CONTINUOUS-TIME GARCH (COGARCH) MODELING OF TURKISH INTEREST RATES

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─Abstract─

We proposed a continuous time GARCH known as COGARCH(p,q) model for modeling the volatility of Turkish interest rates. COGARCH (p,q) models have been statistically proven successful in capturing the heavy-tail behaviour of the interest rates . We demonstrate the capabilities of COGARCH(p,q) model by using Turkish short rate. The Turkish Republic Central Bank's benchmark bond prices are used to calculate the short-term interest rates between the period of 15.07.2006 and 15.07.2008. COGARCH(1,1) model is chosen as best candidate model in modeling the Turkish short rate for the sample period.

Key Words: *interest rate models, continuous-time models, stochastic volatility, Lévy process, GARCH , COGARCH*

JEL Classification: C01, C51

1. INTRODUCTION

Modeling the interest rate dynamics is an expanding research area because of theoretical as well as technical consideration. Economic agents, both private and public, monitor closely the evolution of the interest rate movements in order to make decisions. For this reason, modeling the interest rate dynamics plays important roles in especially macroeconomic policy decision makings, derivative pricing, and hedging and risk management for fixed income securities. The current literature has massive number of studies in this field. Some of them are

Ait-Sahalia (1996:1) (1999:2), Stanton (1997:14), Chapman and Pearson (1998:7), Hong et.al (1996:9), and Pritsker (1998:13).

These studies show some important properties of spot interest rates in developed financial markets, especially in the U.S. markets. According to the studies, there exist a significant mean reverting for the U.S. interest rates, although existence of a non-linear drift is inconclusive. Chan et.al. (1992:6) and Hong et.al. (1996:9) find that the interest rate volatility tends to be higher when the interest rate level is higher, which is often characterized by a constant eleasticity variance (CEV) specification. On the other hand, Gray (1996:8) points out that regime switching and jump models help capturing volatilty clustering and particularly the excess kurtosis and heavy tails of spot interest rates.

Moreover, it is also important to capture conditional heteroskedasticity of interest rates by stochastic volatility / GARCH models. In this paper, we move into this area by modeling the interest rate volatility with continuous-time GARCH (COGARCH) which was introduced by Klüpperg by using Turkish benchmark bond rate.

2. METHODOLOGY
2.1. Related Work
In financial econometrics, the majority of the volatility models are discrete-time GARCH models. The models have been widely used in modeling the stocks returns, currencies, interest rates and similar financial assets. These discrete-time models have been successful in capturing the characteristics of such financial data, for example, heavy tails, volatility clustering and dependencies without correlation. There have been several attempts to model these financial data by using continuous-time models. The continuous-time models have advantage over discrete-time models that they allow closed form solutions for options and other derivatives pricing.

One of the notable studies among these attempts is the GARCH diffusion approximation of Nelson (1990:12). Even though, GARCH process is driven by a single noise sequence, the diffusion limit is driven by two independent Brownian motions $(W_t^1)t \geq 0$ and $(W_t^2)t \geq 0$. The volatility process σ_t^2 and G_t take the form of;

$$dG_t - \sigma_t dW_t^1$$
$$d\sigma_t^2 - \theta(\gamma - \sigma_t^2) + \rho \sigma_t^2 dW_t^2, \ t \geq 0$$

As Nelson's diffusion limit, another important study in this area is Barndorff-Nielsen and Shephard's (2001:3) (2001:4) stochastic volatility model in which volatility process σ_t^2 is an Ornstein-Uhlenbeck (O-U) process driven by a nondecreasing (subordinator) Lévy process. More precisely, let $(L_t) t \geq 0$ be a subordinator and $\alpha > 0$. Then the volatility process $(\sigma_t^2) t \geq 0$ is defined by the stochastic differential equation

$$d\sigma_t^2 = -\alpha \sigma_t^2 dt + dL_t, t \geq 0$$

Where σ_t^2 is a finite random variable independent of $(L_t) t \geq 0$ and $\sigma_t = \sqrt{\sigma_t^2}$. The G_t satisfies the equation of the form

$$dG_t = (\mu + b\sigma_t^2)dt + \sigma_t dW_t, \quad t \geq 0, G_0 = 0$$

where μ and b are constants and $(W_t) t \geq 0$ is standard Brownian motion. As in Nelson's model, the process G is again driven by two independent noise processes and the volatility process σ_t^2 evolves independently of the process W in the equation for G.

2.2. COGARCH(1,1) Model

The models of Nelson, and Barndorff-Nielsen and Shepherd have two independent sources of uncertainty, whereas the discrete-time GARCH process is driven by a single white noise process. As Klüppelberg et. Al (2004:11) pointed out, in GARCH models, the idea of large innovations in the price processes are almost immediately manifested as innovations in the volatility process, but these feedback mechanism is lost in models such as the Nelson's continuous time version. Klüppelberg et.al (2004:10) introduced a different approach to a continuous-time model. The COGARCH model , is based on a single background driving Lévy process. Their construction is based on taking a limit of an explicit representation of the discrete-time GARCH(1,1) process to obtain a continuous-time analog.

The COGARCH process $(G_t)_{t \geq 0}$ is defined in terms of its stochastic differential dG, such that

$$dG_t = \sigma_t + dL_t \quad t \geq 0$$

$$d\sigma_t^2 = (\beta - \eta \sigma_{t-}^2)dt + \varphi \sigma_{t-}^2 d[L,L]_t^{(d)}, \quad t > 0$$

where $\beta > 0$, $\eta \geq 0$, and $\varphi \geq 0$ are constants.

$[L,L]_t^{(d)}$ is the quadratic variation process of L which is defined as

$$[L,L]_t^{(d)} = \sum_{0 < s < t}(\Delta L_s)^2 = \sum_{t=1}^{N_t} V_t$$

where $\Delta L_t = L_t - L_{t-1}$ for $t \geq 0$.

The process G 'jumps' at the same time as L does, and has jump sizes

$$\Delta G_t = \sigma_t \Delta dL_t \qquad t \geq 0$$

Klüppelberg (2004:10) shows the identity as

$$\sigma_t^2 = \beta t + \log(\delta)\int_0^2 \sigma_s^2 \, ds + \varphi \sum_{0 < s \leq t} \sigma_s^2 (\Delta dL_t)^2 + \sigma_0^2 \, t \geq 0$$

Deriving a recursive and deterministic approximation for the volatilities at the jump times we get

$$\sigma_t^2 = \sigma_{t-1}^2 - \beta + \eta \int_0^t \sigma_s^2 \, ds + \varphi \sum_{0 < s \leq t} \sigma_s^2 (\Delta dL_t)^2$$

since σ_s is latent and ΔL_s is usually not observable, hence using Euler approximation for the integral we get

$$\int_0^t \sigma_s^2 \, ds \approx \sigma_{t-1}^2$$

$$\sum_{0 < s \leq t} \sigma_s^2 (\Delta dL_t)^2 \approx (G_t - G_{t-1})^2$$

therefore for the volatility estimation we end up with

$$\sigma_t^2 = \beta + (1 - \eta)\sigma_{t-1}^2 + \varphi(G_t - G_{t-1})^2$$

The bivariate process $(\sigma_t, G_t)_{t \geq 0}$ is Markovian. If $(\sigma_t^2)_{t \geq 0}$ is the stationary version of the process with $\sigma_0^2 = \sigma_\infty^2$, then $(G_t)_{t \geq 0}$ is a process with stationary increments. (Klüppelberg, 2004:10).

3. DATA ANALYSIS

The data we are going to use in this paper are the daily series of Turkish Republic Central Bank's benchmark bond rates between the period of 15.07.2006 and 15.07.2008 (729) observations. The interest rates are calculated by using the benchmark bond prices of the sample period. The following figure exhibits the moving tendency of the Turkish benchmark bond rates for the sample period.

Figure 1: Time Series Plot of the TCMB Benchmark Bond Rates
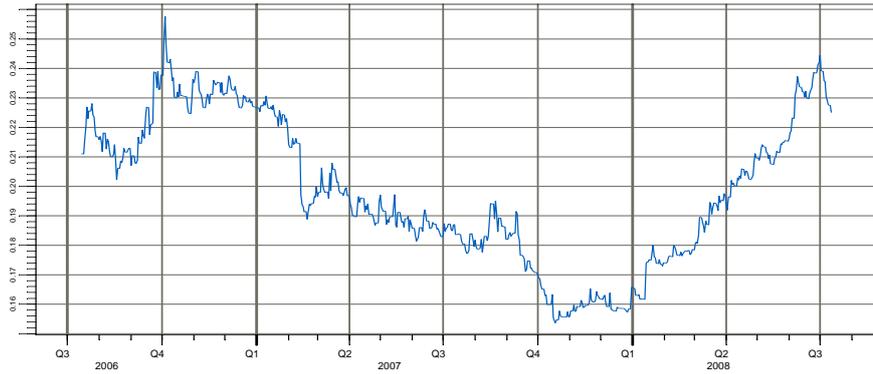
Source: Turkish Republic Central Bank's Web Page http://www.tcmb.gov.tr/

The figure 1 implies that the series are not stationary moreover the Augmented Dickey-Fuller test results confirm that by not rejecting the null hypothesis of the unit-root (Table 1).

Table 1: Unit-Root Test

		t-Statistic	Prob.*
Augmented Dickey-Fuller test statistic		0.092674	0.7118
Test critical values:	1% level	-2.568151	
	5% level	-1.941260	
	10% level	-1.616406	

Source: Own study

Since the original series have a unit-root, we need to transform them into stationary series by taking the first difference of them. The figure 2 shows the first difference of the series that we are going to use in our analysis.

Figure 2: Time Series Plot of the Benchmark Bond Rates (First Differences)
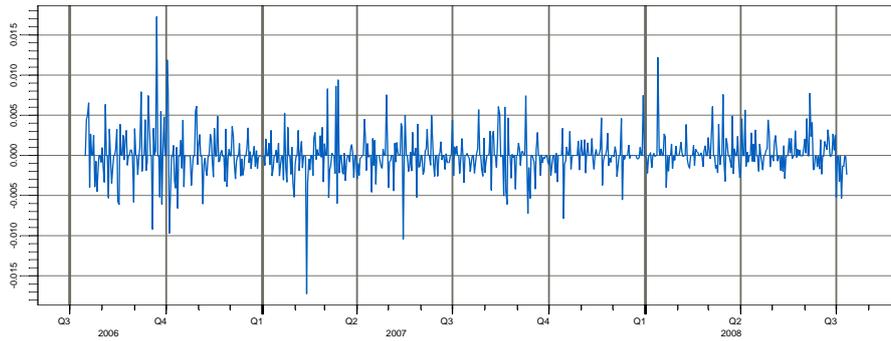
4. EMPIRICAL RESULTS

Before building a continuous-time GARCH model, a discrete-time GARCH model must be estimated. The Akaike's information criteria (AIC) suggests a GARCH(1,1) model for the series which satisfies the mean and variance equations of the form

$$r_t = c + a_t$$
$$\sigma_t^2 = \beta + \lambda a_{t-1}^2 + \delta \sigma_{t-1}^2$$

The table 2 shows the output for the estimated GARCH(1,1) model.

Table:2 Estimation Coefficients of GARCH(1,1) Model

	Value	Std.Error	t value	Pr(>|t|)
C	4.256e-005	9.346e-005	0.4554	0.64899197
A	2.972e-007	1.064e-007	2.7931	0.00535838
ARCH(1)	3.892e-002	9.217e-003	4.2228	0.00002721
GARCH(1)	9.162e-001	2.405e-002 3	8.0889	0.0000000
AIC(4) = -6633.159				
BIC(4) = -6614.798				

The estimated model obeys the non- negativity conditions of the GARCH process as all the coefficients are positive. The model also satisfies the covariance stationary condition that sum of coefficients is less than 1. Also, GARCH(1,1) coefficients are statistically significant in terms of t-value. The Ljung-Box serial correlation test with p-value 0.8031, and ARCH LM test with p-value 0.9759 indicate that there are no serial correlation and ARCH effect in the residuals.

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GARCH(1,1) model is therefore a satisfactory model for our sample data. The candidate model therefore reads as

$$r_t = 0.00004256 + a_t$$
$$\sigma_t^2 = 0.000002972 + 0.03892a_{t-1}^2 + 0.9162\sigma_{t-1}^2$$

After estimating an appropriate discrete-time GARCH process, the next step in the analysis would be estimating a continuous-time model by using discrete model parameters. The COGARCH(1,1) model which takes the form of

$$dG_t = \sigma_t + dL_t \quad t \geq 0$$
$$d\sigma_t^2 = (\beta - \eta\sigma_{t-}^2)dt + \varphi\sigma_{t-}^2 d[L,L]_t^{(d)} , \quad t > 0$$

The parameters of the continuous model are equal to

$$\beta = \beta, \quad \eta = -\ln\delta, \quad \varphi = \frac{\lambda}{\delta}.$$

The COGARCH(1,1) model parameters in this case are

$$\beta = 0.000002972, \quad \eta = 0.08752, \quad \varphi = 0.04247.$$

To start the simulation we use numerical solutions for G_t and σ_t^2 also use a Lévy process driven by Cauchy process. The figure 3 below shows the time plot of the simulated values of COGARCH(1,1) process. And the figure 4 shows the volatility processes generated by GARCH(1,1) and COGARCH(1,1) models compared to real volatility. As we can infer from the figure 4 that both models are successful in capturing the volatility clustering as both volatility processes mimic the real volatility.

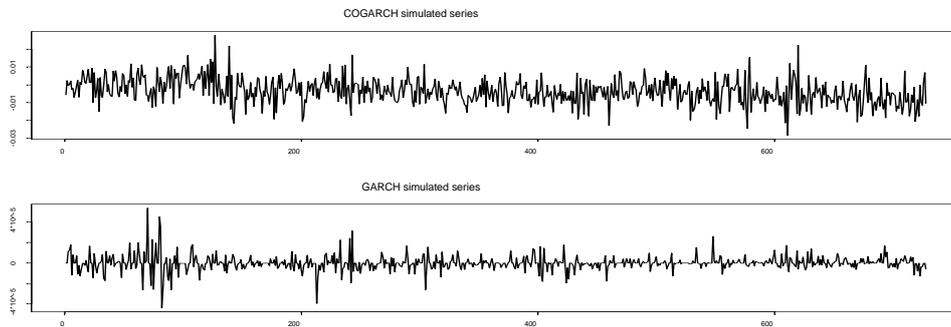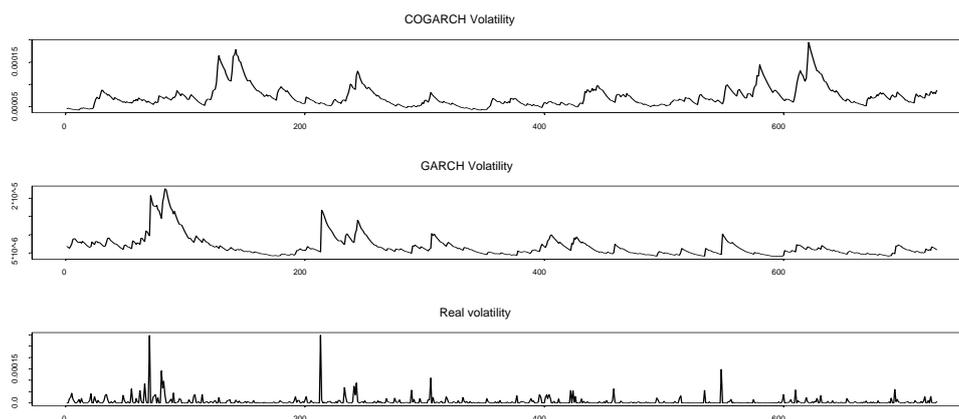Figure 3: COGARCH(1,1) and GARCH(1,1) Simulated Series

Figure 4: Real Volatility vs. GARCH(1,1) and COGARCH(1,1) Volatility

5. CONCLUSION

This paper is an analysis of the interest rate volatility with continuous-time GARCH processes. We have used daily data of the Turkish benchmark bond rate as sample data between the period of 15.07.2006 and 15.07.2008. The COGARCH(1,1) model is applied as canditate model for the sample data and it has been proven as a parsimonuous model in analyzing the characteristics of the Turkish interest rate volatility. Although, COGARCH(1,1) model is successful in capturing the stylized characteristics of the interest rate data, continuous-time models would be more suitable for modeling the asset returns and foreign exchange rates, because of the availability of high-frequency data for these financial series.

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