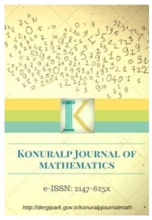
Konuralp Journal of Mathematics, 10 (2) (2022) 355-367

Konuralp Journal of Mathematics

Research Paper

Journal Homepage: *www.dergipark.gov.tr/konuralpjournalmath*

e-ISSN: 2147-625X



Extreme-Fixed Points of the Vandermonde Polynomial when Optimized over p-Normed Surfaces Defined by Univariate Polynomials

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Abstract

The concept of extreme-fixed point is coined from the facts of zeros of polynomials, extreme/optimal points on optimization of a given function and fixed points of a given continuous function. In this article, we establish the close relations between zeros, extreme, fixed points, and also what we define as extreme-fixed points. We illustrate this result with the Vandermonde polynomial (or determinant) when optimized over a given $p-$norm surface expressed by univariate polynomial(s). It is further, established that indeed the coordinates of the extreme-fixed points on such a surface like a p-sphere are given as roots of some classical orthogonal polynomials.

Keywords: *Vandermonde Determinant, Orthogonal Polynomials, Extreme Points, Fixed Points, p-Norm.*

2010 Mathematics Subject Classification: *15A60, 33C52, 65F35.*

1. Introduction and Preliminaries

The Vandermonde matrix is one of the most famously used matrix in both mathematical sciences and numerical analysis. It is named after Alexandre Théophile Vandermonde (1735-1796), [1, 30] and occurs in various applications mainly in curve-fitting using splines [33], D-optimal design [9, 14] random matrix theory [20], charge distribution [7, 12] and sphere parking [6, 10, 32]. Szegő [38] proved in one of his results that the coordinate points obtained when the Vandermonde polynomial is optimized on a sphere given by the zeros of the Hermite classical orthogonal polynomial and these were referred to as extreme points. Otherwise, the points obtained by global optimization of the Vandermonde determinant are famously known as Fekete points [8]. More other related studies on Fekete points can be obtained in [27, 31, 37, 39]. The concept of extreme points of Vandermonde determinant and related applications have been investigated in various fields including modelling power exponential functions [16, 17, 18, 19], optimization of density functions of sphere and applications to financial mathematics [21, 22], general algebraic properties in symmetric functions [23, 24, 25, 26], Jordan algebras and application to symmetric cones [27, 28, 29]. In the recent past there has been a study of approximating solutions of matrix equations via fixed point techniques [35]. In this study, we will consider an $n \times n$ square Vandermonde matrix with n-tuple entries
$\mathbf{x} = (x_1, \cdots, x_n)$ in $\mathbb{R}^{\ltimes}$ of the form $x_{ij} = \left\{x_i^{j-1}\right\}_{i,j=1}^{n}$ so that

$$V_n(\mathbf{x}) = \begin{bmatrix} 1 & x_1 & \cdots & x_1^{n-1} \\ 1 & x_2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & \cdots & x_n^{n-1} \end{bmatrix}. \tag{1.1}$$

In many applications, it is not the Vandermonde matrix that is most useful, but its multivariate polynomial given by the determinant of the matrix in (1.1). The determinant of the Vandermonde matrix is often called the Vandermonde determinant, or simply referred to as the Vandermonde polynomial or Vandermondian [32]. The determinant of the $n-$square Vandermonde matrix can be written using an exceptionally simple formula as the product of the $\frac{1}{2}n(n-1)$ difference factors, $x_i - x_j$ with $x_i \neq x_i$ for two distinct points $x_i, x_j \in \mathbf{x}$, [38].

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Theorem 1.1 ([32, 38]). *The Vandermonde determinant, $v_n(x_1,\dots,x_n)$, is given by*

$$v_n(x_1,\dots,x_n) = \prod_{1\le i<j\le n}(x_j - x_i). \tag{1.2}$$

The proof of Theorem 1.1 has been provided in various texts as related to interpolation and approximation [8], optimization and optimal nodes [19, 31], determinants and applications to mathematical physics [32, 37], orthogonal polynomials [38] and algorithm for computing Fekete points [39].
In this study, we investigate the connection between the fixed points of a given polynomial function, its roots or zeros and the coordinate points when optimized over a given surface. We shall define these points as extreme-fixed points. The method of optimization is employed by use of Lagrange multipliers. We also consider the zeros of some classical orthogonal polynomials. Then we connect these aspect with fixed point particularly on the unit sphere.

2. Vandermonde Metric induced by p-norms

This section illustrate the Vandermone determinant as a metric or distance measure expressed by a polynomial in several variables. This is to further, connect the Vandermonde determinant to the concept of contraction and fixed point.
Considering the polynnomial or determinantal expression given in (1.2) and for brevity, we adopt the following simple notations:

$$\begin{aligned} &v_n(\mathbf{x}) := v(\mathbf{x}) \text{ to denote the n-dimension determinant} \\ &(x,y) := \{(x_i,x_j) : \text{for all } x_i,x_j \in \mathbf{x},\ \ i\neq j, \text{ and } i,j=1,2,\dots,n\} \end{aligned} \tag{2.1}$$

so that the determinant equation in (1.2) can be simply expressed as

$$v(\mathbf{x}) \equiv p(x,y) = \prod_{x\neq y}|x-y|. \tag{2.2}$$

Taking logarithms in (2.2), we obtain

$$f(\mathbf{x}) \equiv \log[v(\mathbf{x})] = \sum_{x\neq y}|x-y|. \tag{2.3}$$

The difference factors $|x-y|$ in both the product and summation parts of (2.2) and (2.3)
respectively represent a distance measure of the form

$$d(x,y) = |x-y|, \text{ for all } x\neq y,\ x,y\in\mathbf{x}, \tag{2.4}$$

and the points (x,y) are as defined in (2.1).
The distance measure in (2.4) is indeed a metric space.

Definition 2.1 ([15], Metric space, metric). *A metric space is a pair (X,d), where X is a set and d is a metric on X (or distance function on X), that is, a function defined as $d: X\times X \to R_+$ such that for all $x,y,z\in X$ we have:*

$$\begin{aligned} &M_1 : d(x,y)\ge 0,\ \ \textit{is real, finite, nonnegative}, && \textit{(positvity)} \\ &M_2 : d(x,y) = 0,\ \ \iff\ \ x=y, && \textit{(definiteness)} \\ &M_3 : d(x,y) = d(y,x), && \textit{(symmetry)} \\ &M_4 : d(x,y)\le d(x,z)+d(z,y) && \textit{(Triangular inequality)} \end{aligned}$$

where d is the distance measure defined in (2.4).

Combining the ideas expressed in (2.2), (2.3), (2.4) and Definition 2.1, leads us to our first important result on the Vandermonde polynomial metric space as stated in the Lemma 2.2 below.

Lemma 2.2. *The space of Vandermonde determinants (or polynomials) defined in (1.2) forms metric space (X,d), where $X\subset\mathbb{R}^n$ is a set and d is a distance measure as given in (2.4).*

Proof. We will show that the space of functions represented by $v(\mathbf{x})$ and $f(\mathbf{x})$ in (2.2) and (2.3) respectively equipped with distance measure given in (2.4) satisfy metric properties $\mathrm{M}_1,\mathrm{M}_2,\mathrm{M}_3$ and M_4.
For M_1, taking $n=2$ so that with $x_i\neq x_j$ for all $i\neq j$ (2.3) and (x,y) as defined in (2.1). gives

$$\sum_{x\neq y}|x-y| \equiv \sum_{1\le i<j\le 2}|x_i-x_j| = |x_1-x_2| = d(x_1,x_2)\ge 0. \tag{2.5}$$

Proceeding in a similar way in higher dimension leads to M_1, that is, $d(x,y)\ge 0$.
For M_2, using (2.5) with $x=y$ where (x,y) are as defined in (2.1), so that $x_i=x_j$ for all $i,j=1,2$ gives

$$\sum_{x\neq y}|x-y| \equiv \sum_{1\le i<j\le 2}|x_i-x_i| = |x_1-x_1| = d(x_1,x_1) = 0.$$

Continuing this way leads to the result of M_2, that is, $d(x,y)=0,\ \iff\ x=y$.

For M_3, taking $n = 2$ so that with $x_i \neq x_j$ for all $i \neq j$ (2.3) and (x, y) as defined in (2.1), gives

$$\sum_{x \neq y} |x-y| \equiv \sum_{1 \le i < j \le 2} |x_i - x_j| = |x_1 - x_2| = d(x_1, x_2) = d(x, y)$$
$$\sum_{x \neq y} |y-x| \equiv \sum_{1 \le i < j \le 2} |x_j - x_i| = |x_2 - x_1| = d(x_2, x_1) = d(y, x)$$

This proves M_3, that is, $d(x,y) = d(y,x)$.
For M_3, taking $n = 3$ and set $x = x_i, y = x_j, z = x_k$ so that with $x_i \neq x_j$ for all $i \neq j$, then from (2.3), we have that

$$\begin{aligned}
\sum_{x \neq y} |x-y| &= \sum_{x \neq y} |(x-z)+(z-y)| \\
&\le \sum_{x \neq z} |(x-z)| + \sum_{z \neq y} |(z-y)| \\
\sum_{1 \le i < j \le 3} |x_i - x_j| &\le \sum_{1 \le i < k \le 2} |x_i - x_k| + \sum_{2 \le k < j \le 3} |x_k - x_j| \\
d(x_1, x_3) &\le d(x_1, x_2) + d(x_2, x_3).
\end{aligned}$$

Thus, it follows that M_3 is satisfied, that is, $d(x_1, x_3) \le d(x_1, x_2) + d(x_2, x_3)$.
With all the properties M_1, M_2, M_3 and M_4 of the metric satisfied, hence proved. □

Since the completeness of Vandernode determinant (or polynomial) is not entirely guaranteed [15]. Thus, the completeness can be attained by looking at the powers, $p \ge 1$, of the determinant so as to able to have a variety of applications. Based on the concepts the ℓ^p to be able to characterize the Vandermonde $p-$ induced normed space.

Definition 2.3 ([15], p-norm). *The p- norm of the vector space $\mathbf{x} \in \mathbb{R}^n$ also denoted by $\|\mathbf{x}\|_p$ is defined as*

$$\|\mathbf{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}, \quad \text{for} \quad p \ge 1. \tag{2.6}$$

Let $p \ge 1$ be a fixed real number, then by Definition 2.3, each element in the space ℓ^p is a sequence

$$\mathbf{x} = (x_1, x_2, \ldots, x_n)$$

with $0 < x_i < 1$, such that

$$|x_1|^p + |x_2|^p + \ldots$$

converges.
Thus, for a fixed $p \ge 1$ we can write

$$\sum_{i=1}^{\infty} |x_i|^p < \infty. \tag{2.7}$$

Thus, we express the Vandermonde determinant $p-$ induced normed space as follows:

Theorem 2.4. *The p-norm over the the metric space (X, d) with $x, y, z \in X$, expressed as*

$$\|f\|_p \equiv d_p(x,y) = \|x-y\|_p, \; p \ge 1. \tag{2.8}$$

where $\|f\|_p$ is the $p-$norm of the Vandermonde determinant (or polynomial), f, given in (2.3) and

$$d_p(x,y) = \left(\sum_{i=1}^{\infty} |x_i - x_j|^p \right)^{\frac{1}{p}} \tag{2.9}$$

with (x, y) are as defined in (2.1), is a called Vandermonde p-norm satisfying

$N_1 : \|f\|_p \ge 0,$ *(positvity)*
$N_2 : \|f\|_p = 0 \iff f = 0$ *(definiteness)*
$N_3 : \|\alpha f\|_p = |\alpha| \|f\|_p,$ *(homogeneity)*
$N_4 : \|f + g\|_p \le \|f\|_p + \|g\|_p,$ *(Triangular inequality)*

where $\|g\|_p \equiv d_p(z,y) = \|z-y\|_p = \left(\sum_{k \neq j} |x_k - x_j|^p \right)^{\frac{1}{p}}$ *and* $\alpha \in \mathbb{R}(or\ \mathbb{C})$.

Proof. We prove Theorem 2.4 by verifying that (2.8) indeed satisfies the four properties of norm, N_1, N_2, N_3 and N_4. Thus, from (2.6) and (2.8) it follows that

$$d_p(x,y) = \|x-y\|_p = \left(\sum_{1\le i<j\le n} |x_i - x_j|^p\right)^{\frac{1}{p}}. \tag{2.10}$$

Expanding for a few terms say $(n=2, p=1)$, $(n=3, p=2)$ and $n=n, p=\infty$ respectively, yields

$$d_1(x,y) = \|x-y\|_1 = \left(\sum_{1\le i<j\le 2} |x_i - x_j|\right) = |x_1 - x_2| \tag{2.11}$$

$$\begin{aligned} d_2(x,y) = \|x-y\|_2 &= \left(\sum_{1\le i<j\le 3} |x_i - x_j|^2\right)^{\frac{1}{2}} \\ &= \sqrt{(x_2-x_1)^2+(x_3-x_1)^2+(x_3-x_2)^2} \\ &= \sqrt{2(x_1^2+x_2^2+x_3^2-x_1x_2-x_1x_3-x_2x_3)} \\ &\le \left(\sum_{i=1}^{3} 2x_i^2\right)^{\frac{1}{2}} = \sqrt{2}\|\mathbf{x}\|_p, \end{aligned} \tag{2.12}$$

since $x_i \neq x_j \neq 0$, for all $i \neq j$, and $i, j = 1, 2, 3$.

$$d_\infty(\mathbf{x},\mathbf{y}) = \|\mathbf{x}-\mathbf{y}\|_\infty = \max_{1\le i<j\le n} |x_i - x_j| \tag{2.13}$$

Using the expansions in (2.11), (2.12) and (2.13), we can easily verify N_1, N_2, N_3 and N_4.
For N_1, it follows from (2.11), (2.12) and (2.13) that

$$\begin{aligned} &d_1(x,y) = \|x-y\|_1 = |x_1 - x_2| \ge 0, \quad \Rightarrow \ \|f\|_p \ge 0. \\ &d_2(x,y) = \|x-y\|_2 \le \sqrt{2}\|\mathbf{x}\|_p, \quad \Rightarrow \ \|f\|_p \ge 0. \\ &d_\infty(x,y) = \|x-y\|_\infty = \max_{1\le i<j\le n} |x_i - x_j| \ge 0, \quad \Rightarrow \ \|f\|_p \ge 0. \end{aligned}$$

where (x,y) are as defined in (2.1).
For N_2, it follows from (2.3) that $f=0$ only if $x=y$, thus with (2.11), (2.12) and (2.13), then

$$\begin{aligned} &d_1(x,y) = \|x-y\|_1 = |x_1 - x_2| = 0 \text{ for } x_1 = x_2, \ \Rightarrow \ \|f\|_p = 0. \\ &d_2(x,y) = \|x-y\|_2 \le \sqrt{2}\|\mathbf{x}\|_p \text{ for } x_1 = x_2 = 0, \ \Rightarrow \ \|f\|_p = 0. \\ &d_\infty(x,y) = \|x-y\|_\infty = \max_{1\le i<j\le n} |x_i - x_j| \ge 0, \text{ for all } x_i = x_j \ \Rightarrow \ \|f\|_p \ge 0. \end{aligned}$$

where (x,y) are as defined in (2.1).
For N_3, it follows from (2.11), (2.12) and (2.13), then

$$\begin{aligned} d_1(\alpha x, \alpha y) &= \|\alpha x - \alpha y\|_1 = |\alpha x_1 - \alpha x_2| \\ &= |\alpha||x_1 - x_2| = |\alpha|\|x-y\|, \ \Rightarrow \ \|\alpha f\|_p = |\alpha|\|f\|_p. \\ d_2(\alpha x, \alpha y) &= \|\alpha x - \alpha y\|_2 \\ &= \sqrt{2\alpha^2(x_1^2+x_2^2+x_3^2-x_1x_2-x_1x_3-x_2x_3)} \\ &= |\alpha|\sqrt{2(x_1^2+x_2^2+x_3^2-x_1x_2-x_1x_3-x_2x_3)}, \\ &\ \Rightarrow \ \|\alpha f\|_p = |\alpha|\|f\|_p. \\ d_\infty(\alpha x, \alpha y) &= \|\alpha x - \alpha y\|_\infty = \|\alpha\| \max_{1\le i<j\le n} |x_i - x_j| \text{ for all } x_i \neq x_j, \\ &\Rightarrow \ \|\alpha f\|_p \ge |\alpha|\|f\|_p. \end{aligned}$$

where (x,y) are as defined in (2.1).
For N_4, based on (2.3), we define

$$f(\mathbf{x}) = P(x,z) = \prod_{x\neq z}(x-z) \quad \text{and} \quad g(\mathbf{x}) = P(z,y) = \prod_{z\neq y}(x-z),$$

where (x,y) are as defined in (2.1), so that

$$\|f\|_p = \left(\sum_{i\neq k} |x_i - x_k|^p\right)^{\frac{1}{p}} \quad \text{and} \quad \|g\|_p = \left(\sum_{k\neq j} |x_k - x_j|^p\right)^{\frac{1}{p}}.$$

Applying the Minkowski inequality for sums [15], that is,

$$\left(\sum_{j=1}^{n}|x_j+x_j|^p\right)^{\frac{1}{p}} \leq \left(\sum_{j=1}^{n}|x_j|^p\right)^{\frac{1}{p}} + \left(\sum_{j=1}^{n}|y_j|^p\right)^{\frac{1}{p}},$$

it follows that

$$\begin{aligned}
d_p(x,y) &= \left(\sum_{\{1\leq i<j\leq n\}}|x_i-x_j|^p\right)^{\frac{1}{p}} \\
&\leq \left(\sum_{\{1\leq i<j\leq n\}}\left[|x_i-x_k|+|x_k-x_j|\right]^p\right)^{1/p} \\
&\leq \left(\sum_{\{1\leq i<k\leq n\}}|x_i-x_k|^p\right)^{1/p} + \left(\sum_{\{1\leq k<j\leq n\}}|x_k-x_j|^p\right)^{1/p} \\
&\leq \|f\|_p+\|g\|_p.
\end{aligned}$$

Therefore, (2.8) normed space since it satisfies all the four properties of the normed space. □

Lemma 2.5. *The Vandermonde p-nom defined in (2.8) is complete.*

Proof. Completeness follows immediately using the steps for the proof of completeness in ℓ^p, see [15] for details. □

It follows that the Vandermonde p-norn we can also state as follows:

Lemma 2.6. *Based on* (2.8)*, the following norms:*

(i) $\|f\|_1 = d_1(x,y) = \|x-y\|_1$,
(ii) $\|f\|_2 = d_2(x,y) = \|x-y\|_2$,
(iii) $\|f\|_\infty = d_\infty(x,y) = \|x-y\|_\infty$

are equivalent.

Proof. Following exactly the workings in (2.11), (2.12) and (2.13), in 2-dimensions, that is $n=2$, and with (x,y) as defined in (2.1), we obtain

$$d_1(x,y) = \|x-y\|_{p=1} = \left(\sum_{\{1\leq i<j\leq 2\}}|x_i-x_j|\right) = |x_1-x_2| \tag{2.14}$$

$$\begin{aligned}
d_2(x,y) = \|x-y\|_{p=2} &= \left(\sum_{\{1\leq i<j\leq 2\}}|x_i-x_j|^2\right)^{\frac{1}{2}} \\
&= \sqrt{(x_2-x_1)^2} = \sqrt{x_1^2+x_2^2-2x_1x_2} \\
&\leq \left(\sum_{i=1}^{2}x_i^2\right)^{\frac{1}{2}} = \|\mathbf{x}\|_p,
\end{aligned} \tag{2.15}$$

since $x_ix_j \neq 0,$ for $i\neq j,\ i,j=1,2.$

$$d_\infty(x,y) = \|x-y\|_{p=\infty} = \max_{1\leq i<j\leq 2}|x_i-x_j| \tag{2.16}$$

The simplifications in (2.14), (2.15) and (2.16) respectively represents an inner unit square, circle of radius $r=1$ and a square at ∞ as illustrated in Figure 2.1(a).

In 3-dimensions, that is $n=3$, and (x,y) are as defined in (2.1), we have

$$\begin{aligned}
d_1(x,y) = \|x-y\|_1 &= \left(\sum_{1\leq i<j\leq 3}|x_i-x_j|\right) \\
&= |x_2-x_1|+|x_3-x_1|+|x_3-x_2|.
\end{aligned} \tag{2.17}$$

$$\begin{aligned}
d_2(x,y) = \|x-y\|_2 &= \left(\sum_{1\leq i<j\leq 3}|x_i-x_j|^2\right)^{\frac{1}{2}} \\
&= \sqrt{(x_2-x_1)^2+(x_3-x_1)^2+(x_3-x_2)^2} \\
&= \sqrt{2(x_1^2+x_2^2+x_3^2-x_1x_2-x_1x_3-x_2x_3)} \\
&\leq \left(2\sum_{i=1}^{3}x_i^2\right)^{\frac{1}{2}} = \sqrt{2}\|\mathbf{x}\|_p,
\end{aligned} \tag{2.18}$$

since $x_i x_j \neq 0$, for all $i \neq j$, and $i, j = 1, 2, 3$.

$$d_\infty(x,y) = \|x-y\|_\infty = \max_{\{1\leq i<j\leq 3\}} |x_i - x_j| \tag{2.19}$$

Based on the above cases in (2.14), (2.15), (2.16), (2.17), (2.18) and (2.19) respectively, represents p- norm surfaces for $n=2$ and $n=3$ as illustrated in Figures 2.1(a) and 2.1(b). □

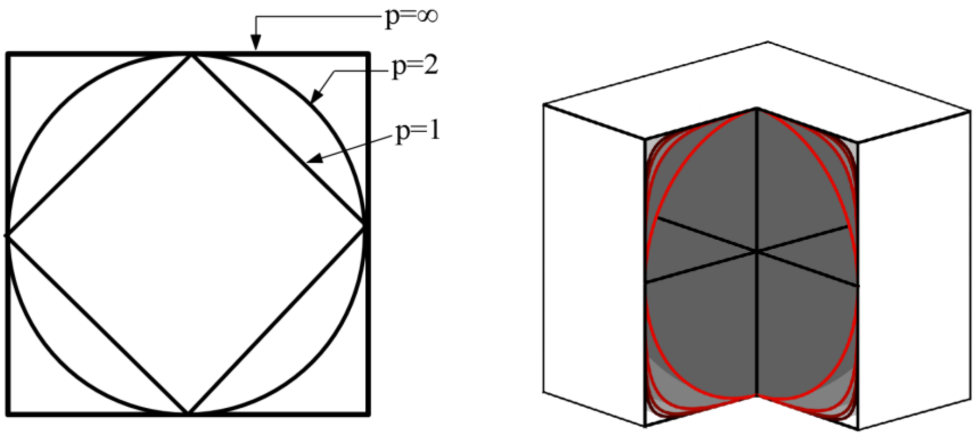

Figure 2.1: In (a) the set of figures inscribed inside each other represent $\|f\|_p = r$ for $n=2, p=1$, $p=2$ and $p=\infty$ with $\mathbf{x} \in \mathbb{R}^2$ while in (b) the set given by $\|f\|_p = r$ for $n=3, p=2, p=4, p=6, p=8$, and $p=\infty$ with $\mathbf{x} \in \mathbb{R}^3$ respectively .

Therefore, for Vandermonde p- norm we can generate various surfaces onto which there can be defined different continuous mappings as illustrated in the Figure 2.1 and thus define for Vandermonde p-normed distance measure as follows.
Based on result of (2.14), we generalize to define for Vandermonde $p=1$ normed distance as stated below:

Definition 2.7. *The Vandermonde $p=1$ normed distance of $\mathbf{x} \in \mathbb{R}^n$, $\|x-y\|_1$ is is defined as*

$$\|f\|_1 = \sum_{1\leq i<j\leq n} |x_i - x_j|. \tag{2.20}$$

Based on the result expressed in (2.19), we generalize to define for Vandermonde $p=\infty$ normed distance as stated below:

Definition 2.8. *The Vandermonde $p=\infty$ normed distance of $\mathbf{x} \in \mathbb{R}^n$, $\|f\|_\infty$ is is expressed as*

$$\|f\|_\infty = \sup_{1\leq i<j\leq n} |x_i - x_j|. \tag{2.21}$$

Based on the result expressed in (2.18), we generalize to define for general Vandermonde p-normed distance or sphere as stated below:

Definition 2.9. *The Vandermonde p-normed distance or sphere given by the standard p-norm, $S_p^{n-1}(r)$, $r \geq 0$, is the set of all $f \in \mathbb{R}$ such that*

$$\|f\|_p \equiv S_p^{n-1}(r) = \sum_{1\leq i<j\leq n} |x_i - x_j|^p \leq r^p \tag{2.22}$$

where $\sum_{i=1}^{n} |x_i|^p = r^p$ *and* $0 \leq r < \infty$.

3. Extreme Points, Fixed Points and Contraction on Surfaces defined by Vandermonde p-norm

This section gives the link between the extreme points, the zeros and the fixed point. It further outlines the principle of fixed point on a p-sphere, which motivates us to the main result of extreme-fixed point.

Definition 3.1. *Let x, y be two points in $\mathbb{R}^n$. The line x and y is given parametrically as*

$$[x,y] = \{x + \beta(y-x) : \ \beta \in [0,1]\} = \{(1-\beta)x + \beta y : \ \beta \in [0,1]\}, \tag{3.1}$$

is called a segment with the endpoints x, y. A subset S of $\mathbb{R}^n$ is called convex, if it contains along with any pair of its points x, y and the entire segment $[x,y]$, that is,

$$[x,y] \subset S, \text{ for evey } x, y \in S. \tag{3.2}$$

Definition 3.2. *A function $F:\mathbb{R}^n \to \mathbb{R}$ is convex if*

$$f((1-\beta)x+\beta y) \leq (1-\beta)f(x)+\beta f(y) \tag{3.3}$$

holds for any $x,y \in \mathbb{R}^n$ and $\beta \in [0,1]$. For a differentiable function f with gradient $\bigtriangledown f(\mathbf{x})$, a necessary and sufficient condition for convexity is given by,

$$f(\mathbf{y}) \geq f(\mathbf{x})+(\mathbf{y}-\mathbf{x})^\top \bigtriangledown f(\mathbf{x}), \tag{3.4}$$

which has to hold for any $\mathbf{x},\mathbf{y} \in \mathbb{R}^n$.

Let x,x_1,x_2 be points in $S \subset \mathbb{R}^n$ such that x lies between x_1 and x_2, $x_1 \neq x_2$ and there exists $\beta, 0 \leq \beta \leq 1$, so that

$$x = \beta x_1 + (1-\beta)x_2. \tag{3.5}$$

Else, the point $x_0^\star \in S$ is called an extreme point on S if it does not lie between any two distinct points $x_1,x_2 \in S$.

Definition 3.3. *A fixed point of continuous mapping $F: X \to X$ of a set X into itself is an $x \in X$ which is mapped onto itself, that is,*

$$f(x) = x. \tag{3.6}$$

The image $f(x)$ coincides with x.

Definition 3.4. *Let x_0 be an arbitrary point in a given set, then an iteration is a recursive relation*

$$x_{n+1} = f(x_n), \quad n = 0,1,2,\dots \tag{3.7}$$

that computes a sequence of points

$$x_1 = f(x_0),\ x_2 = f(x_1) = f^2(x_0),\ x_3 = f^3(x_0),\dots,x_n = f^n(x_0). \tag{3.8}$$

Definition 3.5 ([15])**.** *Let $X=(X,d)$ be a metric space. A mapping $F: X \to X$ is called a contraction on X if there is a positive real number $\alpha < 1$ such that for all $x,y \in X$, then*

$$d(f(x),f(y)) \leq \alpha d(x,y), \quad \textit{for } \alpha < 1. \tag{3.9}$$

Based on the above Definitions 3.1 through to 3.5, we shall define an extreme-fixed point as follows.

Definition 3.6. *Let $x_0^\star$ be a fixed point of X a subset of $\mathbb{R}^n$. Let f be function such that $f: X \to X$. Then, $x_0^\star \in S$ is an extreme-fixed point of X if we have*

$$x_0^\star = \max_{x \in X} f(x). \tag{3.10}$$

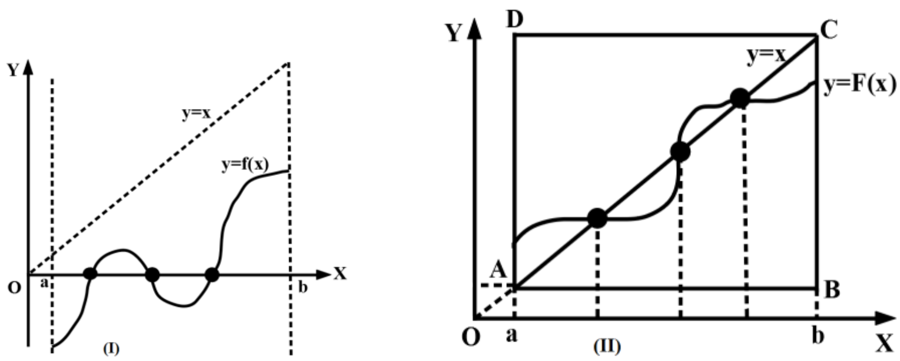

Figure 3.1: In (I) we illustrate the function $y=f(x)$ and would give zeros when $f(x)=0$ while in (b) is the function $y=F(x)$ where $F(x)=f(x)+x$ and would give zeros as fixed points when $F(x)=x$.

Following theorems are useful to establish the next lemma which is useful for further results.

Theorem 3.7 ([2])**.** *Let X be an arbitrary normed space, and let $F: X \to X$ be a completely continiuous linear operator. Then, either,*

(a) the equation

$$0 = x - F(x)$$

has a non-trivial solution, or

(b) the equation

$$y = x - F(x)$$

has a uniques solution for each $y \in X$.

Theorem 3.8 ([13])**.** *Let $F : X \to X$ be a completely continuous operator (that is, a map that is restricted to a bounded set in F is compact).Let*

$$\varepsilon(F) = \{x \in X : x = \lambda F(x) \text{ for some } 0 < \lambda < 1\}.$$

Then, either the set $\varepsilon(F)$ is unbounded or F has a fixed point.

Based on the Definition 3.6 of extreme-fixed point, leads to connection between the zeros of a function f and optimal points, the result stated in Lemma 3.9 given below.

Lemma 3.9. *Let $f : X \to X$ be a continuous function defined on the domain $[a,b]$ such that $f(a) < f(b)$. Then, there exists a number c satisfying the condition $f(a) < c < f(b)$, such that*

$$f(x_0^\star) = c \equiv \max_{x\in\mathbb{R}^n} f(x),$$

and when $x_0^\star = c$ we have an extreme-fixed point.

The proof of Lemma 3.9 employs important results of Theorem 3.7 and Theorem 3.8 which are respectively referred to as Fredholm alternative [2] and as Leray–Schauder alternative [13].

Proof. To prove Lemma 3.9, we begin by applying the principle of intermediate value theorem as geometrically illustrated in the Figure 3.1. In Figure 3.1(I), it can be noticed that for the function f, there exists a point $x_0 \in [a,b]$ for which

$$f(x_0) = c, \tag{3.11}$$

and if $c = 0$, then the solution $\{x_0\}$ to (3.11) will be the roots of the polynomial function.
Again from Figure 3.1(II), and combining the ideas of Theorem 3.7 and Theorem 3.8, we let $F : X \to X$ be a continuous function defined on the domain $[a,b]$ with $F(a) < F(b)$, given by

$$y = F(x) \equiv \lambda f(x) + x,$$

where $\lambda \in \mathbb{R}\backslash\{0\}$ is chosen appropriately. Then, it should be noted that there exists a point $x_0^\star \in [a,b]$ so that

$$F(x_0^\star) \equiv \lambda f(x_0^\star) + x_0^\star = x_0^\star. \tag{3.12}$$

Combining (3.11) and (3.12), it follows immediately that

$$f(x_0^\star) = 0, \text{ where } (c = 0). \tag{3.13}$$

Thus, the function $F(x)$ has fixed points, when $x_0^\star \equiv x_0$.
Now, for $c \neq 0$, it follows from (3.12) that

$$F(x_0^\star) \equiv \lambda f(x_0^\star - c) + x_0^\star = x_0^\star. \tag{3.14}$$

Thus, it follows from Definition 3.6, the solution points $x_0^\star$ of (3.14) that maximizes the function $f(x)$ are the extreme-fixed points. □

The concept of the fixed points on the unit sphere has been proved in several texts based on the property of contraction on a ball (see [3, 4, 5, 36]). Some of the related fixed point theorems on sphere are as stated below.
The Banach fixed point theorem for the existence and uniqueness of fixed points of give mappings. This also describes a constructive procedure for obtaining improved and better approximations to the fixed point by method of iteration.

Theorem 3.10 (Banach Fixed Point Theorem [15])**.** *Consider a metric space $X = (X,d)$, where $X \neq \emptyset$. Suppose that X is complete and let $T : X \to X$ be a contraction on X. Then, T has precisely one fixed point.*

The extreme fixed points on a p-sphere can also be supported by the ideas expressed in theorem of Brouwer and other related thorems of non-retract.

Theorem 3.11 ([36])**.** *For $n \geq 1$, S^{n-1} is not a retraction of B^n where $B^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ is a closed unit ball and $S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$ is a unit sphere.*

The Theorem 3.11 is further complemented by the results of Theorem 3.12 below.

Theorem 3.12 ([11])**.** *Every continuous mapping $F : B^{n+1} \to R^{n+1}$ has at least one of the following properties:*

(i) F has a fixed point,
(ii) there is an $x \in S^n$ such that $x = \lambda F(x)$ for some $0 < \lambda < 1$.

A combination of the ideas of Theorem 3.11 and 3.12 yields Theorem 3.13 below.

Theorem 3.13 (Brouwer Fixed Point Theorem, [4, 5])**.** *A continuous map from $n-$simplex to itself has a fixed point.*

Using the ideas of Theorem 3.10, Theorem 3.11, Theorem 3.12, Theorem 3.13 and the general Banach fixed point theorems stated in [15], we establish a connection between the extreme-fixed point by optimizing the Vandermonde determinant on the sphere.

Theorem 3.14. *Let F be a mapping of a complete metric space $X = (X,d)$ and $F : X \to X$ a contraction on a closed ball*

$B(x_0^\star, r) = \{x \in X : d(x, x_0^\star) \leq r\}$,

that is, F satisfies (3.9) for all $x, y \in B(x_0^\star, r), r > 0$. Moreover, assume that

$$d(x_0^\star, Fx_0^\star) < (1-\alpha)r. \tag{3.15}$$

Then, the iterative sequence (3.8) converges to an $x \in Y$. This $x_0^\star$ is an extreme-fixed point of F in Y.

Proof. The proof of Theorem 3.14, follows through the same steps as for the case of [15]. □

Combining the results of Lemma 3.9 and Theorem 3.7 through to Theorem 3.14, we can be able formulate an optimization problem for maximizing the square of Vandermonde determinant on given surface defined by a unit sphere that aims to evaluate the point $x_0^\star$ defiend as the extreme fixed point so that

$$\mathbf{x}_0^\star \in \begin{cases} \max\limits_{\mathbf{x}\in\mathbb{R}^n} f(\mathbf{x}) \\ \text{subject to} \quad g(\mathbf{x}) - c = 0, \end{cases} \tag{3.16}$$

where $g(\cdot)$ is a function representing a surface defined in p-norn as illustrated in Figure 2.1(b) and $f(\cdot)$ is any function, in our case the Vandermonde determinant. The whole argument is as illustrated in Figure 3.2.

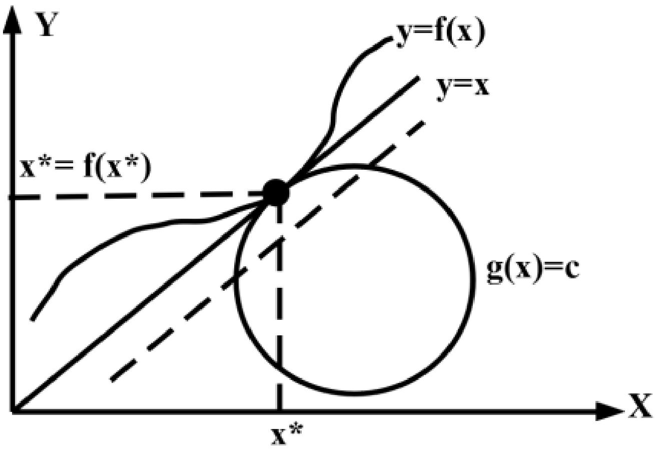

Figure 3.2: An illustration of the extreme fixed point for the function $f(\mathbf{x})$ optimized over a given surface $g(\mathbf{x}) = c$.

4. The Vandermonde Extreme Fixed Point

In this section, we state and prove our main result of the extreme-fixed point. Here, we show that the coordinates obtained when the Vandermonde polynomial is optimized on a surface resented by the p-norm sphere are also expressed by the zeros of classical orthogonal polynomial and also evaluated as the fixed points on the sphere. Thus, the name extreme-fixed points on that same surface.

Combining the ideas and results expressed in Lemma 2.2, Theorem 2.4, Lemma 2.6, Lemma 3.9, Theorem 3.11, Theorem 3.12, Theorem 3.8, Theorem 3.13, and Theorem 3.14, in the previous sections we can give the following general result(s):

Theorem 4.1. *Consider $X = (X,d)$, a complete metric space with $X \neq \emptyset$. Let f be a fixed continuous mapping such that $f : X \to X$. Then, the solution point(s) $\mathbf{x}_0^\star \in X$ of the following optimization problem*

$$\mathbf{x}_0^\star = \begin{cases} \max\limits_{\mathbf{x}\in\mathbb{R}^n} & f(\mathbf{x}) \\ s.t & g(\mathbf{x}) \end{cases} \tag{4.1}$$

where

$$f(\mathbf{x}) = \prod_{\{1\leq i<j\leq n\}} |x_i - x_j|^2, \tag{4.2}$$

the square of the Vandermonde polynomial (or determinant), and

$$g(\mathbf{x}) = \sum_{i=1}^{n} x_i^2 = 1, \tag{4.3}$$

the unit n-dimensional p- unit sphere $S_p^{n-1}(\mathbf{1})$, is called extreme-fixed point(s) for which
$\mathbf{x}_0^\star = F(\mathbf{x}_0^\star)$ and is given by the zeros of the Hermite polynomial $H_n(x)$ where

$$H_n(x) = n! \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k}{k!} \frac{(2x)^{n-2k}}{(n-2k)!}. \tag{4.4}$$

The outline of the proof of Theorem 4.1 is structured as follows:

(i) X, a complete metric space,
(ii) f, a contraction on X,
(iii) $x_0^\star$, a fixed point of f, that is, $f(x_0^\star)=x_0^\star$,
(iv) $x_0^\star$, an extreme-fixed point so that $x_0^\star=\max f(x)$,
(v) the points $x_0^\star$ can be exproximated by the zeros of some classical univariate polynomials.

Proof. (i) To prove for completeness of X, let (x_n) be a general sequence of point:

$x_0^\star, x_1, x_2, \ldots, x_n, \ldots,$

in X. Defining the iterative sequence (x_n) by

$$x_0^\star, x_1=f(x_0^\star), x_2=f(x_1)=f^2(x_0^\star), x_3=f^3(x_0^\star), \ldots, x_n=f^n(x_0^\star). \tag{4.5}$$

This generates a sequence of images of $x_0^\star$ under repeated application of f on the points x_i, x_j in X and for all $i\neq j$, and $i,j=1,2,\ldots,n$.
Now, we show that (x_n) is Cauchy. Using (3.9) and (4.5), and for all the pairs (x_i,x_j),
$i\neq j,\ i,j=1,2,\ldots,m$, it follows that

$$\begin{aligned} d(x_{m+1},x_m) &= d\left(f(x_m),f(x_{m-1})\right)\leq \alpha d\left(x_m,x_{m-1}\right) \\ &= d\left(f(x_{m-1}),f(x_{m-2})\right)\leq \alpha^2 d\left(x_{m-1},x_{m-2}\right) \\ &\quad \cdots \\ &= \alpha^{m-1} d\left(x_2,x_1\right)\leq \alpha^m d\left(x_1,x_0^\star\right), \end{aligned} \tag{4.6}$$

where $m << \frac{1}{2}n(n-1)$ the number of the product of difference factors of the form $|x_i-x_j|,\ i\neq j,\ i,j=1,2,\ldots,m$ that make the Vandermonde determinant of the n-dimensional Vandermonde matrix in (1.1).
Thus, by the extended triangle inequality and the sum of the geometric progression we obtain for $n>m$ so that

$$\begin{aligned} d(x_m,x_n) &\leq d(x_m,x_{m+1})+d(x_{m+1},x_{m+2})+\ldots+d(x_{n-1},x_n) \\ &\leq (\alpha^m+\alpha^{m+1}+\ldots+\alpha^{n-1})d(x_0^\star,x_1) \\ &= \alpha^m\left(\frac{1-\alpha^{n-m}}{1-\alpha}\right)d(x_0^\star,x_1). \end{aligned}$$

Since $0<\alpha<1$, in the numerator we can have $1-\alpha^{n-m}<1$. Consequently,

$$d(x_m,x_n)\leq \frac{\alpha^m}{1-\alpha}d(x_0^\star,x_1). \tag{4.7}$$

On the right, $0<\alpha<1$, and $d(x_0^\star,x_1)$ is fixed, so that we can make the right-hand side small as we please, by taking m sufficiently large (and $n>m$). This proves that (x_n) is Cauchy. Since X is complete, (x_n) converges, say $x_n\to x_0^\star$.
Again, we show that the limit $x_0^\star$ is an extrme-fixed point of the continuous mapping $F:X\to X$.
From the triangle inequality and (3.9), we have

$$\begin{aligned} d\left(x_0^\star,f(x_0^\star)\right) &\leq d\left(x_0^\star,x_m\right)+d\left(x_m,f(x_0^\star)\right) \\ &\leq d\left(x_0^\star,x_m\right)+\alpha d\left(x_{m-1},x_0^\star\right) \end{aligned} \tag{4.8}$$

and we can make the sum in the (4.8) to be smaller than any preassigned $\varepsilon>0$ because $x_m\to x_0^\star$. We conclude that $d\left(x_0^\star,f(x_0^\star)\right)=0$, so that $f(x_0^\star)=x_0^\star$.
The point $x_0^\star$ is the only fixed point of f because from $f(x_0^\star)=x_0^\star$ and $f(\tilde{x}_0^\star)=\tilde{x}_0^\star$, we obtain by (3.9) that

$$d(x_0^\star,\tilde{x}_0^\star)=d\left(f(x_0^\star),f(\tilde{x}_0^\star)\right)\leq \alpha d(x_0^\star,\tilde{x}_0^\star),$$

which implies that $d(x_0^\star,\tilde{x}_0^\star)=0$ since $\alpha<1$. Hence, $\tilde{x}_0^\star=\tilde{x}_0^\star$ by (M_2) of Definition 2.1.
Next, we prove that the point $x_0^\star$ is also the extreme point when the Vandermonde determinant is optimized over the surface defined by say a p-sphere.
Applying the Lagrange multiplier, in order to find the maximum (or minimum) of a function $f(\mathbf{x}))$ subjected to the equality constraint $g(x)=0$, forms the Lagrangian function

$$\mathscr{L}(\mathbf{x},\lambda)=f(\mathbf{x})-\lambda g(\mathbf{x}).$$

From (4.2) and (4.3), it follows for all $x_i\neq x_j$, and $i,j=1,\ldots,n$ that

$$\frac{\partial v(\mathbf{x})}{\partial x_j}=\lambda\frac{\partial\left\{g(\mathbf{x})-1\right\}}{\partial x_j} \iff \sum_{\substack{i=1\\ i\neq j}}^{n}\frac{v_n(\mathbf{x})}{x_j-x_i}=2\lambda\sum_{i=0}^{n}x_i \tag{4.9}$$

for some $\lambda\in\mathbb{R}$.

Consider the n-degree monic polynomial with its zeros as x_i for $i = 1,\dots,n$,

$$f(x) = \prod_{i=1}^{n-1}(x - x_i) \tag{4.10}$$

it follows that

$$\frac{1}{2}\frac{f''(x_j)}{f'(x_j)} = \sum_{\substack{i=1\\ i\neq j}}^{n} \frac{1}{x_j - x_i}. \tag{4.11}$$

Thus, combining (4.9) and (4.11) for each of the points x_j, gives a general expression

$$f''(x_j) - 2\lambda x_j f'(x_j) = 0, \quad j = 1,2,\dots,n. \tag{4.12}$$

Since each x_j is one of the roots of the function $f(x)$ in (4.10), can be seen that actually the left hand side of the Equation (4.12), would be an n-degree polynomial with roots the same as those of $f(x)$. Thus, we can now generalize that for any x, we obtain

$$f''(x) - 2\lambda x f'(x) - 2nf(x) = 0 \tag{4.13}$$

The solution to the ordinary differential equation is given by Hermite polynomial $H_n(x)$ stated in . Thus the values of $x_0^\star \in \mathbb{R}$ such that $H_n(x_0^\star) = 0$ will give the extreme-fixed points of the Vandermonde polynomial (or determinant) when optimized on the unit sphere. □

Lemma 4.2. *Let f be a mapping of a complete metric space $X = (X,d)$ into itself. Then, f given in (4.2) is a contraction on a closed ball,*

$$Y = \{x \in X : d(x, x_0^\star) \leq 1,$$

that is, f satisfied (3.5). Moreover, assuming that

$$d(x_0^\star, f(x_0^\star)) < (1-\alpha), \ 0 < \alpha < 1. \tag{4.14}$$

Then, the iterative scheme (3.8) converges to an $x_0^\star \in Y$. This $x_0^\star$ is an extreme-fixed point of f in Y.

Proof. The statement of proof follows closely the steps for the proof of Theorem 4.1. Here we further show that the sequence of points $\{x_m\}$ its limit point x lie on the closed unit ball Y. Setting $m = 0$ and replacing n with m in (4.7), we obtain

$$d(x_0^\star, x_m) \leq \frac{1}{1-\alpha} d(x, x_0^\star). \tag{4.15}$$

Combining (4.16) with (4.14), it follows that

$$d(x_0^\star, x_m) \leq \frac{1}{1-\alpha} d(x_0^\star, f(x_0^\star)) \leq 1 \tag{4.16}$$

It follows immediately that every neighbourhood of $x_0^\star$ contains a finite number of x_m in the radius $r = 1$, thus all x_n's and $x_0^\star$ are contained in Y. Therefore, $x_0^\star$ is an extreme fixed point of f contained in Y. □

5. Extreme-Fixed Points For Generalized to higher n and even p-norm

This section gives the illustration using the results proved in [28] for the general polynomials whose zeros are actually the extreme points of Vandermonde determinant when optimized on a general p-sphere and these at the same time these being the extreme-fixed points.
The results of Theorem 4.1 can further be extended for higher dimensional n and even p- norms, that is, for positive and even integers, and $n > p$. Using the same method of [28] where the coordinates of extreme points of the Vandermonde determinant constrained to S_p^{n-1}, are given as the roots of a polynomial.
The optimisation problem in 4.1 can be rewritten using differential equation as follows.

Lemma 5.1 ([28, 34]). *Let n and p be even positive integers. Consider the unit sphere given by the p-norm, in other words the surface given by*

$$S_n^p = \left\{ (x_1,\dots,x_n) \in \mathbb{R}^n \middle| \sum_{i=1}^{n} x_i^p = 1 \right\}.$$

There exists a second order differential equation

$$P_n^{p\prime\prime}(x) - \frac{a_{p-2}}{n} x^{p-1} P_n^{p\prime}(x) + Q_n^p(x) P_n^p(x) = 0, \tag{5.1}$$

where $P_n^p(x)$ and $Q_n^p(x)$ are polynomials of the forms

$$P_n^p(x) = x^{2n} + \sum_{i=0}^{\frac{1}{2}n-1} c_{2i} x^{2i}$$

and

$$Q_n^p(x) = -a_{p-2} x^{p-2} + \sum_{i=0}^{\frac{1}{2}p-2} (-1)^i a_{2i} x^{2i}.$$

$P_{2p}^2(x) = x^2 - \frac{1}{2}\left(2^{(p-1/p)}\right),\ p \geq 1,$
$P_{2p}^3(x) = x^3 - \frac{1}{2}\left(2^{(p-1/p)}\right)x,\ p \geq 1,$
$P_2^4(x) = x^4 - \frac{1}{2}x^2 + \frac{1}{48}, P_4^4(x) = x^4 - \frac{\sqrt{6}}{3}x^2 + \frac{1}{12},$ $P_6^4(x) = x^4 - \frac{1}{4}(\sqrt{33}+1)^{\frac{1}{3}}x^2 + \frac{1}{96}\left(9-\sqrt{33}\right)(\sqrt{33}+1)^{\frac{2}{3}}$ $P_8^4(x) = x^4 - \frac{\sqrt{6}}{6}(30\sqrt{5}-30)^{\frac{1}{4}}x^2 + \frac{1}{120}\left(\sqrt{5}-5\right)\sqrt{30\sqrt{5}-30}$
$P_2^5(x) = x^5 - \frac{1}{4}x, P_4^5(x) = x^5 - \frac{2\sqrt{5}}{5}x^3 + \frac{3}{20}x, P_6^5(x) = x^5 - \frac{10^{\frac{1}{3}}}{2}x^3 + \frac{10^{\frac{2}{3}}}{20}x$ $P_8^5(x) = x^5 - \frac{\sqrt{10}}{10}(50\sqrt{13}+10)^{\frac{1}{4}}x^3 + \frac{1}{1800}\left(5\sqrt{13}-55\right)\sqrt{50\sqrt{13}+10}$
$P_2^6(x) = x^6 - \frac{1}{2}x^4 + \frac{1}{20}x^2 - \frac{1}{1800}$ $P_4^6(x) = x^6 - \frac{\sqrt{50+20\sqrt{5}}}{10}x^4 + \frac{\sqrt{5}}{10}x^2 - \frac{(-4+2\sqrt{5})\sqrt{50+20\sqrt{5}}}{600}$
$P_2^7(x) = x^7 - \frac{1}{2}x^5 + \frac{5}{84}x^3 - \frac{5}{3528}$ $P_4^7(x) = x^7 - \frac{\sqrt{1050+84\sqrt{109}}}{42}x^5 + \left(\frac{1}{21} + \frac{\sqrt{109}}{42}\right)x^3 - \frac{(-16+2\sqrt{109})\sqrt{1050+84\sqrt{109}}}{10584}$
$P_2^8(x) = x^8 - \frac{1}{2}x^6 + \frac{15}{224}x^4 - \frac{15}{6272}x^2 + \frac{15}{1404928},$ $P_4^8(x) = x^8 - \frac{\sqrt{140+42\sqrt{6}}}{14}x^6 + \left(\frac{3}{28} + \frac{3\sqrt{6}}{28}\right)x^4$ $- \left(\frac{-(140+42\sqrt{6})^{\frac{3}{2}}}{16464} + \frac{29\sqrt{140+42\sqrt{6}}}{2352}\right)x^2 - \frac{3}{3136} + \frac{\sqrt{6}}{1568}$

Table 1: The Polynomials, P_{2p}^n for $p = 2,3,5$ and $n = 2,3,4,5,6,7,8$, whose zeros give the extreme-fixed points of the Vandermonde polynomial (or determinant) on the $2p$-norm $n-$dimensional unit sphere.[Table is adpoted from [28, 34]].

There is also a relation between the coefficients of P_n^p and Q_n^p given by

$$2j(2j-1)c_{2j} + \left(\sum_{k=0}^{j-1} a_{2k}c_{2(j-k-1)}\right) + \frac{n+p-2j}{n}a_{p-2}c_{2j-p} = 0 \tag{5.2}$$

where $1 \leq j \leq \frac{n+p-2}{2}$ *and* $c_n = 1$*,* $c_k = 0$ *for all values of* $k \notin \{0,2,4,\ldots,n\}$ *and* $a_k = 0$ *for all values of* $k \notin \{0,2,4,\ldots,p-2\}$.

Proof. The detailed of the this Lemma is outlined in [28, 34]. □

Lemma 5.2. *The zeros of the polynomials generated by the recursive scheme of Lemma 5.1 will be the extreme-fixed points of the Vandermonde polynomial (or determinant) when optimized on thep-norm unit sphere.*

Proof. Using the polynomials generated in [28], and as shown in Table 1, it can be seen that for case $n = 4, p = 2$, we have $P_2^4(x) = x^4 - \frac{1}{2}x^2 + \frac{1}{48}$ we have the zeros as

$$x_1 = \frac{\sqrt{3}\sqrt{3+\sqrt{6}}}{6}, x_2 = -\frac{\sqrt{3}\sqrt{3+\sqrt{6}}}{6}, x_3 = \frac{\sqrt{3}\sqrt{3-\sqrt{6}}}{6}, x_4 = -\frac{\sqrt{3}\sqrt{3-\sqrt{6}}}{6}.$$

It can be easily seen that the zero x_1, x_2, x_3 and x_4 lie on the sphere of dimension $n = 4$, that is, $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$. □

6. Conclusion

In this research article, we have been able to establish the close relationship between the zeros of a given polynomial, optimal points obtained by optimizartion and the fixed points on a given surface. These points are in general called the extreme-fixed points and are in general evaluated from the zeros of a given univariate polynomial expression and are also the points that maximize (or minimize) Vandermonde polynomial when optmimized on given surface say a general p-unit sphere. This techniques of extreme-fixed point can be applied in various mathematical, numerical and statistical computations including optimal control, data smoothing and approximation with high degree of accuracy.

Acknowledgements

The author would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Funding

There is no funding for this work.

Availability of data and materials

Not applicable.

Competing interests

There are no competing interests.

Author's contributions

The author contributed to the writing of this paper. The author read and approved the final manuscript.

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