variable x; once again Theorem 5.4 applies showing that $\mathsf{A}_{i,j}$ is a finitely generated right A_j-module.

This completes the proof of the corollary. □

5.3. Multivariate matrix skew polynomial extensions by a free commutative term-ordered monoid. There seems to be no way one can iterate the strategy of Corollary 5.5 without imposing stronger conditions on $\mathbb{X}$ than it is necessary in the case of the matrix polynomial ring $R[\boldsymbol{E},\mathbb{X}]$ in commuting variables. This is not a surprise because (as pointed out just before Theorem 3.12), even a monoid ring extension of a noetherian commutative algebra by a left noetherian monoid S needs not be left noetherian. However the obstructions may be overcome over a division $\mathbb{k}$-algebra when S also has nice combinatorial properties; we shall record this last fact for matrix skew polynomial extensions. For our purpose, we need to recall the following notions.

Definition 5.6 ([16, Definition 4.8, Note 4.9])**.** (a) A term-ordering on a monoid Γ is a strict well-ordering '$<$' on Γ such that for all $\lambda,\lambda',\tau \in \Gamma$, the following implication holds:

$$\lambda < \lambda' \implies \tau\lambda < \tau\lambda' \text{ and } \lambda\tau < \lambda'\tau.$$

One denotes by '$\leq$' the associated large ordering, and calls Γ (or more precisely, $(\Gamma,<)$) a term-ordered monoid.

(b) A (not necessarily graded) skew monoid ring extension of R by a term-ordered monoid $(\Gamma,<)$ is any R-ring $\boldsymbol{R}$ freely generated as a left R-module by Γ, and the induced *maximal term-function* $\mathsf{T} : \boldsymbol{R}\backslash\{0\} \longrightarrow (\Gamma,<)$ satisfies the property that

$$\text{for all } f,g \in \boldsymbol{R} \text{ with } fg \neq 0,\ \ \mathsf{T}(fg) \leq \mathsf{T}(f)\mathsf{T}(g),$$

where the term $\mathsf{T}(f)$ is the smallest $\tau \in \Gamma$ with $f \in R\{\lambda \in \Gamma : \lambda \leq \tau\}$. Obviously, for all $f,g \in \boldsymbol{R}$ with $f,g,f+g \neq 0$, it holds that $\mathsf{T}(f+g) \leq \max(\mathsf{T}(f),\mathsf{T}(g))$. Hence, the maximal term-function is an instance of a *pseudo-valuation* on $\boldsymbol{R}$. Correspondingly there are automatically a *leading coefficient* map lc and a *maximal monomial* map M such that for all $0 \neq f \in \boldsymbol{R}$,

$$\mathrm{M}(f) = \mathrm{lc}(f)\mathsf{T}(f), \text{ and } f - \mathrm{M}(f) \text{ is either zero or } \mathsf{T}(f - \mathrm{M}(f)) < \mathsf{T}(f).$$

Further unfolding the very compact definition given by point (b), the multiplication of $\boldsymbol{R}$, written as '$\star$', is described by the following lines. Associated with the pseudo-valuation $\mathsf{T} : \boldsymbol{R}\backslash\{0\} \longrightarrow (\Gamma, <)$, there is for every $\lambda \in \Gamma$: a conjugation map $\alpha_\lambda : R \longrightarrow R$, a twist map $\mathfrak{q}_\lambda : \Gamma \longrightarrow \boldsymbol{R}$, $\tau \longmapsto \mathfrak{q}_{\lambda,\tau}$ and a derivation map $\delta_\lambda : \boldsymbol{R} \longrightarrow \boldsymbol{R}$, satisfying the following *Ore-like rule* for $a \in R$ and $\tau \in \Gamma$:

$$\begin{aligned} &\lambda \star a = \alpha_\lambda(a)\lambda + \delta_\lambda(a) \text{ and } \lambda \star \tau = \mathfrak{q}_{\lambda,\tau} \cdot \lambda\tau + \delta_\lambda(\tau), \\ &\text{with } \mathsf{T}(\delta_\lambda(a)) < \lambda \text{ and } \mathsf{T}(\delta_\lambda(\tau)) < \lambda\tau. \end{aligned} \tag{5.1}$$

One henceforth writes: $\boldsymbol{R} = R[\Gamma; \boldsymbol{\alpha}, \mathfrak{q}, \boldsymbol{\delta}]$, with $\boldsymbol{\alpha} = (\alpha_\lambda)_{\lambda\in\Gamma}$, $\boldsymbol{\delta} = (\delta_\lambda)_{\lambda\in\Gamma}$ and $\mathfrak{q} = (\mathfrak{q}_\lambda)_{\lambda\in\Gamma}$. Refer to each $\mathfrak{q}_{\lambda,\tau}$ as a *twist coefficient.*

Fix a term-ordering '$<$' on $(\boldsymbol{T}(\boldsymbol{x}), \cdot)$ and let our ground ring $\boldsymbol{R} = R[\boldsymbol{T}(x); \boldsymbol{\alpha}, \mathfrak{q}, \boldsymbol{\delta}]$ be a skew polynomial extension of R by the free commutative term-ordered monoid $(\boldsymbol{T}(\boldsymbol{x}), \cdot, <)$. Let us mention that this setting includes (and is clearly not limited to) Kredel-solvable polynomial rings [9,10], [15, §49.4.2], which in their turn, include interesting algebras from Geometry such as Weyl algebras and their quantized versions.

As usual also fix a seed $(\boldsymbol{E}, \mathbb{X})$ with associated labelled quiver $\mathsf{Q}_{\mathbb{X}}$. Here, $\boldsymbol{E}$ is a multiplicatively closed subset of the canonical basis $\mathbb{E}$ of $\mathcal{M}_n(R)$ containing all the elementary idempotents matrices, and $\mathbb{X} \subset \{x\boldsymbol{e}_{i,j} \ : \ x \in \boldsymbol{x},\ 1 \leq i, j \leq n\}$ is an $\boldsymbol{E}$-saturated set of elementary matrix-variables. Thus we can form the matrix skew polynomial algebra extension $\mathsf{A} = R\langle \boldsymbol{E}, \mathbb{X}; \boldsymbol{\alpha}, \mathfrak{q}, \boldsymbol{\delta}\rangle$ of R by the seed $(\boldsymbol{E}, \mathbb{X})$ over $\boldsymbol{R}$ or over the term-ordered monoid $(\boldsymbol{T}(\boldsymbol{x}), \cdot, <)$: A is an R-subring of the full matrix ring $\mathcal{M}_n(\boldsymbol{R})$ generated by $\boldsymbol{E} \cup \mathbb{X}$. We continue to denote by $\mathbb{T}$ the set of elementary matrix-terms (over the commutative monoid $(\boldsymbol{T}(\boldsymbol{x}), \cdot)$): so for all $i, j \in [\![1, n]\!]$,

$$\mathbb{T}_{i,j} = \{\underline{\omega} = x_1 \cdots x_m \text{ with } \omega : i \xrightarrow{x_1} i_2 \cdots i_m \xrightarrow{x_m} j \text{ a path in } \mathsf{Q}_{\mathbb{X}}\}.$$

Recall that $\mathbb{T}_i^{\mathrm{s}}$ is the submonoid of $\mathbb{T}_i$ generated by terms along simple cycles at i. We have the following extension of Theorem 3.12 to the present setting when the coefficient algebra is a division ring.

Theorem 5.7. *Let* $\mathsf{A} = R\langle \boldsymbol{E}, \mathbb{X}; \boldsymbol{\alpha}, \mathfrak{q}, \boldsymbol{\delta}\rangle$ *be a matrix skew polynomial extension of a division* $\Bbbk$*-algebra* R *over the commutative term-ordered monoid* $(\boldsymbol{T}(\boldsymbol{x}), \cdot, <)$*, such* *that each conjugation map is bijective and each twist coefficient is non-zero. Then* A *is left (right, or bilateral) noetherian provided* $\mathbb{X}$ *is finite and for all* $i, j \in [\![1, n]\!]$ *and any simple cycle* $\sigma \in \mathsf{Q}_{\mathbb{X}}(k, k)$ *with* $\mathsf{Q}_{\mathbb{X}}(i, k), \mathsf{Q}_{\mathbb{X}}(k, j) \neq \varnothing$*, a positive power of* $\underline{\sigma}$ *lives in* $\mathbb{T}_i^{\mathrm{s}}$ *(or resp., in* $\mathbb{T}_j^{\mathrm{s}}$ *or* $\mathbb{T}_i^{\mathrm{s}} \cdot \mathbb{T}_j^{\mathrm{s}}$*).*

Proof. We write the proof only for the statement about left noetherianity, leaving the other cases to the reader. Here it shall be sufficient that the conjugation maps are only injective. By virtue of Lemma 3.1, arbitrarily given $i, j \in [\![1, n]\!]$ and any left A_i-submodule $M \subset \mathsf{A}_{i,j}$, it suffices to show that M is finitely generated. First notice that for any non-zero polynomial $f \in M$ with $\mathrm{M}(f) = a\tau$ where $0 \neq a = \mathrm{lc}(f) \in R$ and $\tau = \mathsf{T}(f)$, one gets a monic polynomial $a^{-1}f \in M$ with $\mathrm{M}(f) = \tau$. Next, since by hypothesis each conjugation map $\alpha_\lambda : R \longrightarrow R$ is injective and each twist coefficient $\mathfrak{q}_{\lambda,\tau}$ is non-zero for all $\lambda, \tau \in \boldsymbol{T}(\boldsymbol{x})$, the Ore-rule (5.1) yields that for all $0 \neq f, g \in \boldsymbol{R}$ with $\mathrm{M}(f) = a\lambda$ and $\mathrm{M}(g) = b\tau$: $\mathrm{M}(f \star g) = a\alpha_\lambda(b)\,\mathfrak{q}_{\lambda,\tau}\,\lambda\tau$ and $\mathsf{T}(f \star g) = \mathsf{T}(f)\mathsf{T}(g) = \lambda\tau$. In particular for any path $\omega : i \xrightarrow{x_1} i_2 \cdots i_m \xrightarrow{x_m} j$ from i to j in $\mathsf{Q}_{\mathbb{X}}$, we have:

$$\mathsf{T}(x_1 \star x_2 \star \cdots \star x_m) = x_1 \cdots x_m = \underline{\omega}.$$

Hence the monoid $\mathbb{T}_i = \{\underline{\omega} = x_1 \cdots x_m \text{ with } \omega : i \xrightarrow{x_1} i_2 \cdots i_m \xrightarrow{x_m} i \text{ a path in } \mathsf{Q}_{\mathbb{X}}\}$ consists of all the leading terms $\mathsf{T}(f)$ for $0 \neq f \in \mathsf{A}_i$, while the set $T = \{\mathsf{T}(f) : 0 \neq f \in M\}$ is forcibly a $\mathbb{T}_i$-submodule in $\mathbb{T}_{i,j}$. As illustrated by Example 5.2(b), one is aware that $\mathbb{T}_i$ needs not be contained in A_i at all, the latter is only an R-subring of the skew polynomial subextension $R[\mathbb{T}_i; \boldsymbol{\alpha}, \mathfrak{q}, \boldsymbol{\delta}] \subset \boldsymbol{R}$. The hypothesis on the shape of $\mathbb{X}$ is the precise condition ensuring by virtue of Theorem 3.12 the left noetherianity of the matrix polynomial ring $R[\boldsymbol{E}, \mathbb{X}]$ in commuting variables. Thus $\mathbb{T}_i$ is a noetherian commutative monoid while $\mathbb{T}_{i,j}$ is a finitely generated left $\mathbb{T}_i$-module, so that, the $\mathbb{T}_i$-submodule $T \subset \mathbb{T}_{i,j}$ must be as well generated by a finite set $\mathsf{T}(F) = \{\tau_s = \mathsf{T}(f_s) = \mathrm{M}(f_s) : 1 \leq s \leq N\}$ for some finite subset $F = \{f_1, \ldots, f_N\} \subset M \backslash \{0\}$ consisting of monic polynomials. We claim that F generates the A_i-module M. Assuming the contrary, the set $\{\mathsf{T}(f) : f \in M \backslash (\mathsf{A}_i \star F)\}$ would be a non-empty subset of the term ordered monoid $(\boldsymbol{T}(\boldsymbol{x}), \cdot, <)$, and so, it must contain a smallest element $\tau = \mathsf{T}(f) = \mathrm{M}(f)$ for some monic polynomial $f \in M \backslash (\mathsf{A}_i \star F)$. But for some $s \in [\![1, N]\!]$ and $\lambda \in \mathbb{T}_i$ it holds that $\mathsf{T}(f) = \lambda\tau_s$. Letting $u \in \mathsf{A}_i$ be a monic polynomial with $\mathrm{M}(u) = \lambda$, the polynomial $h = f - \mathfrak{q}_{\lambda,\tau_s}^{-1} u \star f_s$ still belongs to $M \backslash (\mathsf{A}_i \star F)$ because so does f while $\mathfrak{q}_{\lambda,\tau_s}^{-1} u \star f_s \in \mathsf{A}_i \star F$. But then we see that:

$$\mathrm{M}(\mathfrak{q}_{\lambda,\tau_s}^{-1} u \star f_s) = \mathfrak{q}_{\lambda,\tau_s}^{-1} \mathrm{M}(\lambda \star \tau_s) = \mathfrak{q}_{\lambda,\tau_s}^{-1} \mathfrak{q}_{\lambda,\tau_s} \lambda\tau_s = \mathrm{M}(f),$$

showing that $\mathsf{T}(h) < \mathsf{T}(f) = \tau$ and yielding a contradiction to the minimality of τ. Hence, the left A_i-submodule $M \subset \mathsf{A}_i$ is generated by F and this completes the proof that A is left noetherian. □

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