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MORTALITY MODELING WITH LEVY PROCESSES

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─Abstract ─
Mortality and longevity risk is usually one of the main risk components in economic capital models of insurance companies. Above all, future mortality expectations are an important input in the modeling and pricing of long term products. Deviations from the expectation can lead insurance company even to default if sufficient reserves and capital is not held. Thus, Modeling of mortality time series accurately is a vital concern for the insurance industry. The aim of this study is to perform distributional and spectral testing to the mortality data and practiced discrete and continuous time modeling. We believe, the results and the techniques used in this study will provide a basis for Value at Risk formula in case of mortality.

Key Words: *Mortality, Stochastic Modeling, Levy Processes, ARMA, GARCH, COGARCH*
JEL Classification: G17, C15, C53.

1. INTRODUCTION

1.1. Mortality and Longevity as Risk Factor

Mortality risk is defined as the risk of average life to be shorter than expected and vice versa for the longevity. Mortality and longevity risks are one of the main risks that the Life Insurance companies are exposed to. The mortality and longevity risk is a concern for the insurance companies in various forms. First of all, in product development processes, it is a vital decision to use accurate assumptions. However, it is quite a difficult task taking into account the durations of the products reaching 20 or even 30 years. Too loose assumptions can cause life insurance firms to end up in locked-in loss positions while too tight assumptions can lead to unmarketable products.

Furthermore, Mortality and Longevity risks are one of the most important risk components in Economic Capital Models of Life Insurance companies. According to the QIS 5 study, held by European Insurance and Occupational Pensions Authority, for the participating solo Life undertakings, 52% of the underwritings risks are due to longevity and mortality (EIOPA, 2009).

This study aims to start a discussion in the industry on mortality modeling. Mortality rates are taken as a financial time series and analyzed accordingly to come up with certain findings which can be useful for future research.

1.2. Data

Mortality tables for UK between the years of 1922 to 2009 are available for each year in the HMD, Human Mortality Database. In our analysis, we have identified the death rate (q_x) for the age of 55 from the mortality table for each year. We have used the mortality rate for a single age for every year in order to be able to identify the trends in mortalities. The reader should be aware that the results might be different for different ages.

We have not used mortality tables for males or females, rather used the mortality tables for the whole population. This choice is in line with the new legislation in Europe where the gender cannot be used for premiums or benefits.

2. TIME SERIES ANALYSIS

2.1. Analysis in Time Domain

The data on figure 1 is the mortality rates for the age of 55 from 1922 to 2009 for UK. The data is trending downwards (as expected from the medical developments) and includes jumps, biggest in 1940, due to the Second World War.

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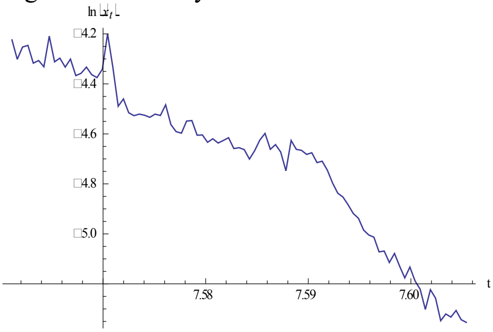
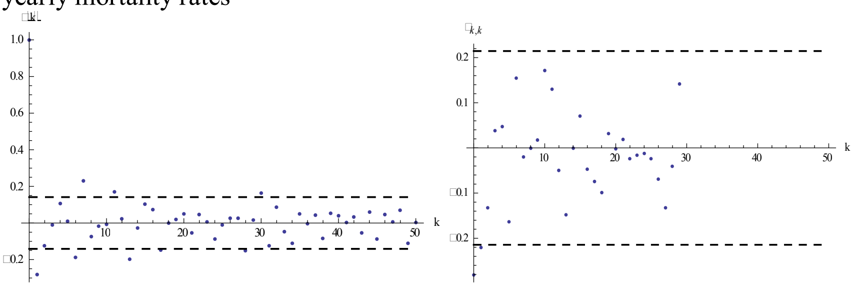
Figure 1: Mortality Rates for the UK from 1922 to 2009 for the age of 60



Due to the trend the series is mean nonstationy.The nonstationarity is also evident in the bahaviour of autocorrelations where the autocorrelations die slowly while the partial autocorrelations sharply converge to zero. To maintain both mean and variance stationarity, we have preferred log differenciencing. According to visual graph, the Autocorrelation, partial autocorrelation and the Augmented Dickey-Fuller test results all indicate that the desired stationarity is achieved. Figure 2 below provides the autocorrelation and partial autocorrleation functions of the transformed data.

Figure 2: Autocorrelation (ACF) and Partial Autocorrelation (PACF) graphs of log difference of yearly mortality rates



The visual representation of the ACF and the PACF function of the data indicate a relationship in the order of one. Furthermore there seems to be significant AC on the order of 6 to 10 and 13. Our expectation is in line with the AC and PAC which is significant on the order of one. We expect to see short term trends in mortality rates due to pandemics or wars. However we do not see any reason to assume a higher order relationship. We apply the Hannan-Rissanen algorithm to decide the mean equation for our time series.

In the Hannan Rissanen Procedure, first a AR model is fitted to the data and the errors are obtained. The estimated errors are used in place of true errors and by least squares regression the ARMA model parameters are fit. We begin with fitting $AR(i)$ models until $i = 10$ by using the Levinson Durbin algorithm. Akaike information criterion (AIC) is calculated for the AR models and the $AR(i)$ model with the smallest AIC is chosen. Then, from the best fitted $AR(i)$ model the residuals are calculated and by using the least squares estimation, $ARMA(p, q)$ coefficients are estimated up to $p = 10$ and $q = 10$. Lastly, model with the lowest Bayesian Information Criterion (BIC) is chosen. The results from best to worst fit can be seen in table 1.

Table 1 – Hannan Rissanen Estimate Results

Model	***Parameters***	***AIC***	**BIC**
AR(1)	$ARModel[\{-0.20\}, 0.002]$	-6.01	-5.99
MA(1)	$MAModel[\{-0.20\}, 0.002]$	-6.00	-5.98
AR(2)	$ARModel[\{-0.20, -0.09\}, 0.002]$	-6.00	-5.96
MA(2)	$MAModel[\{-0.20, 0.04\}, 0.002]$	-6.00	-5.96
ARMA(1,1)	$ARMAModel[\{-0.04\}, \{-0.15\}, 0.002]$	-5.98	-5.94

Although Hannan Rissanen Procedure and the BIC points the AR(1) model; according to AIC the best fit is MA(4). We will go on with AR(1) as estimated by Hannan Rissanen.

The residuals after the AR(1) mean modeling, does not possess any significant AC or PAC. Furthermore we have applied the Portmanteau test to see if there is any autocorrelation is left. Portmanteau tests whether the first h correlations of residuals together have any significant autocorrelation or not. The test statistic is calculated as

$$Q_h = n(n+2)\sum_{k=1}^{h} \hat{\rho}^2(k)/(n-k) \quad (1)$$

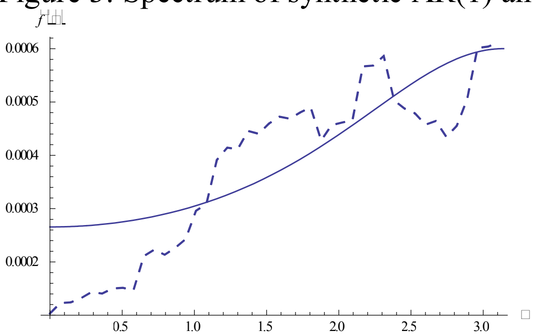
where n is the number of data points and h is the number of autocorrelations we want to evaluate. The portmanteau statistic calculated for AR(1) for the first 35 autocorrelations is 40.85 while the table value (with a degree of freedom equal to h-p-q=35-1-0=34) $\chi^2_{0.95}(34) = 48.60$. Thus, the portmanteau statistics also indicates that we do not have enough evidence to believe there is autocorrelation left in the residuals for AR(1) model.

2.2. Analysis in Frequency Domain

So far all the analysis done are functions of time. One example can be the autocorrelations as function of the time lag. This is called the time series analysis in time domain. In this section of the study, we will do analysis with an alternative domain which is called the frequency domain. It is also called the time series analysis in Fourier space. Working with different domains is simply different representations of the same data. They do end with same the result. However, frequency domain analysis can yield powerfull methods for analysis and can provide new insights.

Figure 3 represents us the smoothed spectrum of the data and the spectrum of the synthetic AR(1) estimated. As seen from the graph, the AR(1) and the actual data follow the same path with some noise in the actual data due to the GARCH effect.

Figure 3: Spectrum of synthetic AR(1) and the Smoothed Spectrum actual data



2.3. Variance Equation Estimation

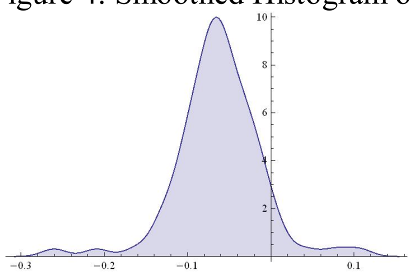
The data provide evidence of dependencies in the variances. The distribution of residuals after mean modeling poses fat tails with a kurtosis of 5.61. Furthermore; there is significant autocorrelation for squared residuals. We have also applied

ArchLM test as a formal test for the existence of a GARCH effect. The calculated LM statistic for the existence of ARCH effect is 8.74 while the $\chi^2_{0.95}(1) = 3.84$ indicating that the null hypothesis that there is no GARCH effect is rejected. For the GARCH modeling, we will apply a GARCH(1,1) model to be used in COGARCH(1,1) estimation as a basis. GARCH(1,1) is chosen due its wide application and practical usage.

2.4. Distribution fit for the errors

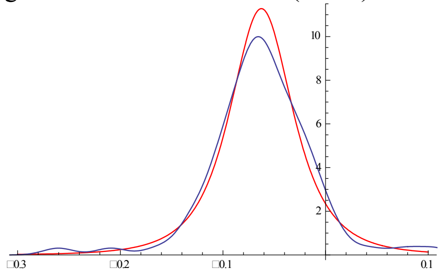
The smoothed histogram of errors left after mean and variance modeling is in figure 4. The histogram has long tails compared to normal. For distribution fitting, we have tested various distributions including normal, Johnson SU, Weibull, Gumbel, Cauchy. The tests done includes controlling the histogram and the distribution PDF's, Checking the Q-Q plots, and conducting distribution fit tests.

Figure 4: Smoothed Histogram of errors



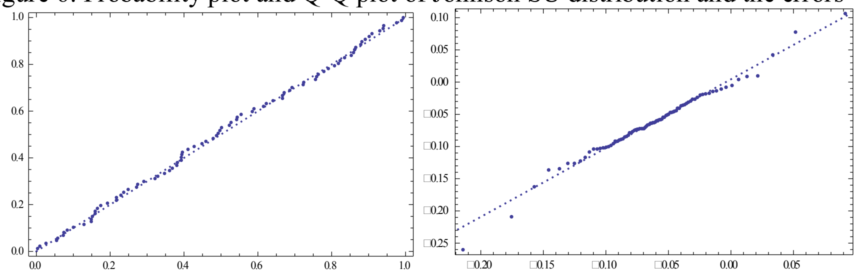
The results of all the distributions will not be presented for simplicity. But the results of the best fit distribution, Johnson SU is presented below. As seen from the graph, Johnson SU has a good performance in fitting the tail behavior of the errors. Johnson SU is a transformation of the normal distribution developed by Johnson (Kleiber et al.,2003)

Figure 5: PD of Johnson SU (in red) and Smoothed Histogram of errors



The fit also can be observed from the Q-Q plot and probability plot graphs.

Figure 6: Probability plot and Q-Q plot of Johnson SU distribution and the errors



We have also conducted various goodness of fit tests. The null hypothesis is that the data is drawn from a population with Johnson SU distribution. All the p-values indicate that the null hypothesis is not rejected at 5%. The results are in Table 2

Table 2 Goodness of Fit test results for Johnson SU

	Statistic	P-Value
Anderson-Darling	0,1186	0,9998
Cramér-von Mises	0,0154	0,9995
Kolmogorov-Smirnov	0,0338	0,9999
Kuiper	0,0657	0,9994
Pearson c2	4,4483	0,9549
Watson U2	0,0144	0,9935

3. METHODOLOGY
3.1. COGARCH(1,1) process

COGARCH(1,1) process is defined as the solution to the following stochastic differential equations.

(2)

$$d\sigma_t^2 = (\beta - \eta\sigma_t^2)dt + \varphi\sigma_t^2 d[L,L]_t^{(d)}, t \geq 0 \quad (3)$$

Where $d[L,L]_t^{(d)} = (\Delta L_t)^2$ *and* $\Delta L_t = L_t - L_{t-1}$ L is Lévy process and $d[L,L]_t^{(d)}$ is discrete part of covariation. (Klüppelberg et al. 2009)

Assuming the data consist of intervals of time length equal to r, the equations 2 and 3 become

$$G_t^{(r)} = G_t - G_{t-r} = \int_{(t-r,t]} \sigma_s \mathbf{dL_s} \quad (4)$$

$$\sigma_{rn}^{2(r)} = \sigma_{rn}^{2(r)} - \sigma_{r(n-1)}^{2(r)} = \beta r - \eta \int_{(r(n-1),rn]} \sigma_s^2 ds + \varphi \int_{(r(n-1),rn]} \sigma_s^2 d[L,L]_s^{(d)} \quad (5)$$

For r = 1;

$$\sigma_n^2 = \sigma_{n-1}^2 + \beta - \eta \int_{(n-1,n]} \sigma_s^2 ds + \varphi \sum_{n-1<s\leq n} \sigma_s^2 (\Delta L_s)^2 \quad (6)$$

We approximate the integral and the sum on the right hand side of the equation. For the integral; we use a simple Euler Approximation

$$\int_{(n-1,n]} \sigma_s^2 ds \approx \sigma_{n-1}^2$$

We can also approximate the sum as;

$$\sum_{n-1<s\leq n} \sigma_s^2 (\Delta L_s)^2 = (G_n - G_{n-1})^2 = \left(G_n^{(1)}\right)^2$$

Thus we end up with an discretized version of equation 3

$$\sigma_n^2 = \beta + (1-\eta)\sigma_{n-1}^2 + \varphi\left(G_n^{(1)}\right)^2 \quad (6)$$

We can clearly see the anologue with the Garch(1,1) which is;

$$\sigma_n^2 = a + b\sigma_{n-1}^2 + c(Y_i)^2 \tag{7}$$

where $Y_i = \sigma_{i-1}\varepsilon_i$

3.2. Application

Discretized model for continuous time model is given by

$$G_{i,n} = G_{i-1,n} + \sigma_{i-1,n}\sqrt{\Delta t_i(n)}\varepsilon_{i,n} \tag{8}$$

$$\sigma_{i,n}^2 = \beta\Delta t_i(n) + \left(1 + \varphi\Delta t_i(n)\varepsilon_{i,n}^2\right)e^{-\eta\Delta t_i(n)}\sigma_{i-1,n}^2 \tag{9}$$

With some simplifications;

$$\sigma_{i,n}^2 = \beta\Delta t_i(n) + e^{-\eta\Delta t_i(n)}\sigma_{i-1,n}^2 + \varphi\Delta t_i(n)e^{-\eta\Delta t_i(n)}\sigma_{i-1,n}^2\varepsilon_{i,n}^2 \tag{10}$$

since the Garch(1,1) can be written as

$$\sigma_n^2 = \beta + \delta\sigma_{n-1}^2 + \lambda(Y_i)^2 \tag{11}$$

Then, with equation 10 and 11 we can find the parameters for the COGARCH(1,1) as follows

$$\beta = \beta \; ; \; e^{-\eta} = \delta \quad => \eta = -\ln(\delta) \; ; \; \varphi e^{-\eta} = \lambda \quad => \quad \varphi\delta = \lambda \quad => \varphi = \frac{\lambda}{\delta}$$

The model we have estimated from our data is

$$X_t = -0.200991X_{t-1} + \varepsilon_t$$

$$\sigma_n^2 = 0.01362 + 0.006847\sigma_{n-1}^2 + 0.419776\varepsilon_{n-1}^2$$

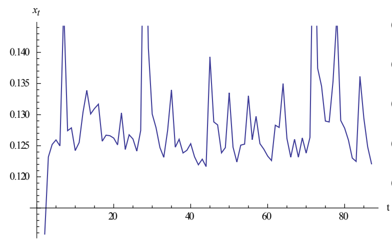
Thus; the continuous model parameters are;

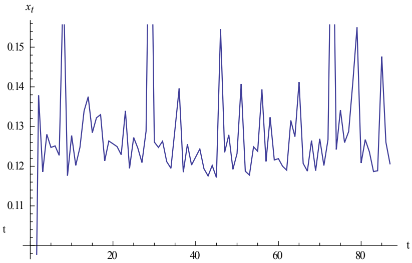
$$\beta = 0.01362 \; ; \; \eta = -\ln(0.006847) \; ; \; \varphi = \frac{0.419776}{0.006847}$$

We have used the above parameters with errors generated from the fitted Johnson SU distribution for simulating the COGARCH(1,1) process for mortality rates.

The results of the volatility process are presented in the table 4

Table 4 GARCH(1,1) and COGARCH(1,1) results





4. CONCLUSION

Mortality rates for UK from 1922 to 2009 was modeled by AR(1),GARCH(1,1). Then, using the parameters estimated for the discrete model; COGARCH(1,1) was applied to the data. The COGARCH was applied by estimating the distribution of errors without using jumps. The results indicate that the COGARCH model generates reasonable volatilities

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